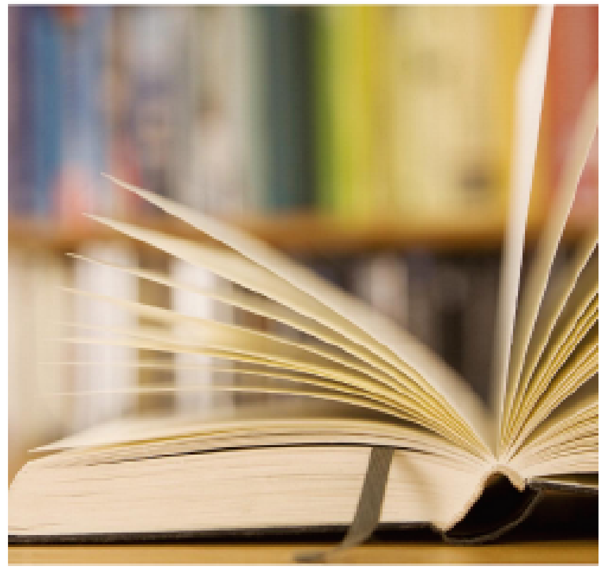
TUTORIAL

ELG3125B:SIGNAL AND SYSTEM ANALYSIS

Chapter (7)

By: Hitham Jleed



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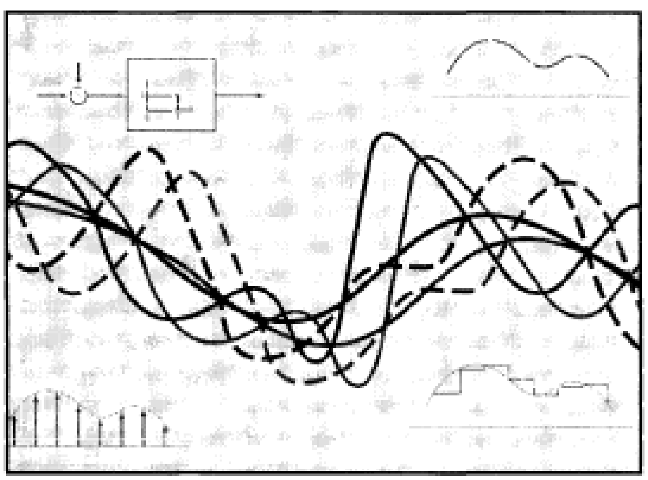
SAMPLING



[1] The sequence $x[n] = (-1)^n$ is obtained by sampling the continuous-time sinusoidal signal $x(t) = \cos\omega_0 t$ at 1-ms intervals, i.e.,

$$\cos(\omega_0 nT) = (-1)^n, \qquad T = 10^{-3}\ \text{s}$$

Determine three *distinct* possible values of ω_0.

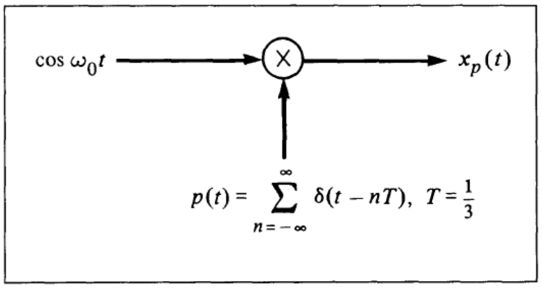
If $\omega_0 = \pi \times 10^3$, then

$$\cos(\omega_0 n \times 10^{-3}) = \cos(\pi n) = (-1)^n$$

Similarly, for $\omega_0 = 3\pi \times 10^{-3}$ and $\omega_0 = 5\pi \times 10^{-3}$,

$$\cos(\omega_0 n \times 10^{-3}) = (-1)^n$$

[2] Consider the system



(a) Sketch $X_p(\omega)$ for $-9\pi \le \omega \le 9\pi$ for the following values of ω_0.

