ELG4177 - DIGITAL SIGNAL PROCESSING
Tutorial 10

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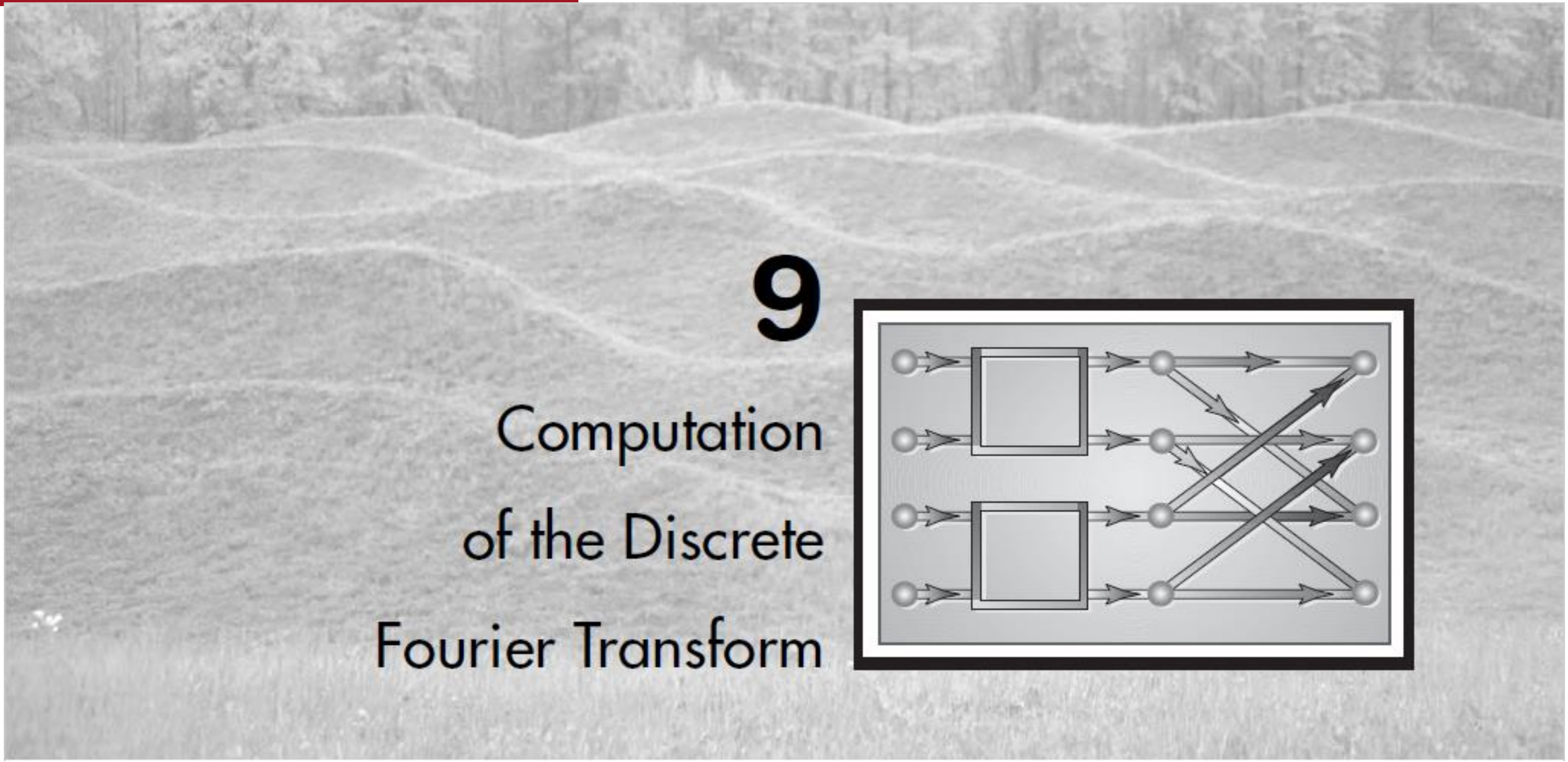

Problems: 9.3, 9.5, 9.6, 9.7, 9.13, 9.15, 9.16, 9.18, 9.19, 9.23, 9.26, 9.28

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9.3. Suppose that you time-reverse and delay a real-valued 32-point sequence $x[n]$ to obtain $x_1[n] = x[32-n]$. If $x_1[n]$ is used as the input for the system in Figure P9.4, find an expression for $y[32]$ in terms of $X(e^{j\omega})$, the DTFT of the original sequence $x[n]$.

