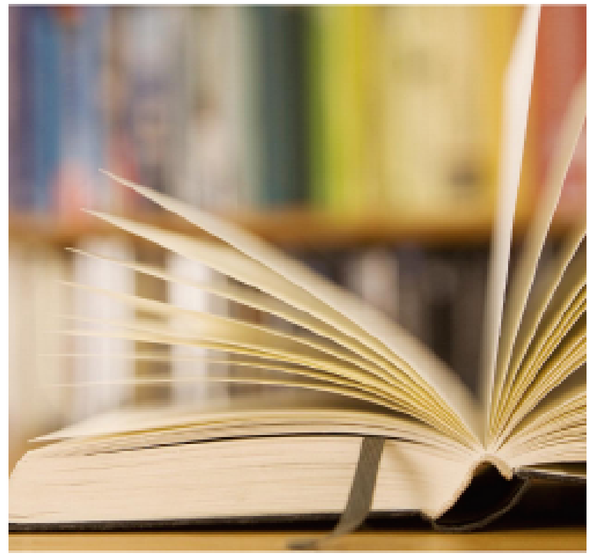
TUTORIAL
ELG3125B:SIGNAL AND SYSTEM ANALYSIS

Chapter (9)



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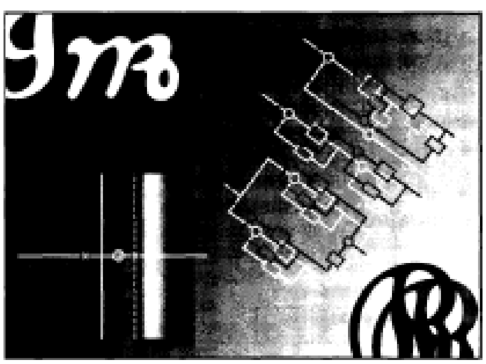
THE LAPLACE TRANSFORM



Laplace Transform

Laplace transform is often a rational of polynomials

$$X(s) = \frac{N(s)}{D(s)} = \text{algebraic expression}; \quad \mathrm{X(s)} = \frac{2s^2 + 5s + 2}{(s^2 + 5s + 10)(s + 2)}$$

- $N(s) = a_n s^n + \ldots + a_1 s + a_0$
- $D(s) = b_m s^m + \ldots + b_1 s + b_0$
- Poles (signularities) : $s \ni D(s) = 0$ (So, $X(s) = \infty$)
- Zeroes : $s \ni N(s) = 0$ (So, $X(s) = 0$)
- Poles and zeroes are complex
- Order of $X(s)$ is the number of poles $= m$
- Poles & Zeros of X(s) :

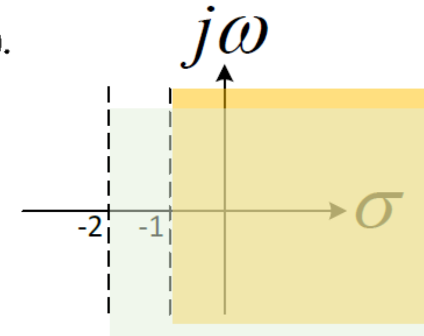
  Completely characterize the algebraic expression of X(s)

|X(s)| will be larger when it is closer to the poles

|X(s)| will be smaller when it is closer to the zeros

example consider a signal that is the sum of two real exponentials:

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t).$$



Solution:

$$e^{-t}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \mathfrak{Re}\{s\} > -1$$

$$e^{-2t}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathfrak{Re}\{s\} > -2$$

Laplace transforms of both terms converge is

Re{s} > **-1,** and thus, combining the two terms, we obtain

$$3e^{-2t}u(t) - 2e^{-t}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{s-1}{s^2+3s+2}, \quad \mathfrak{Re}\{s\} > -1.$$