(i) $\omega_0 = \pi$

(ii) $\omega_0 = 2\pi$

(iii) $\omega_0 = 3\pi$

(iv) $\omega_0 = 5\pi$

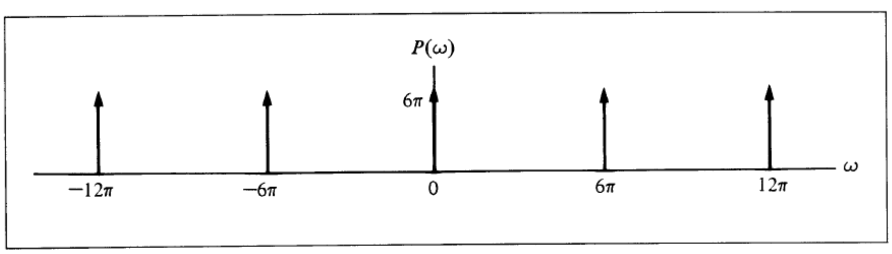
(b) For which of the preceding values of ω_0 is $x_p(t)$ identical?

The sampling function

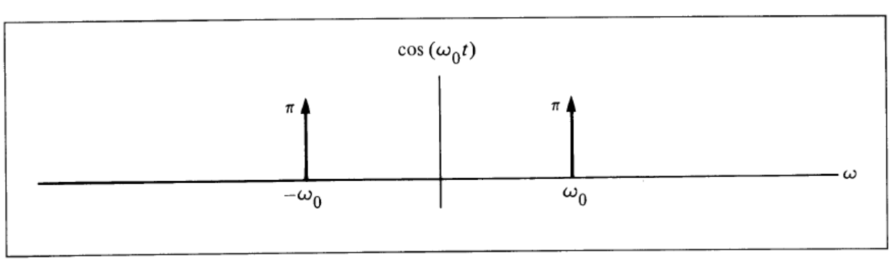
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \qquad T = \tfrac{1}{3},$$

has a spectrum given by

$$P(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$
$$= 6\pi \sum_{n=-\infty}^{\infty} \delta(\omega - 6\pi k),$$



$\cos(\omega_0 t)$ has a spectrum given by $\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$,

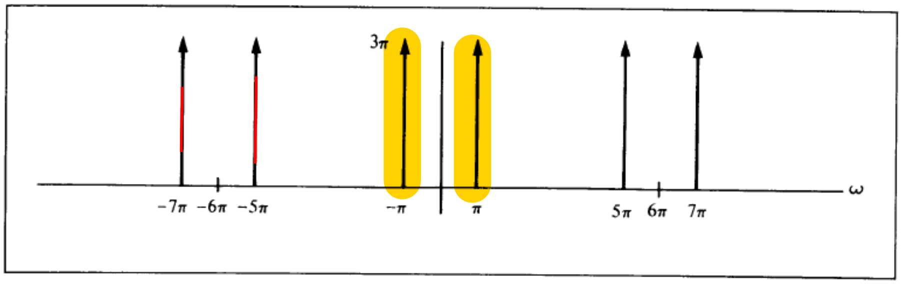


From the convolution theorem

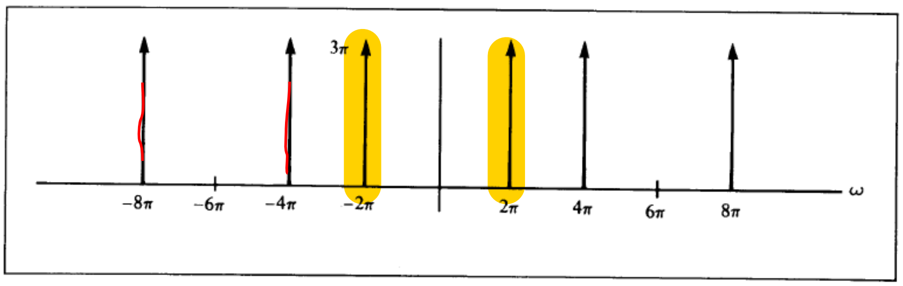
$$X_p(\omega) = \frac{1}{2\pi} P(\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

Hence, it is straightforward to find $X_p(\omega)$.

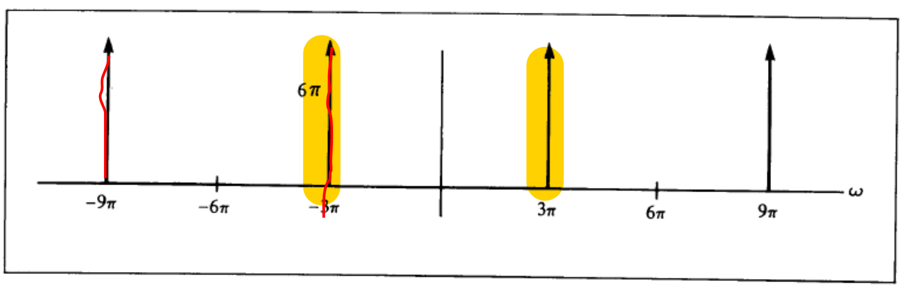
(a) (i) For $\omega_0 = \pi$:



(ii) For $\omega_0 = 2\pi$:



(iii) For $\omega_0 = 3\pi$:



(iv) For $\omega_0 = 5\pi$:

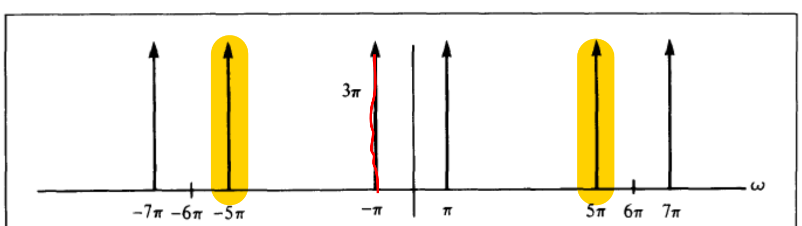


Figure S16.2-6

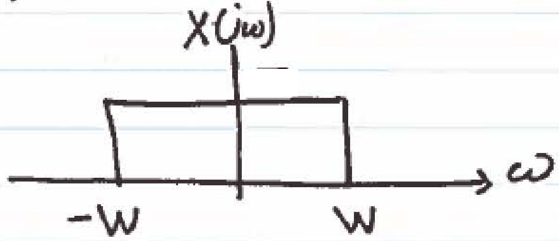
(b) From part (a), it is clear that (i) and (iv) are identical.

Let $x(t)$ be a single with a Nyquist rate of ω_0. Determine the Nyquist rates of the following three signals.

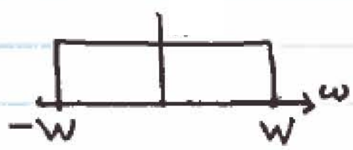
a) $x(t)*x(t)$ (20%)
b) $x(t)\times x(t)$ (20%)
c) $x(t)\times x(t)\times x(t)$ (20%)

Where, $x(t)=\dfrac{\sin Wt}{\pi t}$

Since $x(t)=\dfrac{\sin(Wt)}{\pi t} \Leftrightarrow X(j\omega)=\pi\ \mathrm{rect}\left(\dfrac{\omega}{2(W\pi)}\right)$

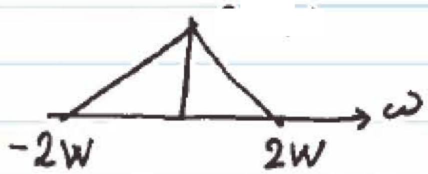


a) $x(t)*x(t) \Leftrightarrow X(j\omega).X(j\omega)$



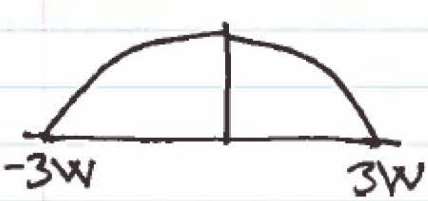
$\omega_{max}=W$
Nyquist rate $=2W$.

b) $x(t).x(t) \Leftrightarrow X(j\omega)*X(j\omega)$



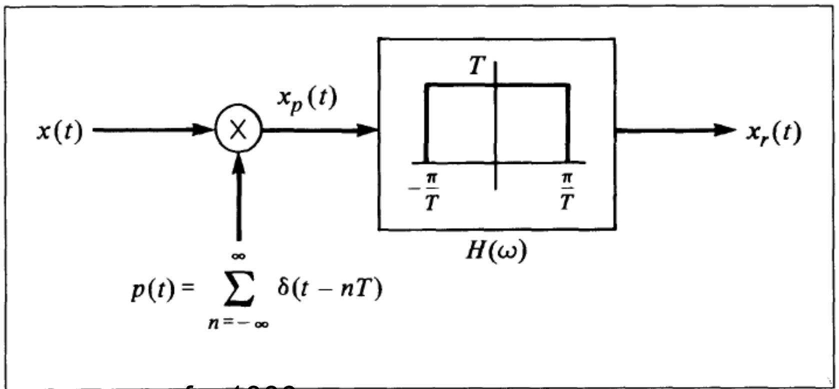
$\omega_{max}=2W$
Nyquist Rate $=4W$

c) $x(t).x(t).\ x(t) \Leftrightarrow X(j\omega)*X(j\omega)*X(j\omega)$



$\omega_{max}=3W$
Nyquist Rate $=6W$

[3] In the system $x(t)$ is sampled with a periodic impulse train, and a reconstructed signal $x_r(t)$ is obtained from the samples by lowpass filtering.