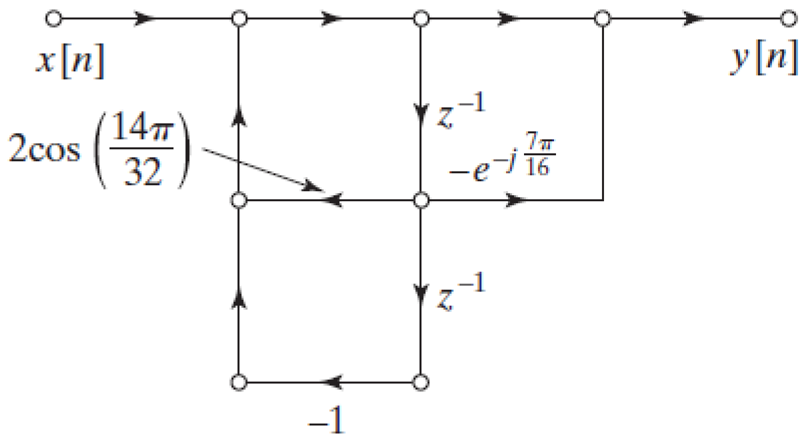

Figure P9.4

First, we derive a relationship between the $X_1(e^{j\omega})$ and $X(e^{j\omega})$ using the shift and time reversal properties of the DTFT.

$$
\begin{aligned}
x_1[n] &= x[32-n] \\
X_1(e^{j\omega}) &= X(e^{-j\omega})e^{-j32\omega}
\end{aligned}
$$

Looking at the figure we see that calculating $y[32]$ is just an application of the Goertzel algorithm with $k = 7$ and $N = 32$. Therefore,

$$
\begin{aligned}
y[32] &= X_1[7] \\
&= X_1(e^{j\omega})\Big|_{\omega=\frac{2\pi 7}{32}} \\
&= X(e^{-j\omega})e^{-j\omega 32}\Big|_{\omega=\frac{7\pi}{16}} \\
&= X(e^{-j\frac{7\pi}{16}})e^{-j(\frac{7\pi}{16})32} \\
&= X(e^{-j\frac{7\pi}{16}})
\end{aligned}
$$

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9.5. Consider the signal flow graph in Figure P9.5. Suppose that the input to the system $x[n]$ is an 8-point sequence. Choose the values of a and b such that $y[8] = X(e^{j6\pi/8})$.

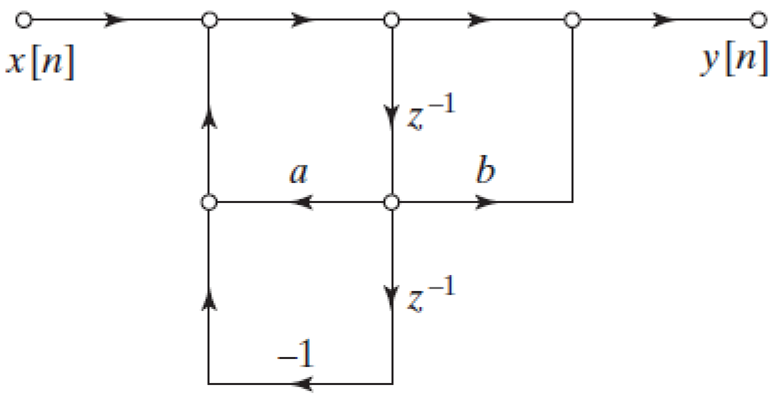


Figure P9.5

. $X(e^{j6\pi/8})$ corresponds to the $k = 3$ index of a length $N = 8$ DFT. Using the flow graph of the second-order recursive system for Goertzel's algorithm,

$$
\begin{aligned}
a &= 2\cos\left(\frac{2\pi k}{N}\right) \\
&= 2\cos\left(\frac{2\pi(3)}{8}\right) \\
&= -\sqrt{2} \\
b &= -W_N^k \\
&= -e^{-j6\pi/8} \\
&= \frac{1+j}{\sqrt{2}}
\end{aligned}
$$

9.6. Figure P9.6 shows the graph representation of a decimation-in-time FFT algorithm for $N = 8$. The heavy line shows a path from sample $x[7]$ to DFT sample $X[2]$.

