On the Asymptotic Behaviour of Circular Fuzzy Cellular Automata

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Fuzzy cellular automata (FCA) are continuous cellular automata where the local rule is defined as the "fuzzification" of the local rule of a corresponding Boolean cellular automaton in disjunctive normal form. In this paper, we consider circular FCA; their asymptotic behaviours had previously been observed through simulation and FCA had been empirically classified accordingly. However, no analytical study previously existed to support these observations.

We now begin the analytical study of circular FCA dynamics by considering a particular class of FCA (*Weighted Average* rules) which includes rules displaying most of the observed dynamics, and we precisely derive their behaviours. We confirm the empirical observations proving that all weighted average rules are periodic in time and space, and we derive their periods.

Key words: Fuzzy cellular automata, asymptotic behaviour, periodic points.

1 INTRODUCTION

Fuzzy cellular automata (FCA) are a particular type of continuous cellular automata, or coupled map lattices, where the local transition rules are the

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"fuzzifications" of the local rules of corresponding Boolean cellular automata in disjunctive normal form (DNF fuzzification)* .

Fuzzy cellular automata were first introduced in [5] as a model to describe the impact that perturbations (e.g. noisy sources, computation errors, mutations, etc.) can have on the evolution of Boolean CA (e.g., see [11]). They have been shown to be useful tools for pattern recognition purposes (e.g., see [13, 14]), and good models for generating images mimicking nature (e.g. [6, 18]). Their asymptotic behaviours have been studied in quiescent backgrounds only, where it has been shown that none of the elementary FCA rules has chaotic dynamics [10, 16, 17]. Recently, DNF fuzzification has been shown to maintain properties of the corresponding Boolean CA [2]. For example, it has been proven that density conservation is preserved through fuzzification, and that additivity of a Boolean CA implies and is implied by self-oscillation of the corresponding fuzzy CA (a peculiar property of the asymptotic behaviour of FCA). In [2], it has also been shown that if a FCA converges to an homogeneous configuration, the convergence value coincides with stable densities of the mean field approximation of the corresponding Boolean CA (i.e., to an estimate of its asymptotic density).

A fundamental problem in the study of Boolean CA, and thus also of their fuzzy counter-parts, is their classification. The first attempt to classify cellular automata was done by Wolfram in [20] where they were grouped according to the observed behaviour of their space-time diagrams. Although not formally precise, this classification captures important distinctions between cellular automata. Several other criteria for classifying CA have followed: some based on observable behaviours, some on intrinsic properties of CA rules, some on topological properties (e.g., see [4, 7, 8, 12, 15, 19]). Since for most classifications class membership is undecidable, observation of the evolution of a CA starting from (possibly all) initial configurations becomes crucial to understanding its dynamics. The evolution of CA is usually observed either for finite initial configurations in zero backgrounds (*quiescent background*), or for infinite but periodic initial configurations (*circular CA*).

Fuzzy cellular automata have been empirically classified according to various criteria. For example, fuzzy cellular automata in Boolean backgrounds have been grouped according to the level of "spread" of fuzziness in [5]. More recently, an empirical classification for elementary circular FCA was proposed based on experimental results from random initial configurations

* These are not to be confused with a variant of cellular automata, also called fuzzy cellular automata, where the fuzziness refers to the choice of a deterministic local rule (e.g., see [1])

[9]. The observed results suggested that all elementary rules have asymptotic periodic behaviour but, interestingly, with periods of only certain lengths: 1,2,4, and n (where n is the size of the circular lattice).

Intrigued by these recent observations and by the surprising resulting classification of [9], we now begin the study of the asymptotic behaviour of circular FCA with the goal of understanding and explaining the empirical findings. We start by considering *Weighted Average rules*, where the value of cell i at time $t+1$ can be seen as the weighted average of two of the values in x_i's neighbourhood (or their negations) at time t. From experiments, these rules seem to display most of the observed dynamics: fixed points, periods of length 2, and periods of length 4. We analytically study the asymptotic behaviour of weighted average rules and we prove that they all have spatial and temporal periodic behaviour from arbitrary initial configurations thus confirming the experimental findings. While analyzing their asymptotic behaviour, we discover an interesting density property for rule 29 which holds both in the Boolean and in the fuzzy domains: rule 29 is density conserving at alternate iterations, where density is defined as the number of 1s in the case of the Boolean model, and as the sum of the values in the fuzzy case. More precisely: the initial density is preserved at even iterations, while at odd iterations the density is the complement of the initial one.

This study is a first step towards the analytical support of the classification of FCA in terms of periodicity, and it opens interesting research directions. In light of the results of [2] on the probabilistic interpretation of FCA, of particular interest would be to understand the meaning of non-homogeneous periodic points of weighted average rules (and of other FCA rules) in terms of estimates of the asymptotic densities of their corresponding Boolean CA. Studies in this direction are underway.

2 DEFINITIONS

A one dimensional circular Boolean cellular automaton is a collection of cells arranged in a circular linear array. Cells have Boolean values and they synchronously update their values according to a local rule applied to their neighbourhood. A configuration $\mathbf{X}^t = (x_0^t, x_,^t \ldots, x_{n-1}^t)$ is a description of all cell values at a given time t. The neighbourhood of a cell consists of the cell itself and its r left and right neighbours, thus the local rule has the form: $\delta : \{0,1\}^{2r+1} \to \{0,1\}$. The global dynamics of a one dimensional cellular automaton composed of n cells is then defined by the global rule (or transition function): $\Delta : \{0,1\}^n \to \{0,1\}^n \quad s.t. \quad \forall \mathbf{X} \in \{0,1\}^n, \forall i \in$

$\{0, \ldots, n-1\}, \Delta(X)_i = \delta(x_{i-r} \ldots, x_i, \ldots, x_{i+r})$, where all operations on indices are modulo n. (Alternatively, one can think of the structure of a circular CA as an infinite array containing a periodic configuration.) A cellular automaton is *elementary* if it has radius and dimension one. In this paper, we focus only on circular elementary cellular automata and we will omit the term circular.

The local rule $\delta : \{0,1\}^3 \to \{0,1\}$ of an elementary Boolean CA is typically given in tabular form by listing the 8 binary tuples corresponding to the 8 possible local configurations a cell can detect in its direct neighbourhood, and mapping each tuple to a binary value:

$$(000, 001, 010, 011, 100, 101, 110, 111) \to (b_0, b_1, \cdots, b_7).$$

The binary representation $(b_0, \cdots, b_7)$ is often converted into the decimal representation $\sum_i b_i 2^i$, and this value is typically used as the rule number. Let us denote by d_i the tuple mapping to b_i. The local rule can also be canonically expressed in *disjunctive normal form* (DNF) as follows:

$$\delta(v_0, v_1, v_2) = \bigvee_{i<8} b_i \bigwedge_{j=0:2} v_j^{d_{ij}}$$

where d_{ij} is the j-th digit, from left to right, of d_i, and v_j^0 (resp. v_j^1) stands for $\neg v_j$ (resp. v_j).

A *fuzzy cellular automaton* (FCA) is a particular continuous cellular automaton where the local rule is obtained by *DNF-fuzzification* of the local rule of a classical Boolean CA. The fuzzification consists of a continuous extension of the Boolean operators AND, OR, and NOT in the DNF expression of the Boolean rule. Depending on which fuzzy operators are used, different types of fuzzy cellular automata can be defined. Among the various possible choices, we consider the following: $(a \vee b)$ is replaced by $max\{1, (a+b)\}^\dagger$, $(a \wedge b)$ by (ab), and $(\neg a)$ by $(1-a)$. Whenever we talk about fuzzification, we will be referring to the *DNF-fuzzification* defined above. The resulting local rule becomes a real function that generalizes the canonical representation of the corresponding Boolean CA:

$$f : [0,1]^3 \to [0,1] \;\; by \;\; f(v_0, v_1, v_2) = \sum_{i<8} b_i \prod_{j=0:2} l(v_j, d_{ij}) \qquad (1)$$

where $l(a, 0) = 1 - a$ and $l(a, 1) = a$.

† note that, in the case of FCA, $max\{1, (a+b)\} = (a+b)$

Example: Rule 18. Consider, for example, elementary rule 18 whose local transition rule in tabular form is given by:

$$(000, 001, 010, 011, 100, 101, 110, 111) \to (0, 1, 0, 0, 1, 0, 0, 0).$$

The local transition rule in DNF form is the following:

$$\delta(v_{-1}, v_0, v_1) = (\neg v_{-1} \wedge \neg v_0 \wedge v_1) \vee (v_{-1} \wedge \neg v_0 \wedge \neg v_1).$$

The corresponding fuzzification is:

$$f(v_{-1}, v_0, v_1) = (1 - v_{-1})(1 - v_0)v_1 + v_{-1}(1 - v_0)(1 - v_1).$$

Throughout this paper, we will denote local rules of Boolean CA by δ and their fuzzifications for the corresponding FCA by f. The corresponding global rules are denoted by Δ and F, respectively.