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The sampling period T is 1 ms, and $x(t)$ is a sinusoidal signal of the form $x(t) = \cos(2\pi f_0 t + \theta)$. For each of the following choices of f_0 and θ, determine $x_r(t)$.

(a) $f_0 = 250$ Hz, $\theta = \pi/4$

(b) $f_0 = 750$ Hz, $\theta = \pi/2$

(c) $f_0 = 500$ Hz, $\theta = \pi/2$

The signal $x(t) = \cos(\omega_0 t + \theta)$, where $\omega_0 = 2\pi f_0$, can be written as

$$x(t) = \tfrac{1}{2}e^{j\theta}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\theta}e^{-j\omega_0 t}$$

and the spectrum of $x(t)$ is given by

$$X(\omega) = \pi e^{j\theta}\delta(\omega - \omega_0) + \pi e^{-j\theta}\delta(\omega + \omega_0)$$

The spectrum of $p(t)$ is given by

$$P(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$$

Therefore, the spectrum of $x_p(t)$ is

$$X_p(\omega) = \frac{1}{2\pi}\left(\frac{2\pi^2}{T}\right)\left[\sum_{k=-\infty}^{\infty} e^{j\theta}\delta\left(\omega - \frac{2\pi k}{T} - \omega_0\right) + e^{-j\theta}\delta\left(\omega - \frac{2\pi k}{T} + \omega_0\right)\right]$$

and the spectrum of $X_r(\omega)$ is given by

$$X_r(\omega) = H(\omega)X_p(\omega)$$

(a) $\omega_0 = 2\pi \times 250, \quad \theta = \frac{\pi}{4}, \quad T = 10^{-3},$

$$X_p(\omega) = \frac{\pi}{T}\sum_{k=-\infty}^{\infty}[e^{j\theta}\delta(\omega - 2\pi \times 10^3 k - 2\pi \times 250) + e^{-j\theta}\delta(\omega - 2\pi \times 10^3 k + 2\pi \times 250)]$$

Hence, only the $k = 0$ term is passed by the filter:

$$X_r(\omega) = \pi[e^{j\theta}\delta(\omega - 2\pi \times 250) + e^{-j\theta}\delta(\omega + 2\pi \times 250)]$$

and

$$x_r(t) = \frac{1}{2}e^{j\theta}e^{j2\pi\times 250t} + \frac{1}{2}e^{-j\theta}e^{-j2\pi\times 250t} = \cos(2\pi \times 250t + \theta) = \cos\left(2\pi \times 250t + \frac{\pi}{4}\right)$$

(b) $\omega_0 = 2\pi \times 750$ Hz, $T = 10^{-3}$,

$$X_p(\omega) = \frac{\pi}{T} \sum_{k=-\infty}^{\infty} [e^{j\theta}\delta(\omega - 2\pi \times 10^3 k - 2\pi \times 750) + e^{-j\theta}\delta(\omega - 2\pi \times 10^3 k + 2\pi \times 750)]$$

Only the $k = \pm 1$ term has nonzero contribution:

$$X_r(\omega) = \frac{\pi}{T}[e^{j\theta}\delta(\omega + 2\pi \times 250) + e^{-j\theta}\delta(\omega - 2\pi \times 250)]$$