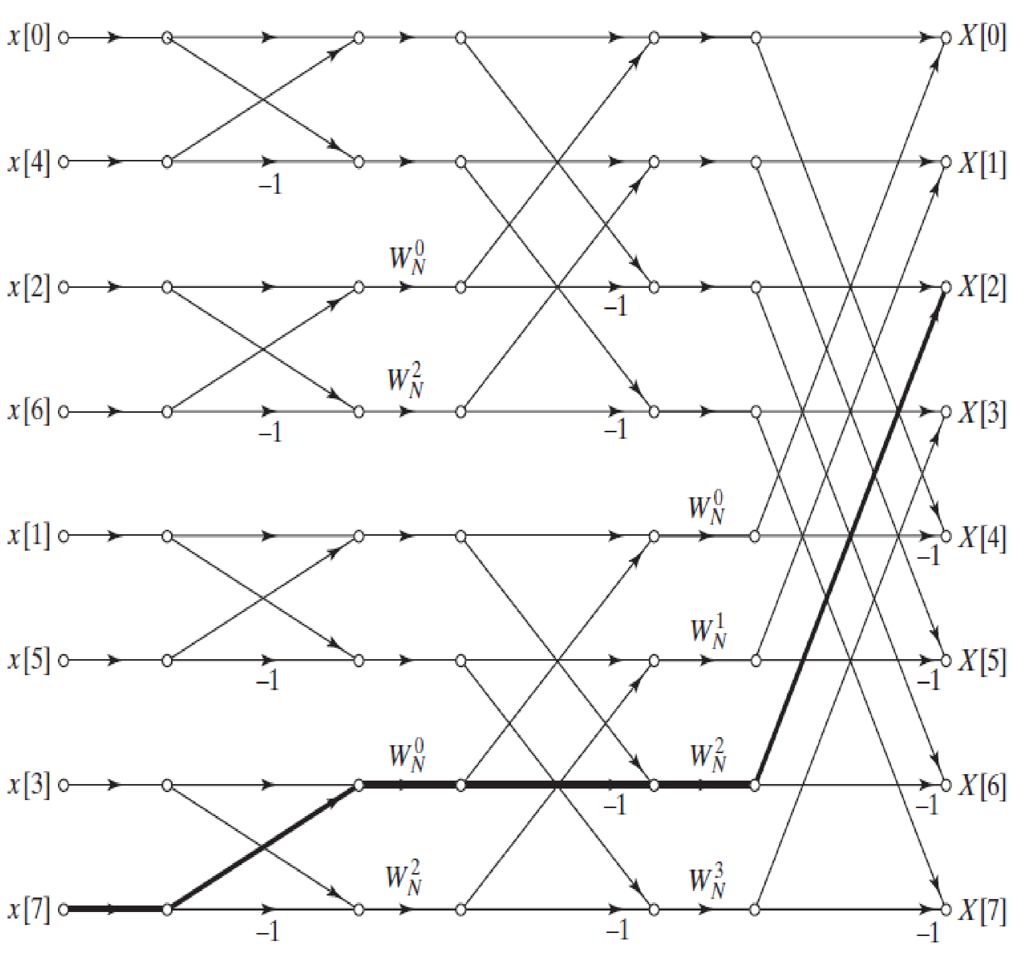


Figure P9.6

(a) What is the "gain" along the path that is emphasized in Figure P9.6?

(b) How many other paths in the flow graph begin at $x[7]$ and end at $X[2]$? Is this true in general? That is, how many paths are there between each input sample and each output sample?

(c) Now consider the DFT sample $X[2]$. By tracing paths in the flow graph of Figure P9.6, show that each input sample contributes the proper amount to the output DFT sample; i.e., verify that

$$X[2] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)2n}.$$

(a) The "gain" along the emphasized path is $-W_N^2$.

(b) In general, there is only one path between each input sample and each output sample.