We define a *fixed point* $\mathbf{P} \in [0, 1]^n$ for a circular FCA with global transition rule $F : [0, 1]^n \to [0, 1]^n$ as a configuration $\mathbf{P}$ such that $F(\mathbf{P}) = \mathbf{P}$.

Also, a CA is said to be *Temporally Periodic* with period τ if $\exists T$ such that $\forall t > T$: $F(\mathbf{X}^t) = F(\mathbf{X}^{t+\tau})$. Similarly, a rule is *Spatially Periodic* with period ω if $\exists T$ such that $\forall t > T$, $\forall i$: $x_i^t = x_{i+\omega}^t$.

Definition 1. *A rule is* Asymptotically Periodic in Time *(or Asymptotically Temporally Periodic) with period τ if, for any initial configuration in $(0, 1)$ and $\forall \epsilon > 0$, $\exists T$ such that $\forall t > T$ and $\forall i$: $|x_i^t - x_i^{t+\tau}| < \epsilon$.*

Definition 2. *A rule is* Asymptotically Periodic in Space *(or Asymptotically Spatially Periodic) with period ω if, for any initial configuration in $(0, 1)$ and $\forall \epsilon > 0$, $\exists T$ such that $\forall t > T$ and $\forall i$: $|x_i^t - x_{i+\omega}^t| < \epsilon$.*

A rule that is asymptotically periodic in space with period 1 will be called *asymptotically homogeneous*. A rule which is asymptotically periodic in time with period 1 will be said to be *convergent to a fixed point*.

When a configuration at time t is spatially periodic with smallest period $m < n$, we shall indicate it as $X^t = (\alpha)^m$, where α is the sequence of values corresponding to the smallest period. For example, given a FCA with even size n and a spatially periodic configuration of period 2, $\mathbf{X} = (a, b, a, b, a, b, \ldots)$, we will write $\mathbf{X}$ as $(ab)^{\frac{n}{2}}$. When $\mathbf{X}$ is a homogeneous point, we will omit the parentheses, e.g. $\mathbf{X} = (a, a, \cdots, a) = a^n$

3 ASYMPTOTIC BEHAVIOUR OF WEIGHTED AVERAGE RULES

Observation suggests that elementary FCA have simpler dynamics than elementary Boolean CA. In fact, a recent classification based on observations from random initial configurations has grouped circular elementary FCA into four categories [9]: FCA converging to a fixed point, periods of length two, four and n.

Period 1 (Quiescent)	0, 8, 24, 32, 36, 40, 44, 56, 72, 74, 104, 128, 136, 152, 160, 164, 168 , 200
Period 1 (Homogeneous)	6, 9, 22, 25, 26, 30, 33, 35, 37, 38, 41, 45, 54, 57, 60, 61, 62, 73, 90, 105, 106, 110, 122, 126, 134, 150, 154, **172**.
Period 1 (Heterogeneous)	4, 12, 13, 28, 76, 77, 108, 132, 140, 156, 204, 232
Period 1 for n even (Heterogeneous)	**78**, 94
Period 2 for all n	$1, 5, 19, 23, \mathbf{27}, 50, 51178$
Period 2 for n even	18, **29**, **58**, 146, **184**
Period 4 for n multiple of 4	**46**
Period n (Shifts)	2, 3, 7, 10, 11, 14, 15, 34, 42, 43, 130, 138, 142, 162,170

TABLE 1
Observed dynamics of Circular Elementary Cellular Automata. Rules equivalent under conjugation, reflection and both are not indicated.

We now begin the analytical study of FCA by focusing on a group of rules called Weighted Average rules, which includes rules in most categories above (the rules in bold in the table). We will confirm the observed temporal periodicity for these rules and we will also prove the spatial periodicities of their convergence points.

We will begin by defining a generalized fuzzy circular cellular automaton where the rule at each point need not be identical but will have the same form. We will then show that under certain conditions weighted average rules on generalized fuzzy CA will converge to a fixed point. We will proceed to describe each of the rules highlighted above as a weighted average rule. Rule 172 will be shown to be an example of a generalized fuzzy CA which meets the conditions for convergence. For each subsequent rule, we will show that it is conjugate to a generalized weighted average CA either in several steps or when taken as the cross product of two separate generalized circular CA or as a combination of both and will therefore converge to a regular pattern which is described in detail.

3.1 Preliminaries

We first introduce the notion of a weighted average and of a type of generalized fuzzy CA.

A weighted average of two numbers α and β is an equation of the form $\mu = (1-\gamma)\alpha + \gamma\beta$ where $\gamma \in [0, 1]$, (the usual mean occurs when $\gamma = \frac{1}{2}$). Weighted averages have the following useful properties.

Property 1. *If* $\mu = (1-\gamma)\alpha+\gamma\beta$ then $(1-\mu) = (1-\gamma)(1-\alpha)+\gamma(1-\beta)$.

Property 2. *If* $\gamma_1 < \gamma_2$ *and* $\alpha < \beta$ then $(1-\gamma_1)\alpha+\gamma_1\beta < (1-\gamma_2)\alpha+\gamma_2\beta$.

A *generalized fuzzy CA* is a lattice where every cell updates its state according to a different local rule. In this paper, we will consider generalized fuzzy CA with neighbourhood 1 ($r = 1$) and with rules of the type defined below.

Definition 3. *A* generalized fuzzy cellular automaton with weighted average rules *($GWCA$) is a generalized fuzzy CA where the local rule has the following form:*

$$x_i^{t+1} = \gamma_i^t x_i^t + (1-\gamma_i^t)x_{i+1}^t \tag{2}$$

$$(\text{or } \; x_i^{t+1} = \gamma_i^t x_i^t + (1-\gamma_i^t)x_{i-1}^t) \tag{3}$$

with bounded weights. That is, there exists $0 < \gamma < \frac{1}{2}$ *such that* $\gamma_i^t \in [\gamma, 1-\gamma]$ *for all* i *and for all* t.

In other words, in a $GWCA$ the state of a cell i at time $t+1$ takes the average of the state of the cell itself and of one of its neighbours at time t weighted by a value in $(0, 1)$ that varies from cell to cell. In this section, we consider only $GWCA$ of form (2); the proofs for averaging with the neighbour on the left as in equation (3) are analogous.

3.2 A General Convergence Theorem

We now prove a convergence result for $GWCA$ when the initial configurations are in the open set $(0, 1)^n$. We wish to show that repeated weighted averaging of a circular array results in the values in the array converging to a fixed point, as stated in the general theorem below.

Theorem 1. General Theorem
Consider a GWCA starting from configuration $\mathbf{X}^0 = (x_0^0, \cdots, x_{n-1}^0)$ *with* $x_i^0 \in (0,1)$ *for all* i*. Then for some* $p \in (0,1)$*,* $x_i^t \to p$ *for all* i *as* $t \to \infty$*.*

We will prove the theorem with a sequence of lemmas.

Lemma 1. *Consider the sequence*

$$\min_i\{x_i^0\}, \min_i\{x_i^1\}, \ldots, \min_i\{x_i^t\}, \ldots.$$

This sequence converges to some value l_m*. Furthermore, the sequence*

$$\max_i\{x_i^0\}, \max_i\{x_i^1\}, \ldots, \max_i\{x_i^t\}, \ldots$$

converges to some value l_M*.*

PROOF Since each value is averaged with its neighbour, the result is always between these two values. Assume that $x_i^t \leq x_{i+1}^t$. Then,

$$x_i^{t+1} = \gamma_i^t x_i^t + (1-\gamma_i^t)x_{i+1}^t \geq \gamma_i^t x_i^t + (1-\gamma_i^t)x_i^t \geq x_i^t$$

with equality if and only if $x_i^t = x_{i+1}^t$. In particular, the previous holds when x_i^t is the minimum value at time t. Since all values at time $t+1$ are the weighted averages of values greater than or equal to the minimum value at time t, all values at time $t+1$ must be greater than or equal to the minimum at time t. Thus the sequence is increasing and bounded above by 1 and therefore convergent.

The analogous proof holds for the sequence of max_i. ∎

We wish to show that $l_m = l_M$. We will proceed by showing that if there is a difference in the maximum and minimum values in the configuration at some time t, then in $n-1$ iterations (i.e., at time $t+n-1$), the minimum must increase by at least an amount proportional to the difference of the limits.

Lemma 2. *Given a configuration of length* n*, if at any time* t*,* $\max_i\{x_i^t\} - \min_i\{x_i^t\} \geq \delta$ *then* $\min_i\{x_i^{t+n-1}\} - \min_i\{x_i^t\} \geq \gamma^{n-1}\delta$ *where* $\gamma < \gamma_i^t < 1-\gamma$*.*

PROOF Let $m = \min_i \{x_i^t\}$ and $M = \max_i \{x_i^t\}$ so that $M - m \geq \delta$. Renumbering if necessary, we may assume that the maximum occurs at x_{n-1}^t, so that:

$$x_i^t \geq \begin{cases} m & \text{for } 0 \leq i < n-1 \\ M \geq m + \gamma^0\delta & \text{for } i = n-1 \end{cases} \tag{4}$$

with $x_i^t = m$ for at least one i and $x_{n-1}^t = M$.