Example: $x(t) = e^{-2t}u(t) - e^{-t}u(-t)$

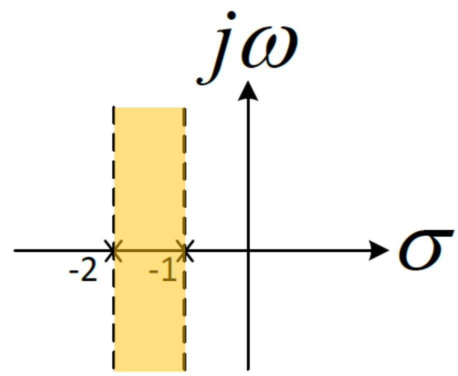
Solution:

$$e^{-2t}u(t) \leftrightarrow \frac{1}{2+s}, \mathrm{Re}\{s\} > -2$$

$$-e^{-t}u(-t) \leftrightarrow \frac{1}{1+s}, \mathrm{Re}\{s\} < -1$$

$$X(s) = \frac{1}{2+s} + \frac{1}{1+s}, -2 < \mathrm{Re}\{s\} < -1$$

As can be seen that ROC is a strip parallel to the $j\omega$ axis.

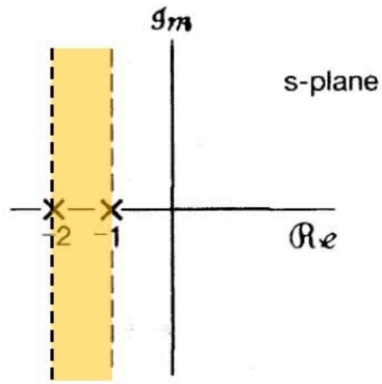


Example : If the ROC is $-2 < \text{Re}\{s\} < -1$, what is the inverse LT?

Solution:

Based on $-e^{-at}u(-t) \overset{L}{\longleftrightarrow} \frac{1}{a+s}, \text{Re}\{s\} < -a$, we have

$$-e^{-t}u(-t) \overset{L}{\longleftrightarrow} \frac{1}{s+1}, \text{Re}\{s\} < -1$$



Based on $e^{-at}u(t) \overset{L}{\longleftrightarrow} \frac{1}{a+s}, \text{Re}\{s\} > -a$, we have

$$e^{-2t}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+2}, \text{Re}\{s\} > -2$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}, -2 < \text{Re}\{s\} < -1$$

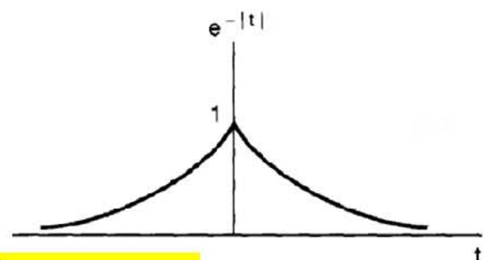
$$\rightarrow x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$$

Example: Consider a system with impulse response.

$$h(t) = e^{-|t|}$$

Since $h(t) \neq 0$ for $t < 0$, this system is not causal.

$$h(t) = e^{-t|} = e^{\ t}u(t) + e^{t}u(-t)$$



Considering

$$e^{-t}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+1}, \text{Re}\{s\} > -1 \quad -e^{t}u(-t) \overset{L}{\longleftrightarrow} \frac{1}{s-1}, \text{Re}\{s\} < 1$$

The LT transform of the system function:

$$H(s) = \frac{1}{s+1} - \frac{1}{s-1}, -1 < \text{Re}\{s\} < 1$$

$$= -\frac{2}{s^2-1}, -1 < \text{Re}\{s\} < 1$$

Thus, $H(s)$ is rational and has an ROC that is *not* to the right of the rightmost pole, the system is not causal.

Example: A continuous-time causal LTI system has a transfer function given below :

$$H(s)=\frac{\frac{101}{100}(s+j10)(s-j10)}{s^2+2s+101}$$

(a) Find the poles and the zeros.

(b) Indicate the poles and the zeros on the s-plane. Indicate the region of convergence (ROC).

(c) Write the differential equation of the system.

(d) Prove that the gain of the system at dc is unity (or the gain is 1 at dc).