(c) $x[0]$ to $X[2]$: The gain is 1.
$x[1]$ to $X[2]$: The gain is W_N^2.
$x[2]$ to $X[2]$: The gain is $-W_N^0 = -1$.
$x[3]$ to $X[2]$: The gain is $-W_N^0 W_N^2 = -W_N^2$.
$x[4]$ to $X[2]$: The gain is $W_N^0 = 1$.
$x[5]$ to $X[2]$: The gain is $W_N^0 W_N^2 = W_N^2$.
$x[6]$ to $X[2]$: The gain is $-W_N^0 W_N^0 = -1$.
$x[7]$ to $X[2]$: The gain is $-W_N^0 W_N^0 W_N^2 = -W_N^2$, as in Part (a).
Now

$$\begin{aligned} X[2] &= \sum_{n=0}^{7} x[n] W_8^{2n} \\ &= x[0] + x[1]W_8^2 + x[2]W_8^4 + x[3]W_8^6 + x[4]W_8^8 + x[5]W_8^{10} + x[6]W_8^{12} \\ &\quad + x[7]W_8^{14} \\ &= x[0] + x[1]W_8^2 + x[2](-1) + x[3](-W_8^2) + x[4](1) + x[5]W_8^2 \\ &\quad + x[6](-1) + x[7](-W_8^2) \end{aligned}$$

Each input sample contributes the proper amount to the output DFT sample.

9.7. Figure P9.7 shows the flow graph for an 8-point decimation-in-time FFT algorithm. Let $x[n]$ be the sequence whose DFT is $X[k]$. In the flow graph, $A[\cdot]$, $B[\cdot]$, $C[\cdot]$, and $D[\cdot]$ represent separate arrays that are indexed consecutively in the same order as the indicated nodes.

(a) Specify how the elements of the sequence $x[n]$ should be placed in the array $A[r]$, $r = 0, 1, \ldots, 7$. Also, specify how the elements of the DFT sequence should be extracted from the array $D[r]$, $r = 0, 1, \ldots, 7$.

(b) Without determining the values in the intermediate arrays, $B[\cdot]$ and $C[\cdot]$, determine and sketch the array sequence $D[r], r = 0, 1, \ldots, 7$, if the input sequence is $x[n] = (-W_N)^n, n = 0, 1, \ldots, 7$.

(c) Determine and sketch the sequence $C[r]$, $r = 0, 1, \ldots, 7$, if the output Fourier transform is $X[k] = 1, k = 0, 1, \ldots, 7$.

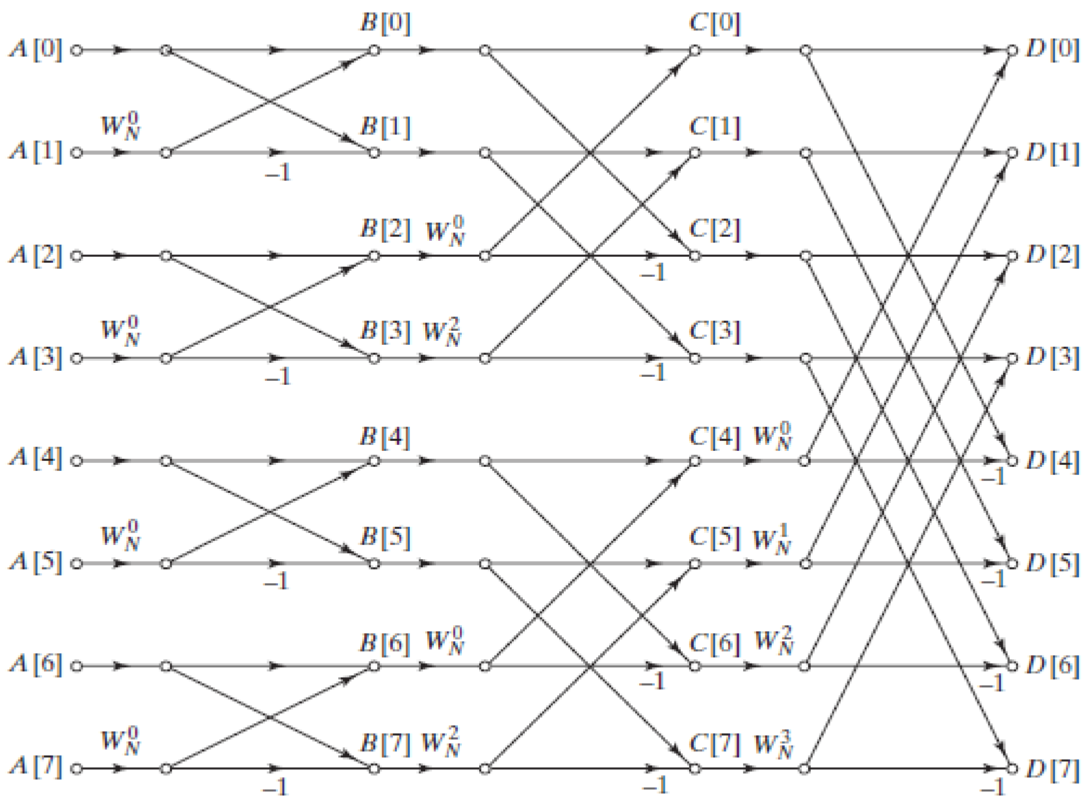


Figure P9.7

(a) The input should be placed into A[r] in bit-reversed order.