We wish to show by induction that for $0 \le s \le n-1$,

$$x_i^{t+s} \ge \begin{cases} m & \text{for } 0 \le i < n-s-1 \\ m + \gamma^s \delta & \text{for } n-s-1 \le i < n. \end{cases}$$

This is true when $s = 0$ by equation (4). Now assume it is true for x_i^{t+s} with $s \le n-2$, and consider x_i^{t+s+1}.

We prove this separately for $i = n-s-2$, for $i = n-1$, and for $0 \le n-s-1 < i < n-1$. In each case, we obtain a new lower bound by averaging the current lower bounds, giving the highest weight, $(1-\gamma)$, to the lower of the two values being averaged, as in Property 2. For $i = n-s-2$, we have that:

$$\begin{aligned} x_{n-s-2}^{t+s+1} &= (1-\gamma_{n-s-2}^{t+s})x_{n-s-2}^{t+s} + \gamma_{n-s-2}^{t+s} x_{n-s-1}^{t+s} \text{ (by definition)} \\ &\ge (1-\gamma_{n-s-2}^{t+s})m + \gamma_{n-s-2}^{t+s}(m+\gamma^s\delta) \text{ (induction hypothesis)} \\ &\ge (1-\gamma)m + \gamma(m+\gamma^s\delta) \qquad \text{(by Property 2)} \\ &\ge m + \gamma(m+\gamma^s\delta - m) \ge m + \gamma^{s+1}\delta. \end{aligned}$$

In the case of $i = n-1$, we are averaging m and $m+\gamma^s\delta$, hence $x_{n-1}^{t+s+1} \ge (1-\gamma)m + \gamma\gamma^s\delta \ge m + \gamma^{s+1}\delta$. For $0 < i < n-s-2$, we are averaging values greater than or equal to m, and thus the result will be greater than or equal to m. Finally, for $n-s-1 < i < n-1$, we have weighted averages of values greater than $m+\gamma^s\delta$ and therefore, $x_i^{t+s+1} \ge m+\gamma^s\delta > m+\gamma^{s+1}\delta$.

As a consequence, when $s = n-1$, that is, at time $t+n-1$, we have that $x_i^{t+n-1} \ge m + \gamma^{n-1}\delta$ for all i. Hence $\min_i\{x_i^{n-1}\} - \min_i\{x_i^0\} \ge (m+\gamma^{n-1}\delta) - m \ge \gamma^{n-1}\delta$. ∎

We are now ready to prove the General Theorem.

PROOF We will show by contradiction that $l_m = l_M$. Let $l_M - l_m = \delta > 0$. Fix ϵ such that $0 < \epsilon < \gamma^{n-1}\delta$ and let T be such that for all $t > T$, if $m = \min\{x_1^t, \ldots, x_n^t\}$ and $M = \max\{x_1^t, \ldots, x_n^t\}$ then $l_m - m < \epsilon$ and $M - l_M < \epsilon$. Note that $M - m > \delta$.

Let $m^t = \min_i\{x_i^t\}$. Fix $t > T$. By Lemma 2,

$$m^{t+n-1} > m^t + \gamma^{n-1}\delta > m + \epsilon > l_m$$

which is a contradiction. Hence $l_M - l_m = 0$. Let us call this limit p.

For all x_i^t, we have $m^t \le x_i^t \le M^t$. Since as $t \to \infty$, $m^t \to p$ and $M^t \to p$, $x_i^t \to p$ also. ∎

3.3 Weighted Average Rules

Several elementary FCA can be viewed as weighted averages of two values in a neighbourhood (or their negations) weighted by the third value (or its negation), closely resembling $GWCA$. Table 2 lists the elementary FCA with this feature. The weights are the values in parentheses. Note that in the following we will use $(1-x)$ and $\bar{x}$ interchangeably.

Rule	Equation
R_{184} ($\cong R_{226}$)	$(1-x_i)x_{i-1}+(x_i)x_{i+1}$
R_{46} ($\cong R_{116}, R_{139}, R_{209}$)	$(x_i)\bar{x}_{i-1}+(1-x_i)x_{i+1}$
R_{27} ($\cong R_{39}, R_{53}, R_{83}$)	$(x_{i+1})\bar{x}_{i-1}+(1-x_{i+1})\bar{x}_i$
R_{29} ($\cong R_{71}$)	$(x_i)\bar{x}_{i-1}+(1-x_i)\bar{x}_{i+1}$
R_{58} ($\cong R_{114}, R_{163}, R_{177}$)	$(x_{i-1})\bar{x}_i+(1-x_{i-1})x_{i+1}$
R_{78} ($\cong R_{92}, R_{141}, R_{197}$)	$(x_{i+1})\bar{x}_{i-1}+(1-x_{i+1})x_i$
R_{172} ($\cong R_{212}, R_{202}, R_{228}$)	$(1-x_{i-1})x_i+(x_{i-1})x_{i+1}$

TABLE 2
Weighted Average Rules. Rules equivalent under conjugation, reflection and both are indicated in parentheses.

Although Theorem 1 does not apply directly to these rules (except for rule R_{172}), we can determine their asymptotic behaviour by constructing, for each of them, a topologically conjugate system for which the theorem does apply. In this section, we assume that the initial configurations are in the open set $(0,1)^n$. We show convergence beginning with the simplest proof and progressing to the most complex.

We remind the reader that we will indicate by p^n a spatial asymptotic homogeneous configuration $(p,p,\ldots,p)$. We will also indicate a spatially periodic configuration of the form $(p,q,p,q,\ldots)$ by $(pq)^{\frac{n}{2}}$, and analogous notation is used for other periodicities.

Rule 172
The simplest rule to analyze is rule 172 where the state of x_i at time $t+1$ is the average of x_i and x_{i+1} weighted by x_{i-1} at time t.

Theorem 2. *Rule* 172 *converges spatially and temporally to a homogeneous configuration.*

PROOF Rule 172 is the weighted average of x_i^t and x_{i+1}^t, so to apply Theorem 1, we only need to show that there exists a value $0<\gamma<\frac{1}{2}$ such that the weights are bounded by γ and $(1-\gamma)$. Let

$$\gamma = min\left\{x_0^0, \cdots, x_{n-1}^0, 1-x_0^0, \cdots, 1-x_{n-1}^0\right\}.$$

Then, since weighted averaging increases minimum values and decreases maximum values, $\forall t$,

$$\begin{array}{rcl} \gamma & \leq & min\left\{x_0^t, \cdots, x_{n-1}^t, 1-x_0^t, \cdots, 1-x_{n-1}^t\right\} \\ 1-\gamma & \geq & max\left\{x_0^t, \cdots, x_{n-1}^t, 1-x_0^t, \cdots, 1-x_{n-1}^t\right\}. \end{array}$$

Since the weights used by rule 172 are taken from these sets, it follows that $\gamma \leq \gamma_i^t \leq 1-\gamma$ and the conditions of Theorem 1 hold. Hence, there exists a value p such that $x_i^t \to p$ for all i as $t \to \infty$. ∎

Note that any configuration p^n is a fixed point for rule f_{172} since $\forall p$ $f_{172}(p,p,p) = p$, so the convergence value may vary, depending on the initial configuration.

Summary of results for $t \to \infty$

Rule f_{172}	$x_i^{t+1} = (1-x_{i-1}^t)x_i^t + x_{i-1}^t x_{i+1}^t$
Spatial:	$\to p^n$, for some p
Temporal:	fixed point

Rule 78

Consider rule R_{78} as a weighted average of x_i and $1-x_{i-1}$:

$$x_i^{t+1} = (1-x_{i+1}^t)x_i^t + x_{i+1}^t(1-x_{i-1}^t).$$

Theorem 1 does not apply directly to this rule since one of the values is inverted before it is averaged, so we will derive the asymptotic behaviour of FCA 78 by finding a topologically conjugate FCA to which Theorem 1 does apply. In the even case, this will simply involve inverting every other element and modifying the rules at each point appropriately. Accordingly, we would expect spatial periodicity with period 2. In the odd case, we will double the length of the circular CA again inverting every other element so that at every iteration each value and its inverse is calculated.

Theorem 3. *When n is even, rule* 78 *converges temporally to a homogeneous configuration. Furthermore, if* x_0^t *converges to a value p, then for all i,* x_{2i}^t *converges to p, and* x_{2i+1}^t *converges to* $(1-p)$*. When* n *is odd, rule 78 converges to the homogeneous configuration* $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.

PROOF

Let A be a fuzzy cellular automaton following rule 78 and let $f : (0,1)^3 \to (0,1)$ be its local rule and F its global rule.