Hence,

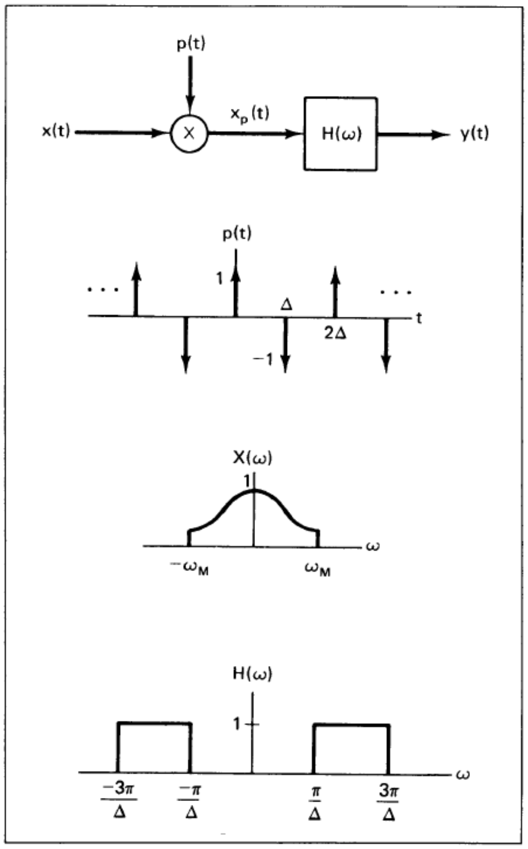
$$x_r(t) = \cos(2\pi \times 250t - \theta) = \cos\left(2\pi \times 250t - \frac{\pi}{2}\right)$$

(c) $\omega_0 = 2\pi \times 500$, $\theta = \frac{\pi}{2}$, $T = 10^{-3}$,

$$X_p(\omega) = \frac{\pi}{T} \sum_{k=-\infty}^{\infty} [e^{j\theta}\delta(\omega - 2\pi \times 10^3 k - 2\pi \times 500) + e^{-j\theta}\delta(\omega - 2\pi \times 10^3 k + 2\pi \times 500)]$$

Since $H(\omega) = 0$ at $\omega = 2\pi \times 500$, the output is zero: $x_r(t) = 0$.

[4] Figure gives a system in which the sampling signal is an impulse train with alternating sign. The Fourier transform of the input signal is as indicated in the figure.



(a) For $\Delta < \pi/2\omega_M$, sketch the Fourier transform of $x_p(t)$ and $y(t)$.

(b) For $\Delta < \pi/2\omega_M$, determine a system that will recover $x(t)$ from $x_p(t)$.

(c) For $\Delta < \pi/2\omega_M$, determine a system that will recover $x(t)$ from $y(t)$.

(d) What is the *maximum* value of Δ in relation to ω_M for which $x(t)$ can be recovered from either $x_p(t)$ or $y(t)$.

(a) $$x_p(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - 2\Delta n) - \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - \Delta - 2\Delta n)$$

$$= x(t)\left[\sum_{n=-\infty}^{\infty} \delta(t - 2\Delta n) - \sum_{n=-\infty}^{\infty} \delta(t - \Delta - 2\Delta n)\right]$$

By the convolution theorem,

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * \frac{2\pi}{2\Delta} \sum_{n=-\infty}^{\infty} \delta\left(\omega - n\frac{2\pi}{2\Delta}\right)$$

$$- \frac{1}{2\pi} X(\omega) * \frac{2\pi}{2\Delta} \sum_{n=-\infty}^{\infty} \delta\left(\omega - n\frac{2\pi}{2\Delta}\right) e^{-j\omega\Delta}$$

$$= X(\omega) * \left[\frac{1}{2\Delta} \sum_{n=-\infty}^{\infty} (1 - e^{-j\pi n})\,\delta\left(\omega - \frac{n\pi}{\Delta}\right)\right]$$

$$= X(\omega) * \left[\frac{1}{2\Delta} \sum_{n=-\infty}^{\infty} (1 - (-1)^n)\,\delta\left(\omega - \frac{n\pi}{\Delta}\right)\right]$$