$$\begin{aligned} A[0] &= x[0] \\ A[1] &= x[4] \\ A[2] &= x[2] \\ A[3] &= x[6] \\ A[4] &= x[1] \\ A[5] &= x[5] \\ A[6] &= x[3] \\ A[7] &= x[7] \end{aligned}$$

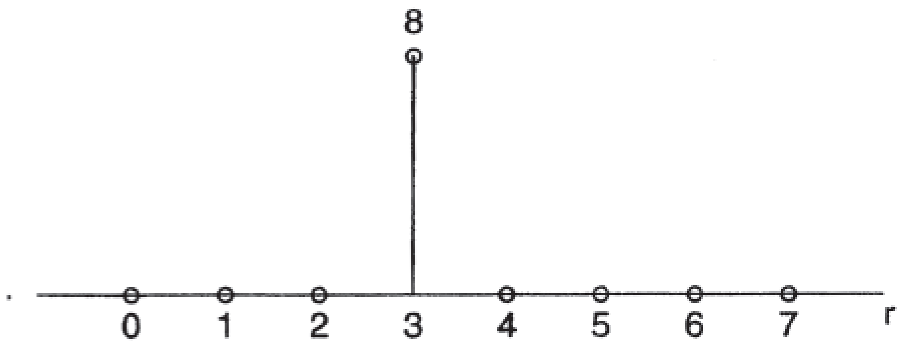
The output should then be extracted from D[r] in sequential order.

$$X[k] = D[k], \quad k = 0, \ldots, 7$$

(b) First, we find the DFT of $(-W_N)^n$ for $N = 8$.

$$\begin{aligned} X[k] &= \sum_{n=0}^{7} (-W_8)^n W_8^{nk} \\ &= \sum_{n=0}^{7} (-1)^n W_8^n W_8^{nk} \\ &= \sum_{n=0}^{7} (W_8^{-4})^n W_8^n W_8^{nk} \\ &= \sum_{n=0}^{7} W_8^{n(k-3)} \\ &= \frac{1 - W^{k-3}}{1 - W_8^{k-3}} \\ &= 8\delta[k-3] \end{aligned}$$

A sketch of D[r] wis provided below.



(c) First, the array $D[r]$ is expressed in terms of $C[r]$.

$$\begin{aligned} D[0] &= C[0] + C[4] \\ D[1] &= C[1] + C[5]W_8^1 \\ D[2] &= C[2] + C[6]W_8^2 \\ D[3] &= C[3] + C[7]W_8^3 \\ D[4] &= C[0] - C[4] \\ D[5] &= C[1] - C[5]W_8^1 \\ D[6] &= C[2] - C[6]W_8^2 \\ D[7] &= C[3] - C[7]W_8^3 \end{aligned}$$

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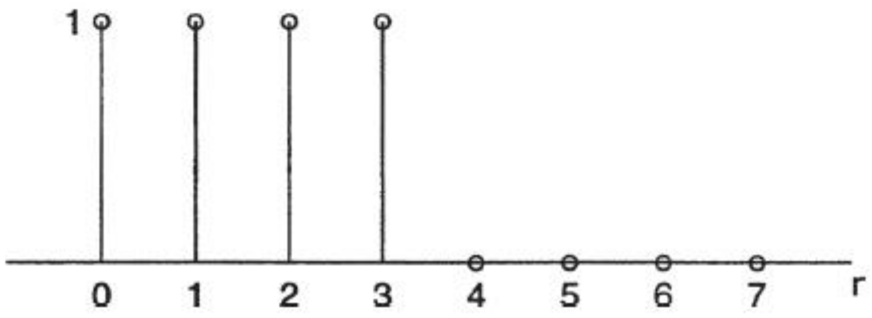
Solving this system of equations for $C[r]$ gives

$$\begin{aligned}
C[0] &= (D[0] + D[4])/2 \\
C[1] &= (D[1] + D[5])/2 \\
C[2] &= (D[2] + D[6])/2 \\
C[3] &= (D[3] + D[7])/2 \\
C[4] &= (D[0] - D[4])/2 \\
C[5] &= (D[1] - D[5])W_8^{-1}/2 \\
C[6] &= (D[2] - D[6])W_8^{-2}/2 \\
C[7] &= (D[3] - D[7])W_8^{-3}/2
\end{aligned}$$

for $r = 0, 1, \ldots, 7$. A sketch of C[r] is provided below.