Assume that n is even. Consider a $GWCA\ A'$ with global rule F' given by two local rules $f'_1, f'_2 : (0,1)^3 \to (0,1)$ defined as follows:

$$\begin{aligned} f'_1(x,y,z) &= zy + (1-z)x \\ f'_2(x,y,z) &= (1-z)y + zx \end{aligned}$$

which are applied in alternation to a configuration $(x_0, \cdots, x_{n-1})$ as follows:

$$F'(x_0, \cdots, x_{n-1}) = (f'_1(x_{n-1}, x_0, x_1), f'_2(x_0, x_1, x_2), \cdots).$$

Let $h : (0,1)^n \to (0,1)^n$ be the homeomorphism defined by:

$$h(x_0, x_1, x_2, \ldots, x_{n-1}) = (x_0, 1-x_1, x_2, \ldots, 1-x_{n-1}).$$

We want to show that F and F' are conjugates, that is that $h \circ F = F' \circ h$. Let $\mathbf{X} = (x_0, x_1, \ldots, x_{n-1})$ be an arbitrary configuration for A.

$$\begin{aligned} & h \circ F(x_0, x_1, \cdots, x_{n-1}) \\ = \;& h((1-x_1)x_0 + x_1(1-x_{n-1}), \cdots, (1-x_0)x_{n-1} + x_0(1-x_{n-2})) \\ = \;& ((1-x_1)x_0 + x_1(1-x_{n-1}), \cdots, \overline{(1-x_0)x_{n-1} + x_0(1-x_{n-2})}) \\ = \;& ((1-x_1)x_0 + x_1(1-x_{n-1}), \cdots, (1-x_0)(1-x_{n-1}) + x_0 x_{n-2}). \end{aligned}$$

$$\begin{aligned} & F' \circ h(x_0, x_1, \cdots, x_{n-1}) \\ = \;& F'(x_0, (1-x_1), \cdots, (1-x_{n-1})) \\ = \;& ((1-x_1)x_0 + x_1(1-x_{n-1}), \cdots, (1-x_0)(1-x_{n-1}) + x_0 x_{n-2}). \end{aligned}$$

Since the weights are bounded as in the proof of Theorem 2, by Theorem 1, the $GWCA\ A'$ converges to some point $(p, p, \cdots p)$. Then the point of convergence A is $h^{-1}(p, p, \cdots, p) = (p, 1-p, p, \cdots, 1-p)$,

$$\mathbf{X}^t \to (p, 1-p, \cdots, p, 1-p).$$

When n is odd, consider a GWCA B of length $2n$ with global rule F' as above, and a homeomorphism $h : A \to B$ defined as:

$$h(x_0, x_1, \cdots, x_{n-1}) = (x_0, \bar{x}_1, \cdots, x_{n-1}, \bar{x}_0, x_1, \cdots, \bar{x}_{n-1}).$$

Functions F and F' are conjugate since $h \circ F = F' \circ h$. In fact:

$$
\begin{aligned}
& F' \circ h(x_0, x_1, \cdots, x_{n-1}) \\
= \; & F'(x_0, \bar{x}_1, \cdots, x_{n-1}, \bar{x}_0, x_1, \cdots, \bar{x}_{n-1}) \\
= \; & (\bar{x}_1 x_0 + x_1 \bar{x}_{n-1}, \bar{x}_2 \bar{x}_1 + x_2 x_0, \cdots, \bar{x}_0 x_{n-1} + x_0 \bar{x}_{n-2}, \\
& \bar{x}_1 \bar{x}_0 + x_1 x_{n-1}, \bar{x}_2 x_1 + x_2 \bar{x}_0, \cdots, \bar{x}_0 \bar{x}_{n-1} + x_0 x_{n-2}),
\end{aligned}
$$

while

$$
\begin{aligned}
& h \circ F(x_0, x_1, \cdots, x_{n-1}) \\
= \; & h(\bar{x}_1 x_0 + x_1 \bar{x}_{n-1}, \bar{x}_2 x_1 + x_2 \bar{x}_0, \cdots, \bar{x}_0 x_{n-1} + x_0 \bar{x}_{n-2}) \\
= \; & (\bar{x}_1 x_0 + x_1 \bar{x}_{n-1}, \bar{x}_2 \bar{x}_1 + x_2 x_0, \cdots, \bar{x}_0 x_{n-1} + x_0 \bar{x}_{n-2}, \\
& \bar{x}_1 \bar{x}_0 + x_1 x_{n-1}, \bar{x}_2 x_1 + x_2 \bar{x}_0, \cdots, \bar{x}_0 \bar{x}_{n-1} + x_0 x_{n-2}).
\end{aligned}
$$

Since f_1' and f_2' are weighted sums of neighbours with bounds as in the proof of Theorem 2, F' must converge to a single value, p. But we also notice that for all i, $x_i^t = (1 - x_{i+n}^t)$ and as $t \to \infty$ both x_i^t and x_{i+n}^t converge to p, so $p = 1 - p$ and therefore, $p = \frac{1}{2}$. The point of convergence of A is then $h^{-1}(\frac{1}{2}, \cdots, \frac{1}{2}) = (\frac{1}{2}, \cdots, \frac{1}{2})$. ∎

Summary of results for $t \to \infty$

Rule f_{78}	$x_i^{t+1} = (1 - x_{i+1}^t) x_i^t + x_{i+1}^t (1 - x_{i-1})^t$
Spatial:	n odd: $\to \frac{1}{2}^n$ n even: $\to (p(1-p))^{\frac{n}{2}}$ for some p
Temporal	fixed point

Rule 27

Consider rule R_{27} as a weighted average of $(1 - x_i^t)$ and $(1 - x_{i-1}^t)$:

$$x_i^{t+1} = (1 - x_{i+1}^t)(1 - x_i^t) + x_{i+1}^t (1 - x_{i-1}^t).$$

In this case, all values are inverted before averaging so we will show that it is topologically equivalent after two steps to a system to which Theorem 1 applies. Effectively, we are inverting and then inverting back thus obtaining period 2 temporally.

Theorem 4. *Rule* 27 *converges spatially to an homogeneous configuration and temporally it has asymptotic periodicity with period 2. Furthermore, if for all i, as* $t \to \infty$ $x_i^{2t} \to p$ *for some* p*, then* $x_i^{2t+1} \to 1 - p$.

PROOF Let A be a fuzzy cellular automaton following rule 27 and let $f : (0,1)^3 \to (0,1)$ be its local rule and F its global rule.

Let B be a fuzzy cellular automaton with global rule $F \circ F$. Consider a $GWCA$ B' with global rule $F'_2 \circ F'_1$ where F'_1 is defined by local rule f'_1 and F'_2 by f'_2. Local rules $f'_1, f'_2 : (0,1)^3 \to (0,1)$ are defined as follows:

$$\begin{aligned} f'_1(x,y,z) &= (1-z)y + zx \\ f'_2(x,y,z) &= zy + (1-z)x. \end{aligned}$$

We wish to show that $F \circ F = F'_2 \circ F'_1$. Let id be the identity homeomorphism and let inv be the negation homeomorphism:

$$\mathrm{inv}(x_0, x_1, \cdots, x_{n-1}) = (\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_{n-1}).$$

We will prove the equality of $F \circ F$ and $F'_2 \circ F'_1$ by showing that $F'_1 = \mathrm{inv} \circ F$ and $F = F'_2 \circ \mathrm{inv}$:

$$\begin{aligned} \mathrm{inv} \quad &\circ \quad F(\cdots, x_{i-1}, x_i, x_{i+1}, \cdots) \\ &= \mathrm{inv}(\cdots, \bar{x}_{i+1}\bar{x}_i + x_{i+1}\bar{x}_{i-1}, \cdots) \\ &= (\cdots, \bar{x}_{i+1}x_i + x_{i+1}x_{i-1}, \cdots) \\ &= F'_1(\cdots, x_{i-1}, x_i, x_{i+1}, \cdots), \end{aligned}$$

$$\begin{aligned} F'_2 \quad &\circ \quad \mathrm{inv}(\cdots, x_{i-1}, x_i, x_{i+1}, \cdots) \\ &= F'_2(\cdots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \cdots) \\ &= (\cdots, \bar{x}_{i+1}\bar{x}_i + x_{i+1}\bar{x}_{i-1}, \cdots) \\ &= F(\cdots, x_{i-1}, x_i, x_{i+1}, \cdots). \end{aligned}$$

Finally,

$$\begin{aligned} F \circ F &= (F'_2 \circ \mathrm{inv}) \circ (\mathrm{inv} \circ F'_1) \\ &= F'_2 \circ F'_1. \end{aligned}$$

Thus $F \circ F$ and $F'_2 \circ F'_1$ are equal. Since both F'_1 and F'_2 are weighted averages of neighbours with bounded weights, Theorem 1 applies, thus B' is converging to a homogeneous configuration. Moreover, every other iteration of A must also be converging.

If we let the limit of the even iterations be p, then

$$\begin{aligned}\lim_{t\to\infty}(x_0^{2t+1},x_1^{2t+1},\cdots,x_{n-1}^{2t+1}) &= \lim_{t\to\infty}F(x_0^{2t},x_1^{2t},\cdots,x_{n-1}^{2t})\\ &= F(\lim_{t\to\infty}x_0^{2t},\lim_{t\to\infty}x_1^{2t},\cdots,\lim_{t\to\infty}x_{n-1}^{2t})\\ &= F(p,p,\cdots,p)\\ &= (1-p,1-p,\cdots,1-p).\end{aligned}$$

So

$$\begin{aligned}\mathbf{X}^{2t} &\to (p,\cdots,p)\\ \mathbf{X}^{2t+1} &\to (1-p,\cdots,1-p)\end{aligned}$$

and the theorem is proven. ∎

Note that any configuration p^n is mapped to $(1-p)^n$ by rule f_{27} since $f_{27}(p,p,p) = 1-p \ \forall p$, so the convergence values p and $(1-p)$ will vary and depend on the initial configuration.