$X_p(\omega)$ is sketched in Figure S16.4-1 and $Y(\omega)$ is sketched in

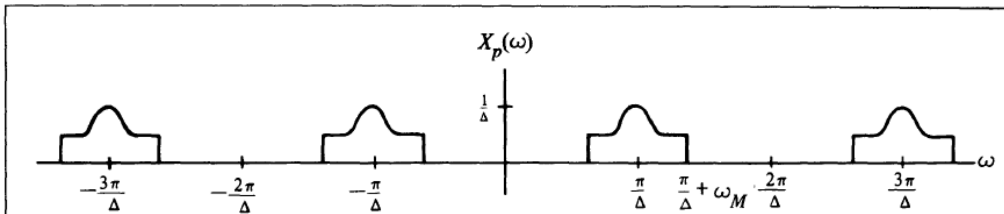
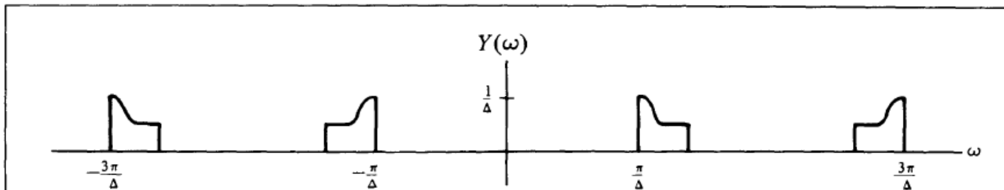
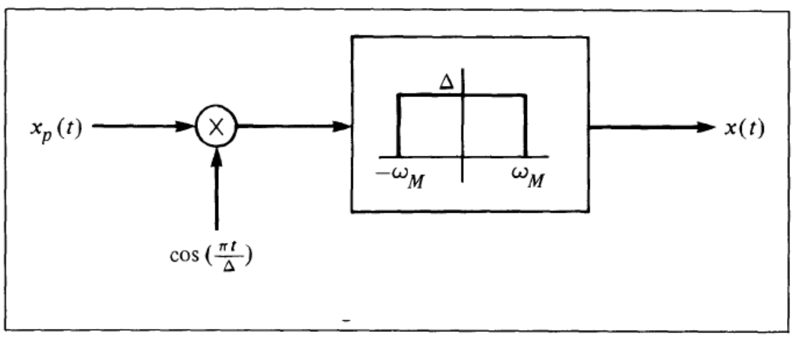


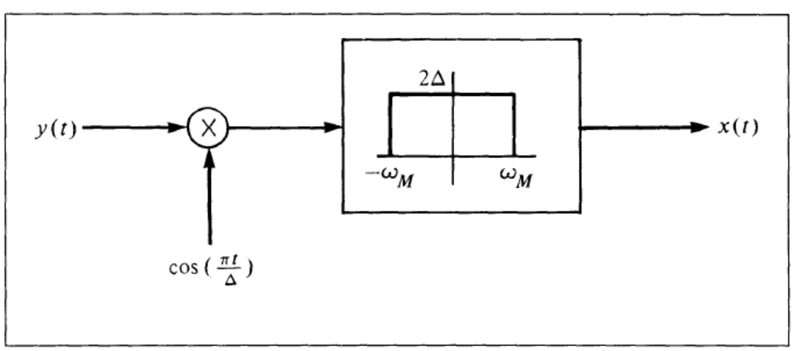
Figure S16.4-1



(b)



(c)



(d) Δ is maximum when π/Δ is minimum. From part (a) we see that aliasing is avoided in $X_p(\omega)$ if $\omega_M \leq \pi/\Delta$. Hence, $\Delta_{max} = \pi/\omega_M$.

Let $x(t)$ be a signal with Nyquist rate ω_0. Determine the Nyquist rate for each of the following signals:

(a) $x(t) + x(t-1)$

Solution:

If the signal x(t) has Nyquist rate of ω_0, then its Fourier transform $X(\omega) = 0$ for $|\omega| > \omega_0/2$

Let signal x(t) have a Fourier transform $X(\omega)$, i.e. $x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(\omega))$

Using the Fourier transform properties,

$$y(t) = x(t) + x(t-1) \stackrel{\mathscr{F}}{\longleftrightarrow} Y(\omega)$$

$$Y(\omega) = X(\omega) + e^{-j\omega}X(\omega) = X(\omega)(1 + e^{-j\omega})$$

We can only guarantee that $Y(\omega) = 0$ for $|\omega| > \frac{\omega_0}{2}$. Therefore, the Nyquist rate for y(t) is also ω_0

(b) $\frac{dx(t)}{dt}$

Solution: Using the Fourier transform properties,

$$y(t) = \frac{d(x(t)}{dt} \stackrel{\mathscr{F}}{\longleftrightarrow} Y(\omega)$$

$$Y(\omega) = j\omega X(\omega)$$

We can only guarantee that $Y(\omega) = 0$ for $|\omega| > \frac{\omega_0}{2}$. Therefore, the Nyquist rate for y(t) is also ω_0.

(c) $x^2(t)$

Solution: Using the Fourier transform properties,

$$y(t) = x^2 \stackrel{\mathscr{F}}{\longleftrightarrow} Y(\omega)$$

$$Y(\omega) = \frac{1}{2\pi}[X(\omega) * X(\omega)]$$

We can only guarantee that $Y(\omega) = 0$ for $|\omega| > \omega_0$. Therefore, the Nyquist rate for y(t) is $2\omega_0$.