9.13. Assume that you wish to sort a sequence $x[n]$ of length $N = 16$ into bit-reversed order for input to an FFT algorithm. Give the new sample order for the bit-reversed sequence.

Reversing the bits (denoted by $\rightarrow$) gives

$$\begin{aligned}
0 &= 0000 \rightarrow 0000 = 0 \\
1 &= 0001 \rightarrow 1000 = 8 \\
2 &= 0010 \rightarrow 0100 = 4 \\
3 &= 0011 \rightarrow 1100 = 12 \\
4 &= 0100 \rightarrow 0010 = 2 \\
5 &= 0101 \rightarrow 1010 = 10 \\
6 &= 0110 \rightarrow 0110 = 6 \\
7 &= 0111 \rightarrow 1110 = 14 \\
8 &= 1000 \rightarrow 0001 = 1 \\
9 &= 1001 \rightarrow 1001 = 9 \\
10 &= 1010 \rightarrow 0101 = 5 \\
11 &= 1011 \rightarrow 1101 = 13 \\
12 &= 1100 \rightarrow 0011 = 3 \\
13 &= 1101 \rightarrow 1011 = 11 \\
14 &= 1110 \rightarrow 0111 = 7 \\
15 &= 1111 \rightarrow 1111 = 15
\end{aligned}$$

The new sample order is 0, 8, 4, 12, 2, 10, 6, 14, 1, 9, 5, 13, 3, 11, 7, 15.

9.15. The butterfly in Figure P9.15 was taken from a decimation-in-frequency FFT with $N = 16$, where the input sequence was arranged in normal order. Note that a 16-point FFT will have four stages, indexed $m = 1, \ldots, 4$. Which of the four stages have butterflies of this form? Justify your answer.

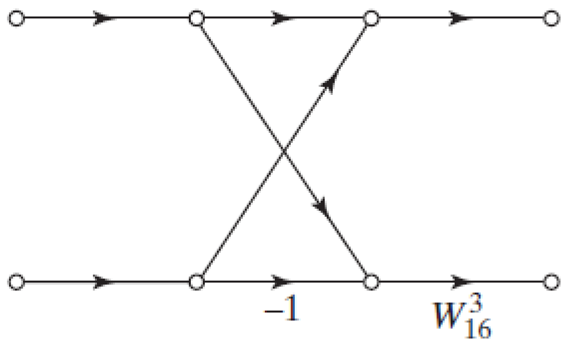


Figure P9.15

Solution

Only the $m = 1$ stage will have this form. No other stage of a $N = 16$ radix-2 decimation-in-frequency FFT will have a W_{16} term raised to an odd power.

9.16. The butterfly in Figure P9.16 was taken from a decimation-in-time FFT with $N = 16$. Assume that the four stages of the signal flow graph are indexed by $m = 1, \ldots, 4$. What are the possible values of r for each of the four stages?

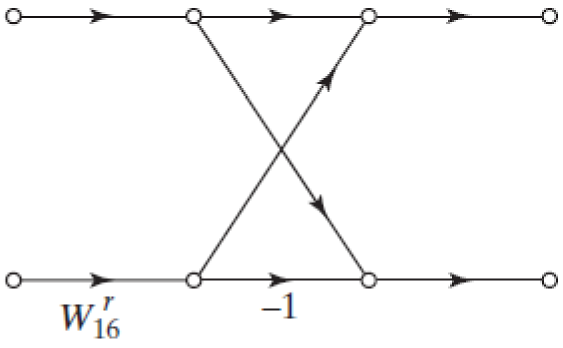


Figure P9.16

Solution

The possible values of r for each of the four stages are

$$\begin{aligned} m &= 1, & r &= 0 \\ m &= 2, & r &= 0, 4 \\ m &= 3, & r &= 0, 2, 4, 6 \\ m &= 4, & r &= 0, 1, 2, 3, 4, 5, 6, 7 \end{aligned}$$

9.18. The butterfly in Figure P9.18 was taken from a decimation-in-time FFT with $N = 16$. Assume that the four stages of the signal flow graph are indexed by $m = 1, \ldots, 4$. Which of the four stages have butterflies of this form?