Summary of results for $t \to \infty$

Rule f_{27}	$x_i^{t+1} = (1-x_{i+1}^t)(1-x_i^t) + x_{i+1}^t(1-x_{i-1}^t)$
Spatial:	$\to p^n$ for some p
Temporal:	$\mathbf{X}^t \to p^n$ implies $\mathbf{X}^{t+1} \to (1-p)^n$

Rule 58

To begin with, we recall rule 58: $x_i^{t+1} = x_{i-1}^t\bar{x}_i^t + (1-x_{i-1}^t)x_{i+1}^t$, which can also be written as: $\bar{x}_i^{t+1} = x_{i-1}^t x_i^t + (1-x_{i-1}^t)\bar{x}_{i+1}^t$.

The proof of the asymptotic behaviour of rule 58 combines both techniques used above since it combines both essential properties: only one value is inverted before averaging so even and odd indices (when the length of the array is even) must be treated separately as for rule 78, and the value of x_i^t itself is inverted in the average so two steps must be taken at a time as in the proof of rule 27. As a result, in the even case we have both spatial and temporal period 2.

Theorem 5. *When n is even, rule 58 is spatially and temporally asymptotic with period 2. Furthermore, if $x_i^{2t} \to p$ then $x_i^{2t+1} \to 1-p$, $x_{i+1}^{2t} \to 1-p$ and $x_{i+1}^{2t+1} \to p$. When n is odd, rule 58 converges to the homogeneous configuration $(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$.*

PROOF Let A be a fuzzy cellular automaton following rule 58 and let $f : (0,1)^3 \to (0,1)$ be its local rule and F its global rule.

Assume that n is even. Let B be a fuzzy cellular automaton with global rule $F \circ F$. Consider a $GWCA$ B' with global rule $F_2' \circ F_1'$ where F_1' and F_2' are given by two local rules $f_1', f_2' : (0,1)^3 \to (0,1)$ defined as follows:

$$\begin{aligned} f_1'(x,y,z) &= (1-x)y + xz \\ f_2'(x,y,z) &= xy + (1-x)z \end{aligned}$$

and where F_1' and F_2' alternate f_1' and f_2' as shown:

$$\begin{aligned} F_1'(x_0, x_1, \cdots, x_{n-1}) &= (f_1'(x_{n-1}, x_0, x_1), f_2'(x_0, x_1, x_2), \cdots), \\ F_2'(x_0, x_1, \cdots, x_{n-1}) &= (f_2'(x_{n-1}, x_0, x_1), f_1'(x_0, x_1, x_2), \cdots). \end{aligned}$$

We want to show that $F \circ F$ and $F_2' \circ F_1'$ are conjugate. Let h be the homeomorphism previously defined for rule 78:

$$h(x_0, x_1, x_2, \ldots, x_{n-1}) = (x_0, 1-x_1, x_2, \ldots, 1-x_{n-1}).$$

Again we need to show that $h \circ F \circ F = F_2' \circ F_1' \circ h$. We will first show that $F_1' \circ h = h' \circ F$, with $h' = h \circ \mathrm{inv}$. Without loss of generality, assume that i is even.

$$\begin{aligned} F_1' \quad \circ \quad & h(\cdots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \cdots) \\ = \quad & F_1'(\cdots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, x_{i+2}, \cdots) \\ & (\cdots, x_{i-1}x_i + \bar{x}_{i-1}\bar{x}_{i+1}, x_i\bar{x}_{i+1} + \bar{x}_i x_{i+2}, \cdots) \end{aligned}$$

$$\begin{aligned} h' \quad \circ \quad & F(\cdots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \cdots) \\ = \quad & h'(\cdots, x_{i-1}\bar{x}_i + \bar{x}_{i-1}x_{i+1}, x_i\bar{x}_{i+1} + \bar{x}_i x_{i+2}, \cdots) \\ = \quad & (\cdots, x_{i-1}x_i + \bar{x}_{i-1}\bar{x}_{i+1}, x_i\bar{x}_{i+1} + \bar{x}_i x_{i+2}, \cdots). \end{aligned}$$

Similarly, $h \circ F = F_2' \circ h'$.
Then

$$\begin{aligned} h \circ F \circ F &= F_2' \circ h' \circ F \\ &= F_2' \circ F_1' \circ h. \end{aligned}$$

Thus $h \circ F \circ F$ and $F_2' \circ F_1' \circ h$ are conjugate. By Theorem 1, B' is converging to a homogeneous configuration of the form $(p, p, \cdots, p)$. Every other iteration of B must also be converging to a configuration of the form $h^{-1}(p, p, \cdots, p) = (p, 1-p, p, \cdots, 1-p)$.

Now for odd time steps we have:

$$\begin{aligned}\lim_{t\to\infty}(x_0^{2t+1}, x_1^{2t+1}, \cdots, x_{n-1}^{2t+1}) &= \lim_{t\to\infty} F(x_0^{2t}, x_1^{2t}, \cdots, x_{n-1}^{2t}) \\ &= F(\lim_{t\to\infty} x_0^{2t}, \lim_{t\to\infty} x_1^{2t}, \cdots, \lim_{t\to\infty} x_{n-1}^{2t}) \\ &= F(p, 1-p, p, \cdots, 1-p) \\ &= (1-p, p, 1-p, \cdots, p).\end{aligned}$$

Summarizing,

$$\begin{aligned}\mathbf{X}^{2t} &\to (p, 1-p, \cdots, p, 1-p) \\ \mathbf{X}^{2t+1} &\to (1-p, p, \cdots, 1-p, p).\end{aligned}$$

We now turn our attention to the case where n is odd. We consider the FCA B described in Theorem 3 and we will show that A with global rule $F \circ F$ is conjugate to B with global rule $F_2' \circ F_1'$ and homeomorphism h as in the proof of Theorem 3 when n is odd.

If we let $h' = h \circ \mathrm{inv}$ as before, we again have that $F_1' \circ h = h' \circ F$ and $h \circ F = F_2' \circ h'$. We give the details for the second equality.

$$\begin{aligned} h &\circ F(x_0, x_1, \cdots, x_{n-1}) \\ &= h(x_{n-1}\bar{x}_0 + \bar{x}_{n-1}x_1, x_0\bar{x}_1 + \bar{x}_0x_2, \cdots, x_{n-2}\bar{x}_{n-1} + \bar{x}_{n-2}x_0) \\ &= (x_{n-1}\bar{x}_0 + \bar{x}_{n-1}x_1, x_0x_1 + \bar{x}_0\bar{x}_2, \cdots, x_{n-2}\bar{x}_{n-1} + \bar{x}_{n-2}x_0, \\ &\quad x_{n-1}x_0 + \bar{x}_{n-1}\bar{x}_1, x_0\bar{x}_1 + \bar{x}_0x_2, \cdots x_{n-2}x_{n-1} + \bar{x}_{n-2}\bar{x}_0).\end{aligned}$$

$$\begin{aligned} F_2' &\circ h'(x_0, x_1, \cdots, x_{n-1}) \\ &= F_2'(\bar{x}_0, x_1, \cdots, \bar{x}_{n-1}, x_0, \bar{x}_1, \cdots, x_{n-1}) \\ &= (x_{n-1}\bar{x}_0 + \bar{x}_{n-1}x_1, x_0x_1 + \bar{x}_0\bar{x}_2, \cdots, x_{n-2}\bar{x}_{n-1} + \bar{x}_{n-2}x_0, \\ &\quad x_{n-1}x_0 + \bar{x}_{n-1}\bar{x}_1, x_0\bar{x}_1 + \bar{x}_0x_2, \cdots, x_{n-2}x_{n-1} + \bar{x}_{n-2}\bar{x}_0).\end{aligned}$$

As in the case of rule 78 with n odd, B must converge to $\frac{1}{2}^n$ so even time steps of A must also converge to $\frac{1}{2}^n$. Odd time steps also will converge to $\frac{1}{2}^n$ since $F(\frac{1}{2}, \cdots, \frac{1}{2}) = (\frac{1}{2}, \cdots, \frac{1}{2})$. ∎

Summary of results for $t \to \infty$

Rule f_{58}	$x_{i-1}^t x_i^t + (1 - x_{i-1}^t)\bar{x}_{i+1}^t$
Spatial:	n odd: $\to (\frac{1}{2})^n$ n even: $\to (p(1-p))^{\frac{n}{2}}$, for some p
Temporal:	Shift

Rule 184
We recall rule 184: $x_i^{t+1} = (1-x_i^t)x_{i-1}^t + x_i^t x_{i+1}^t$. Rules 184, 46, and 29 use x_i^t itself as the weighting factor in the average of its two neighbours. When n is even, even indices are averaged with other even indices, odd with odd, effectively creating two separate weighted averages, one of the even indices, the other of the odd. The weight factors in each case come from the other set of values. We will exploit this structure to determine topologically conjugate FCA where we can apply Theorem 1.