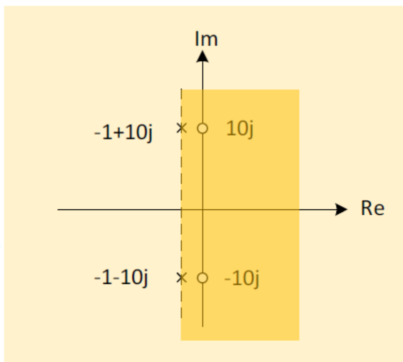
Solution:

a) To fond the poles, we let $s^2+2s+101=0$, we have the solutions given by:

$$s_{p1,p2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{2^2-4\times1\times101}}{2\times1}=\frac{-2\pm\sqrt{-400}}{2}=\frac{-2\pm j20}{2}=-1\pm j10$$

Then we have

$$H(s)=\frac{\frac{101}{100}(s+j10)(s-j10)}{s^2+2s+101}=\frac{\frac{101}{100}(s+j10)(s-j10)}{(s+1+10j)(s+1-10j)}$$



The zeros and the poles are shown in the figure.

b) The ROC is $\mathrm{Re}\{s\}>-1$, considering the system is causal.

c) $$H(s)=\frac{\frac{101}{100}(s+j10)(s-j10)}{s^2+2s+101}=\frac{\frac{101}{100}s^2+101}{s^2+2s+101}$$

$$\frac{Y(s)}{X(s)}==\frac{\frac{101}{100}s^2+101}{s^2+2s+101}$$

$$Y(s)s^2+2sY(s)+101Y(s)=\frac{101}{100}s^2X(s)+101X(s)$$

$$\frac{dy^2(t)}{dt^2}+2\frac{dy(t)}{dt}+101y(t)=\frac{101}{100}\frac{dx^2(t)}{dt^2}+101x(t)$$

d) Let $s=j\omega$, we have $H(j\omega)=\dfrac{\frac{101}{100}(j\omega+j10)(j\omega-j10)}{(j\omega)^2+2j\omega+101}$

At dc, we have $\omega=0$, $H(j0)=\dfrac{\frac{101}{100}(j0+j10)(j0-j10)}{(j0)^2+2j0+101}=1$, the gain at dc is 1.

9.21. Determine the Laplace transform and the associated region of convergence and pole-zero plot for each of the following functions of time:

(a) $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$ **(b)** $x(t) = e^{-4t}u(t) + e^{-5t}(\sin 5t)u(t)$

(c) $x(t) = e^{2t}u(-t) + e^{3t}u(-t)$ **(d)** $x(t) = te^{-2|t|}$

(e) $x(t) = |t|e^{-2|t|}$

(a) $e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \dfrac{1}{s+2}$ $\mathcal{R}e\{s\} > -2.$ & $e^{-3t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \dfrac{1}{s+3}$ $\mathcal{R}e\{s\} > -3.$

(b) $e^{-4t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \dfrac{1}{s+4},$ $\mathcal{R}e\{s\} > -4.$

Also, $e^{-5t}e^{j5t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \dfrac{1}{s+5-j5},$ $\mathcal{R}e\{s\} > -5.$

and $e^{-5t}e^{-j5t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \dfrac{1}{s+5+j5},$ $\mathcal{R}e\{s\} > -5.$

From this we obtain

$$e^{-5t}\sin(5t)u(t) = \frac{1}{2j}[e^{-5t}e^{j5t} - e^{-5t}e^{-j5t}]u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{5}{(s+5)^2+25},$$

where $\mathcal{R}e\{s\} > -5$. Therefore,

$$e^{-4t}u(t) + e^{-5t}\sin(5t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s^2+15s+70}{s^3+14s^2+90s+100}, \quad \mathcal{R}e\{s\} > -4.$$

(c) The Laplace transform of $x(t)$ is

$$\begin{aligned} X(s) &= \int_{-\infty}^{0} (e^{2t} + e^{3t})e^{-st}dt \\ &= [-e^{(s-2)t}/(s-2)]|_{-\infty}^{0} + [-e^{-(s-3)t}/(s-3)]|_{-\infty}^{0} \\ &= \frac{1}{s-2} + \frac{1}{s-3} = \frac{2s-5}{s^2-5s+6} \end{aligned}$$

The region of convergence (ROC) is $\mathcal{R}e\{s\} < 2$.