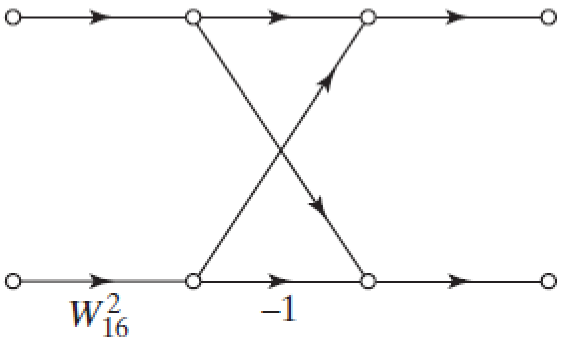


Figure P9.18

Solution

The possible values for r for each of the four stages are

$$
\begin{aligned}
m &= 1, & r &= 0 \\
m &= 2, & r &= 0, 4 \\
m &= 3, & r &= 0, 2, 4, 6 \\
m &= 4, & r &= 0, 1, 2, 3, 4, 5, 6, 7
\end{aligned}
$$

where W_N^r is the twiddle factor for each stage. Since the particular butterfly shown has $r = 2$, the stages which have this butterfly are

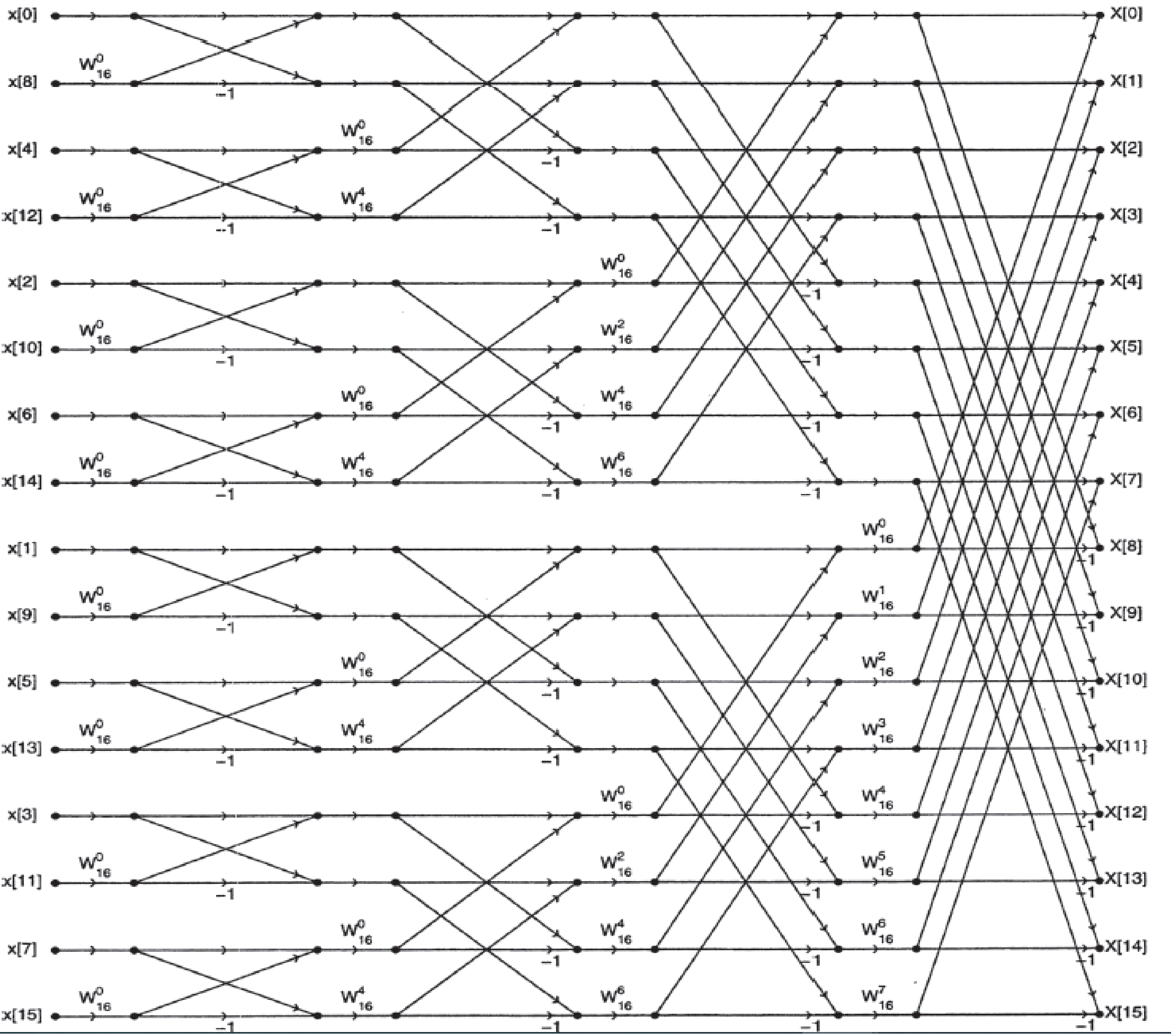
$$m = 3, 4$$

9.19. Suppose you are told that an $N = 32$ FFT algorithm has a "twiddle" factor of W_{32}^{2} for one of the butterflies in its fifth (last) stage. Is the FFT a decimation-in-time or decimation-in-frequency algorithm?

Solution

The FFT is a decimation-in-time algorithm, since the decimation-in-frequency algorithm has only W_{32}^{0} terms in the last stage.

9.23. Construct a flow graph for a 16-point radix-2 decimation-in-time FFT algorithm. Label all multipliers in terms of powers of W_{16}, and also label any branch transmittances that are equal to -1. Label the input and output nodes with the appropriate values of the input and DFT sequences, respectively.


x[0]
x[8]
x[4]
x[12]
x[2]
x[10]
x[6]
x[14]
x[1]
x[9]
x[5]
x[13]
x[3]
x[11]
x[7]
x[15]
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]
X[8]
X[9]
X[10]
X[11]
X[12]
X[13]
X[14]
X[15]
W_{16}^{0}
W_{16}^{1}
W_{16}^{2}
W_{16}^{3}
W_{16}^{4}
W_{16}^{5}
W_{16}^{6}
W_{16}^{7}
–1

9.26. We are given a finite-length sequence $x[n]$ of length 627 (i.e., $x[n] = 0$ for $n < 0$ and $n > 626$), and we have available a program that will compute the DFT of a sequence of any length $N = 2^{\nu}$.

For the given sequence, we want to compute samples of the DTFT at frequencies