Rule 184 is somewhat special because its Boolean version is known to be the only non trivial density conserving elementary rule [3]: the number of 1s in the initial configuration are preserved during the evolution. In [2], it was shown that the same property applies to the corresponding fuzzy rule as well. We now prove the following convergence theorem.

Theorem 6. *When n is even, rule 184 is asymptotically periodic with period 2 both spatially and temporally. Furthermore, if $\forall i$ as $t \to \infty$, $x_{2i}^{2t} \to p$ and $x_{2i+1}^{2t} \to q$, then $x_{2i+1}^{2t+1} \to p$, and $x_{2i}^{2t+1} \to q$ with $p+q = \frac{2}{n}\sum_{i=0}^{n-1} x_i^0$. When n is odd, rule 184 converges spatially and temporally to a homogeneous configuration. Moreover, if $\forall i$ as $t \to \infty$, $x_i^t \to p$, then $p = \frac{1}{n}\sum_{i=0}^{n-1} x_i^0$.*

PROOF Let A be a fuzzy cellular automaton following rule 184 and let $f : (0,1)^3 \to (0,1)$ by $f(x,y,z) = \bar{y}x + yz$ be its local rule and F its global rule.

When n is even, let B be the fuzzy cellular automaton where the global rule G is the left shift of F:

$$G(x_0, \cdots, x_{n-1}) = (f(x_0,x_1,x_2), f(x_1,x_2,x_3), \cdots, f(x_{n-1},x_0,x_1)).$$

Now let S be the shift self-homeomorphism of B:

$$S(x_0, \cdots, x_{n-1}) = (x_{n-1}, x_0, \cdots, x_{n-2}).$$

Notice that $S \circ G = G \circ S$, also that $F = S \circ G$. Now consider F^n: $F^n = (S \circ G)^n = S^n G^n = G^n$. Thus every n-th iteration of A is equal to every n-th iteration of B. Furthermore, in-between steps can be determined by the appropriate number of shifts of B. We will now determine the asymptotic behaviour of B.

Let B' be the cross product of two $GWCA$ of length $m = n/2$ with global rule G' given by:

$$\begin{aligned} & G'((u_0, \cdots, u_{m-1}) \times (v_0, \cdots, v_{m-1})) \\ = \quad & (f(u_0,v_0,u_1), f(u_1,v_1,u_2), \cdots, f(u_{m-1},v_{m-1},u_0)) \\ \times \quad & (f(v_0,u_1,v_1), f(v_1,u_2,v_2), \cdots, f(v_{m-1},u_0,v_0)), \end{aligned}$$

and with γ as in the proof of Theorem 2.

The topological space B is conjugate to B' under the homeomorphism h defined as follows:

$$h(x_0, \cdots, x_{n-1}) = (x_0, x_2, \cdots, x_{n-2}) \times (x_1, x_3, \cdots, x_{n-1}).$$

To prove this we need to show that $h \circ G = G' \circ h$. The i-th position of $G(x_0, \cdots, x_{n-2})$ is $(1 - x_{i+1})x_i + x_{i+1}x_{i+2}$, so

$$\begin{aligned}
& h \circ G(x_0, \cdots, x_{n-1}) \\
= \quad & h((1-x_1)x_0 + x_1x_2, (1-x_2)x_1 + x_2x_3, \cdots, (1-x_0)x_{n-1} + x_0x_1) \\
= \quad & ((1-x_1)x_0 + x_1x_2, \cdots, (1-x_{n-1})x_{n-2} + x_{n-1}x_0) \\
& \times((1-x_2)x_1 + x_2x_3, \cdots, (1-x_0)x_{n-1} + x_0x_1) \\
= \quad & G'((x_0, x_2, \cdots, x_{n-2}) \times (x_1, x_3, \cdots, x_{n-1})) \\
= \quad & G' \circ h(x_0, \cdots, x_{n-1}).
\end{aligned}$$

Both $GWCA$ in the cross product will converge to fixed points by Theorem 1. If $(p, \cdots, p) \times (q, \cdots, q)$ is the point of convergence of B', then the point of convergence of B is $h^{-1}((p, \cdots, p) \times (q, \cdots, q)) = (p, q, p, \cdots, q)$.

Since, by definition, A's global rule is given by the right shift of B's global rule, the theorem follows. In summary, we have,

$$\begin{aligned}
\mathbf{X}^{2t} \quad & \rightarrow \quad (p, q, \cdots, p, q), \\
\mathbf{X}^{2t+1} \quad & \rightarrow \quad (q, p, \cdots, q, p).
\end{aligned}$$

Now the sum of all values at the point of convergence is $\frac{n}{2}(p+q)$. We know from [2] that this must be equal to the sum of values in the initial configuration. Hence, $p + q = \frac{2}{n}\sum_{i=0}^{n-1} x_i^0$.

With n odd, we consider the system B described as above. However, this time we will consider a homeomorphism h given by:

$$h(x_0, x_1, \cdots, x_{n-1}) = (x_0, x_2, \cdots, x_{n-1}, x_1, x_3, \cdots, x_{n-2})$$

and a $GWCA$ B' of size $2n$ with a new global rule G'. Let $m = \frac{n+1}{2}$, then

$$\begin{aligned}
& G'(x_0, x_1, \cdots, x_{n-1}) \\
= \quad & (f(x_0, x_m, x_1), f(x_1, x_{m+1}, x_2), \cdots, f(x_i, x_{m+i}, f_{i+1}), \cdots)
\end{aligned}$$

where f is local rule 184, and let γ be as in the proof of Theorem 2.

The system B' is conjugate to B under h. To prove this we show that $h \circ G = G' \circ h$.

$$\begin{aligned}
& h \circ G(x_0, x_1, \cdots, x_{n-1}) \\
= \;& h((1-x_1)x_0 + x_1x_2, (1-x_2)x_1 + x_2x_3, \cdots, (1-x_0)x_{n-1} + x_0x_1) \\
= \;& (\bar{x}_1x_0 + x_1x_2, \bar{x}_3x_2 + x_3x_4, \cdots, \bar{x}_0x_{n-1} + x_0x_1, \\
& \bar{x}_2x_1 + x_2x_3, \cdots, \bar{x}_{n-1}x_{n-2} + x_{n-1}x_0).
\end{aligned}$$

$$\begin{aligned}
& G' \circ h(x_0, x_1, \cdots, x_{n-1}) G'(x_0, x_2, \cdots, x_{n-1}, x_1, x_3, \cdots, x_{n-2}) \\
= \;& (\bar{x}_1x_0 + x_1x_2, \bar{x}_3x_2 + x_3x_4, \cdots, \bar{x}_0x_{n-1} + x_0x_1, \\
& \bar{x}_2x_1 + x_2x_3, \cdots, \bar{x}_{n-1}x_{n-2} + x_{n-1}x_0).
\end{aligned}$$

Now G' must converge to a single value and since h merely reorders those values, G must also converge to a single value. Finally, since the shift is the identity function on the point of convergence, A must converge to a single value both spatially and temporally. Moreover, if $\forall i$ as $t \to \infty$, $x_i^t \to p$, then $p = \frac{1}{n}\sum_{i=0}^{n-1} x_i^0$ from [2] . ∎

Note that, given the special nature of rule 184, which conserves density, we have been able not only to determine the asymptotic periodicity, but also to derive the convergence point as a function of the initial configuration.

Summary of results for $t \to \infty$

Rule f_{184}	$x_i^{t+1} = (1-x_i^t)x_{i-1}^t + x_i^t x_{i+1}^t$
Spatial:	n odd: $\to p^n$ with $p = \frac{1}{n}\sum_{i=0}^{n-1} x_i^0$ n even: $\to (pq)^{\frac{n}{2}}$, with $p + q = \frac{2}{n}\sum_{i=0}^{n-1} x_i^0$
Temporal:	shift

Rule 29
We now examine rule 29: $x_i^{t+1} = x_i^t(1 - x_{i-1}^t) + (1 - x_i^t)(1 - x_{i+1}^t)$.

Rule 29, analogously to rule 184, has an interesting density conservation property, both in the fuzzy and the Boolean domains, which has never been observed before: the sum S of the values of a configuration at time t is conserved at any time $t + 2i$, $\forall i$; moreover, the sum of the values of a configuration at any time $t + 2i + 1$, $\forall i$, is $n - S$, as we show below.

Lemma 3. *Given a fuzzy CA with rule 29 and initial configuration* $X = (x_0^0, \cdots, x_{n-1}^0)$*, if* $S = \sum_{i=0}^{n-1} x_i^0$*, then* $\sum_{i=0}^{n-1} x_i^{2t} = S$ *and* $\sum_{i=0}^{n-1} x_i^{2t+1} = n - S$ *for all* t.