(d)

$$e^{-2t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \qquad \mathcal{R}e\{s\} > -2.$$

Using an approach along the lines of part (c), we obtain

$$e^{2t}u(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} -\frac{1}{s-2}, \quad \mathcal{R}e\{s\} < 2.$$

From these we obtain

$$e^{-2|t|} = e^{-2t}u(t) + e^{2t}u(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{-4}{s^2-4} \qquad -2 < \mathcal{R}e\{s\} < 2.$$

Using the differentiation in the s-domain property, we obtain

$$te^{-2|t|} \stackrel{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}\left[\frac{-4}{s^2-4}\right] = -\frac{-8s}{(s^2-4)^2}, \qquad -2 < \mathcal{R}e\{s\} < 2.$$

(e)

Using the differentiation in the s-domain property on eq. (S9.21-1),

$$te^{-2t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}\left[\frac{1}{s+2}\right] = \frac{1}{(s+2)^2}, \qquad \mathcal{R}e\{s\} > -2.$$

Using the differentiation in the s-domain property on eq. (S9.21-2),

$$-te^{2t}u(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds}\left[\frac{1}{s-2}\right] = -\frac{1}{(s-2)^2}, \qquad \mathcal{R}e\{s\} < 2.$$

Therefore,

$$|t|e^{-2|t|} = te^{-2t}u(t) + -te^{2t}u(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{-4s}{(s+2)^2(s-2)^2}, \qquad -2 < \mathcal{R}e\{s\} < 2.$$

9.22. Determine the function of time, $x(t)$, for each of the following Laplace transforms and their associated regions of convergence:

(a) $\frac{1}{s^2+9}$, $\mathcal{R}e\{s\} > 0$

(b) $\frac{s}{s^2+9}$, $\mathcal{R}e\{s\} < 0$

(c) $\frac{s+1}{(s+1)^2+9}$, $\mathcal{R}e\{s\} < -1$

(d) $\frac{s+2}{s^2+7s+12}$, $-4 < \mathcal{R}e\{s\} < -3$

(g) $\frac{s^2-s+1}{(s+1)^2}$, $\mathcal{R}e\{s\} > -1$

(a) From Table 9.2, we have

$$x(t) = \frac{1}{3}\sin(3t)u(t).$$

(b) From Table 9.2 we know that

$$\cos(3t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+9}, \qquad \mathcal{R}e\{s\} > 0.$$

Using the time scaling property, we obtain

$$\cos(3t)u(-t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{s}{s^2+9}, \qquad \mathcal{R}e\{s\} < 0.$$

Therefore, the inverse Laplace transform of $X(s)$ is

$$x(t) = -\cos(3t)u(-t).$$

(c) From Table 9.2 we know that

$$e^t\cos(3t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s-1}{(s-1)^2+9}, \qquad \mathcal{R}e\{s\} > 1.$$

Using the time scaling property, we obtain

$$e^{-t}\cos(3t)u(-t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{s+1}{(s+1)^2+9}, \qquad \mathcal{R}e\{s\} < -1.$$

Therefore, the inverse Laplace transform of $X(s)$ is

$$x(t) = -e^{-t}\cos(3t)u(-t).$$

(d) Using partial fraction expansion on $X(s)$, we obtain

$$X(s) = \frac{2}{s+4} - \frac{1}{s+3}.$$

From the given ROC, we know that $x(t)$ must be a two-sided signal. Therefore,

$$x(t) = 2e^{-4t}u(t) + e^{-3t}u(-t).$$

(g) We may rewrite $X(s)$ as

$$X(s) = 1 - \frac{3s}{(s+1)^2}.$$

From Table 9.2, we know that

$$tu(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^2}, \qquad \mathcal{R}e\{s\} > 0.$$

Using the shifting property, we obtain

$$e^{-t}tu(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)^2}, \qquad \mathcal{R}e\{s\} > -1.$$

Using the differentiation property,

$$\frac{d}{dt}[e^{-t}tu(t)] = e^{-t}u(t) - te^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{(s+1)^2}, \qquad \mathcal{R}e\{s\} > -1.$$

Therefore,

$$x(t) = \delta(t) - 3e^{-t}u(t) - 3te^{-t}u(t).$$