$$\omega_k = \frac{2\pi}{627} + \frac{2\pi k}{256}, \qquad k = 0, 1, \ldots, 255.$$

Specify how to obtain a new sequence $y[n]$ from $x[n]$ such that the desired frequency samples can be obtained by applying the available FFT program to $y[n]$ with ν *as small as possible*.

Solution

Let

$$y[n] = e^{-j2\pi n/627}x[n]$$

Then

$$Y(e^{j\omega}) = X(e^{j(\omega+\frac{2\pi}{627})})$$

Let $y'[n] = \sum_{m=-\infty}^{\infty} y[n+256m]$, $0 \le n \le 255$, and let $Y'[k]$ be the 256 point DFT of $y'[n]$. Then

$$Y'[k] = X\left(e^{j(\frac{2\pi k}{256}+\frac{2\pi}{627})}\right)$$

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9.28. You are asked to build a system that computes the DFT of a 4-point sequence

$$x[0], x[1], x[2], x[3].$$

You can purchase any number of computational units at the per-unit cost shown in Table 9.1.

TABLE 9.1

Module	Per-Unit Cost
8-point DFT	$1
8-point IDFT	$1
adder	$10
multiplier	$100

Design a system of the lowest possible cost. Draw the associated block diagram and indicate the system cost.

A. The DFT of the 4-point sequence $x[0], x[1], x[2], x[3]$ is given by

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{4}}, \quad k = 0,1,2,3,$$

where

$$X(e^{j\omega}) = \sum_{n=0}^{3} x[n] e^{-j\omega n}.$$

The 8-point DFT of the sequence $x[0], x[1], x[2], x[3], 0, 0, 0, 0$ is given by

$$\hat{X}[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{8}}, \quad k = 0, \ldots, 7,$$

where $X(e^{j\omega})$ is as above. We see that

$$X[k] = \hat{X}[2k], \quad k = 0,1,2,3.$$

The cost of the system is the cost of the 8-point DFT, which is \$1.

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