PROOF

$$\begin{aligned}
\sum_{i=0}^{n-1} x_i^{t+1} &= \sum_{i=0}^{n-1}[x_i^t(1-x_{i-1}^t)+(1-x_i^t)(1-x_{i+1}^t)] \\
&= \sum_{i=0}^{n-1}[x_i^t - x_i^t x_{i-1}^t + 1 - x_i^t - x_{i+1}^t + x_i^t x_{i+1}^t] \\
&= n - \sum_{i=0}^{n-1} x_{i+1}^t - \sum_{i=0}^{n-1} x_i^t x_{i-1}^t + \sum_{i=0}^{n-1} x_i^t x_{i+1}^t \\
&= n - \sum_{i=0}^{n-1} x_i^t.
\end{aligned}$$

In particular, $\sum_{i=0}^{n-1} x_i^1 = n - S$.

Now at time $t+2$, $\sum_{i=0}^{n-1} x_i^{t+2} = n - \sum_{i=0}^{n-1} x_i^{t+1} = n - (n - \sum_{i=0}^{n-1} x_i^t) = \sum_{i=0}^{n-1} x_i^t$, so every other iteration will have the same density, as required. ∎

This lemma obviously applies as well in the Boolean case.

Corollary 1. *If the initial configuration has length n and sum to d, all subsequent even numbered iterations will sum to d while odd iterations will sum to $n-d$.*

We will show convergence using sub-arrays as in the previous case. However, the rules for each sub-array are averages of the inverted values as for rule 27 so we will need to consider two steps at a time. The convergence lemma allows us to describe more precisely the point of convergence.

Theorem 7. *When n is even, rule 29 is asymptotically periodic with period 2 both spatially and temporally. Furthermore, if $\forall i$ as $t \to \infty$, $x_{2i}^{2t} \to p$, and $x_{2i+1}^{2t} \to q$ then $x_{2i}^{2t+1} \to 1-q$, and $x_{2i+1}^{2t+1} \to 1-p$ and $p+q = \frac{2}{n}\sum_{i=0}^{n-1} x_i^0$. When n is odd, rule 29 converges spatially to a homogeneous configuration, $x_i^{2t} \to p = \frac{1}{n}\sum_{i=0}^{n-1} x_i^0$ and $x_i^{2t+1} \to 1-p$, and is asymptotically periodic with period 2 temporally.*

PROOF We proceed as for rule 184. Let A be a fuzzy cellular automaton following rule 29 and let $f : (0,1)^3 \to (0,1)$ be its local rule and F its global rule.

Let B be a fuzzy cellular automaton with global rule G given by the shift of F:

$$G(x_0, \cdots, x_{n-1}) = (f(x_0, x_1, x_2), f(x_1, x_2, x_3), \cdots, f(x_{n-1}, x_0, x_1)).$$

When n is even, we let B' be the cross product of two $GWCA$ of length $m = n/2$ with global rules G'_1 and G'_2 given by local rules f_1 and f_2 as follows:

$$\begin{array}{rcl} f_1(x,y,z) & = & yx + (1-y)z, \\ f_2(x,y,z) & = & (1-y)x + yz, \end{array}$$

$$\begin{array}{rl} & G'_i((u_0, \cdots, u_{m-1}) \times (u_0, \cdots, u_{m-1})) \\ = & (f_i(u_0, v_0, u_1), f_i(u_1, v_1, u_2), \cdots, f_i(u_{m-1}, v_{m-1}, u_0)) \\ \times & (f_i(v_0, u_1, v_1), f_i(v_1, u_2, v_2), \cdots, f_i(v_{m-1}, u_0, v_0)) \end{array}$$

for $i = 1, 2$.

The topological space B using rule $G \circ G$ is conjugate to B' using rule $G'_2 \circ G'_1$ under the homeomorphism h defined as follows:

$$h(x_0, \cdots, x_{n-1}) = (x_0, x_2, \cdots, x_{n-2}) \times (x_1, x_3, \cdots, x_{n-1}).$$

To show this, we consider a second homeomorphism $h' = h \circ \mathrm{inv}$ which maps B to B' by

$$h'(x_0, \cdots, x_{n-1}) = (\bar{x}_0, \bar{x_2}, \cdots, \bar{x}_{n-2}) \times (\bar{x_1}, \bar{x_3}, \cdots, \bar{x}_{n-1}).$$

We first show that $G'_2 \circ h' = h \circ G$

$$\begin{array}{rcl} G'_2 & \circ & h'(x_0, x_1, x_2, x_3, \cdots) \\ & = & G'_2((\bar{x}_0, \bar{x_2}, \cdots, \bar{x}_{n-2}) \times (\bar{x}_1, \bar{x}_3, \cdots, \bar{x}_{n-1}) \\ & = & (f_2(\bar{x}_0, \bar{x}_1, \bar{x}_2), f_2(\bar{x}_2, \bar{x}_3, \bar{x}_4), \cdots) \\ & \times & (f_2(\bar{x}_1, \bar{x}_2, \bar{x}_3), f_2(\bar{x}_3, \bar{x}_4, \bar{x}_5), \cdots) \\ & = & ((x_1\bar{x}_0 + \bar{x}_1\bar{x}_2), (x_3\bar{x}_2 + \bar{x}_3\bar{x}_4), \cdots) \\ & \times & ((x_2\bar{x}_1 + \bar{x}_2\bar{x}_3), (x_4\bar{x}_3 + \bar{x}_4\bar{x}_5), \cdots), \\ h & \circ & G(x_0, x_1, x_2, x_3, \cdots) \\ & = & h((x_1\bar{x}_0 + \bar{x}_1\bar{x}_2, x_2\bar{x}_1 + \bar{x}_2\bar{x}_3, x_3\bar{x}_2 + \bar{x}_3\bar{x}_4, \cdots)) \\ & = & ((x_1\bar{x}_0 + \bar{x}_1\bar{x}_2), (x_3\bar{x}_2 + \bar{x}_3\bar{x}_4), \cdots) \\ & \times & ((x_2\bar{x}_1 + \bar{x}_2\bar{x}_3), (x_4\bar{x}_3 + \bar{x}_4\bar{x}_5), \cdots). \end{array}$$

Furthermore, $G'_1 \circ h = h' \circ G$:

$$\begin{aligned} G'_1 \quad \circ &\quad h(x_0, x_1, x_2, x_3, \cdots) \\ = &\quad G'_1((x_0, x_2, \cdots x_{n-2}) \times (x_1, x_3, \cdots x_{n-1})) \\ = &\quad (x_1 x_0 + \bar{x}_1 x_2, x_3 x_2 + \bar{x}_3 x_4, \cdots) \\ &\quad \times (x_2 x_1 + \bar{x}_2 x_3, x_4 x_3 + \bar{x}_4 x_5, \cdots) \\ = &\quad h'((x_1 \bar{x}_0 + \bar{x}_1 \bar{x}_2, x_2 \bar{x}_1 + \bar{x}_2 \bar{x}_3, x_3 \bar{x}_2 + \bar{x}_3 \bar{x}_4, x_4 \bar{x}_3 + \bar{x}_4 \bar{x}_5, \cdots) \\ = &\quad h' \circ G(x_0, x_1, x_2, x_3, \cdots). \end{aligned}$$

Finally, we have,

$$\begin{aligned} G'_2 \circ (G'_1 \circ h) &= (G'_2 \circ h') \circ G \\ &= h \circ G \circ G, \end{aligned}$$

as required.

By Theorem 1, both $GWCA$ in B' converge to homogeneous configurations. If $(p, \cdots, p) \times (q, \cdots, q)$ is the point of convergence of B', then $h^{-1}((p, \cdots, p) \times (q, \cdots, q)) = (p, q, p, \cdots, q)$ is the point of convergence of B.

Since, by definition, A's global rule is given by the right shift of B's global rule, the theorem follows and we have,

$$\mathbf{X}^{2t} \to (p, q, \cdots, p, q)$$

and

$$\begin{aligned} \mathbf{X}^{2t+1} &\to F(p, q, \cdots, p, q) \\ &\to (1-q, 1-p, \cdots, 1-q, 1-p). \end{aligned}$$

That $p + q = \frac{2}{n} \sum_{i=0}^{n-1} x_i^0$ follows from Lemma 3.

When n is odd, let B' and h be as in the proof of Theorem 6, the case of rule 184 when n is odd. But let B' transform under $G'_2 \circ G'_1$ where

$$\begin{aligned} &G'_j(x_0, x_1, \cdots, x_{2n-1}) \\ = &\quad (f_j(x_0, x_m, x_1), f_j(x_1, x_{m+1}, x_2), \cdots, f_j(x_i, x_{i+m}, x_{i+1}), \cdots) \end{aligned}$$

for $j = 1, 2$ and f_j defined as for n even above.

Let h' be the negation of h, so that $h' = h \circ \text{inv}$

$$h'(x_0, x_1, \cdots, x_{n-1}) = (\bar{x}_0, \bar{x}_2, \cdots, \bar{x}_{n-1}, \bar{x}_1, \cdots, \bar{x}_{n-2}).$$

Then $h \circ G \circ G = G_2' \circ G_1' \circ h$. Again we prove this by showing that $h \circ G = G_2' \circ h'$ and $h' \circ G = G_1' \circ h$:

$$\begin{aligned} h \quad &\circ \quad G(x_0, x_1, \cdots, x_{n-1}) \\ &= h(x_1\bar{x}_0 + \bar{x}_1\bar{x}_2, x_2\bar{x}_1 + \bar{x}_2\bar{x}_3, \cdots, x_0\bar{x}_{n-1} + \bar{x}_0\bar{x}_1) \\ &= (x_1\bar{x}_0 + \bar{x}_1\bar{x}_2, x_3\bar{x}_2 + \bar{x}_3\bar{x}_4, \cdots, x_0\bar{x}_{n-1} + \bar{x}_0\bar{x}_1, \\ &\quad x_2\bar{x}_1 + \bar{x}_2\bar{x}_3, \cdots, x_{n-1}\bar{x}_{n-2} + \bar{x}_{n-1}\bar{x}_0). \end{aligned}$$

$$\begin{aligned} G_2' \quad &\circ \quad h'(x_0, x_1, \cdots, x_{n-1}) \\ &= G_2'(\bar{x}_0, \bar{x}_2, \cdots, \bar{x}_{n-1}, \bar{x}_1, \cdots, \bar{x}_{n-2}) \\ &= (x_1\bar{x}_0 + \bar{x}_1\bar{x}_2, x_3\bar{x}_2 + \bar{x}_3\bar{x}_4, \cdots, x_0\bar{x}_{n-1} + \bar{x}_0\bar{x}_1, \\ &\quad x_2\bar{x}_1 + \bar{x}_2\bar{x}_3, \cdots, x_{n-1}\bar{x}_{n-2} + \bar{x}_{n-1}\bar{x}_0) \\ &= h \circ G(x_0, x_1, \cdots, x_{n-1}). \end{aligned}$$

$$\begin{aligned} h' \quad &\circ \quad G(x_0, x_1, \cdots, x_{n-1}) \\ &= h(x_1\bar{x}_0 + \bar{x}_1\bar{x}_2, x_2\bar{x}_1 + \bar{x}_2\bar{x}_3, \cdots, x_0\bar{x}_{n-1} + \bar{x}_0\bar{x}_1) \\ &= (x_1 x_0 + \bar{x}_1 x_2, x_3 x_2 + \bar{x}_3 r x_4, \cdots, x_0 x_{n-1} + \bar{x}_0 x_1, \\ &\quad x_2 x_1 + \bar{x}_2 x_3, \cdots, x_{n-1} x_{n-2} + \bar{x}_{n-1} x_0) \\ &= G_1'(x_0, x_2, \cdots, x_{n-1}, x_1, \cdots, x_{n-2}) \\ &= G_1' \circ h(x_0, x_1, \cdots, x_{n-1}). \end{aligned}$$

Since B' satisfies Theorem 1, it converges to a homogeneous configuration $(p, \cdots, p)$ and even time steps of B must also converge to a $(p, \cdots, p)$. At odd time interval,

$$\begin{aligned} \lim_{t\to\infty} x_i^{2t+1} &= \lim_{t\to\infty} f(x_{i-1}^{2t}, x_i^{2t}, x_{i+1}^{2t}) \\ &= f(\lim_{t\to\infty} x_{i-1}^{2t}, \lim_{t\to\infty} x_i^{2t}, \lim_{t\to\infty} x_{i+1}^{2t}) \\ &= f(p,p,p) \\ &= p(1-p) + (1-p)(1-p) \\ &= 1-p. \end{aligned}$$

In other words,

$$\mathbf{X}^{2t} \to (p, \cdots, p)$$

$$\mathbf{X}^{2t+1} \to (1-p, \cdots, 1-p).$$

Again, the value of p is given by Lemma 3 ∎

Summary of results for $t \to \infty$

Rule f_{29}	$x_i^{t+1} = x_i^t(1-x_{i-1}^t) + (1-x_i^t)(1-x_{i+1}^t)$
Spatial:	n odd: $\to p^n$, for $p = \frac{2}{n}\sum_{i=0}^{n-1} x_i^0$ or $1 - \frac{2}{n}\sum_{i=0}^{n-1} x_i^0$ n even: $\to (pq)^{\frac{n}{2}}$, for some p, q where $p + q = \frac{2}{n}\sum_{i=0}^{n-1} x_i^0$
Temporal:	$\mathbf{X}^t \to (pq)^{\frac{n}{2}}$ implies $\mathbf{X}^{t+1} \to ((1-q)(1-p))^{\frac{n}{2}}$

Rule 46

Rule 46 yields the most interesting results of all FCA in this group and it appears to be unique among all elementary FCA. From the experimental observations of [9], it is the only rule with periodic behaviour of length 4. It once again uses the central value as the weight factor, but the resulting rule individuates sub-automata where the rule uses one value directly while the other is inverted resulting in temporally periodic behaviour in each of the $GWCA$ in the cross product.

Theorem 8. *When n modulo 4 is equal to 0, rule 46 is asymptotically periodic with period 4 both spatially and temporally. Furthermore, if $\forall i$ as $t \to \infty$, $x_{4i}^{4t} \to p$, and $x_{4i+1}^{4t} \to q$ then $x_{4i+2}^{4t} \to 1-p$, and $x_{4i+3}^{4t} \to 1-q$. Also, $x_{4i}^{4t+1} \to q$, $x_{4i}^{4t+2} \to 1-p$, and $x_{4i}^{4t+3} \to 1-q$. When n modulo 4 is not equal to 0, rule 46 converges to the homogeneous configuration $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.*

PROOF Let A be a fuzzy cellular automaton following rule 46 and let $f : (0,1)^3 \to (0,1)$ be its local rule and F its global rule.

Assume, to begin with, that n modulo 4 is 0. Let B be a fuzzy CA with global rule G given by the left shift of F:

$$G(x_0, \cdots, x_{n-1}) = (f(x_0, x_1, x_2), f(x_1, x_2, x_3), \cdots, f(x_{n-1}, x_0, x_1)).$$

Let B' be the usual cross product with global rule G' given by local rules f_1 and f_2 as follows:

$$\begin{aligned} f_1(x,y,z) &= yx + (1-y)z, \\ f_2(x,y,z) &= (1-y)x + yz, \end{aligned}$$

$$\begin{aligned} & G'((u_0, \cdots, u_{m-1}) \times (u_0, \cdots, u_{m-1})) \\ = \ & (f_1(u_0, v_0, u_1), f_2(u_1, v_1, u_2), \cdots, f_2(u_{m-1}, v_{m-1}, u_0)) \\ \times \ & (f_2(v_0, u_1, v_1), f_1(v_1, u_2, v_2), f_2(v_2, u_3, v_3) \cdots, f_1(v_{m-1}, u_0, v_0)). \end{aligned}$$

The topological space B using rule $G \circ G$ is conjugate to B' using rule $G' \circ G'$ under the homeomorphism h:

$$h(x_0, \cdots, x_{n-1}) = (x_0, \bar{x}_2, x_4, \cdots, \bar{x}_{n-2}) \times (x_1, \bar{x}_3, x_5, \cdots, \bar{x}_{n-1}).$$

To show this we consider a second homeomorphism $h' = h \circ \mathrm{inv}$ which maps B to B' by

$$h'(x_0, \cdots, x_{n-1}) = (\bar{x}_0, x_2, \bar{x}_4, \cdots, x_{n-2}) \times (\bar{x}_1, x_3, \bar{x}_5, \cdots, x_{n-1}).$$

We first show that $G' \circ h = h' \circ G$:

$$\begin{array}{ccl}
G' & \circ & h(x_0, x_1, x_2, x_3, \cdots) \\
 & = & G'(x_0, \bar{x}_2, x_4, \cdots, \bar{x}_{n-2}) \times (x_1, \bar{x}_3, x_5, \cdots, \bar{x}_{n-1}) \\
 & = & (f_1(x_0, x_1, \bar{x}_2), f_2(\bar{x}_2, \bar{x}_3, x_4), \cdots) \\
 & \times & (f_2(x_1, \bar{x}_2, \bar{x}_3), f_1(\bar{x}_3, x_4, x_5), \cdots) \\
 & = & ((x_1 x_0 + \bar{x}_1 \bar{x}_2), (x_3 \bar{x}_2 + \bar{x}_3 x_4), \cdots) \\
 & \times & ((x_2 x_1 + \bar{x}_2 \bar{x}_3), (x_4 \bar{x}_3 + \bar{x}_4 x_5), \cdots), \\
h' & \circ & G(x_0, x_1, x_2, x_3, \cdots) \\
 & = & h'((x_1 \bar{x}_0 + \bar{x}_1 x_2, x_2 \bar{x}_1 + \bar{x}_2 x_3, x_3 \bar{x}_2 + \bar{x}_3 x_4, x_4 \bar{x}_3 + \bar{x}_4 x_5, \cdots)) \\
 & = & ((x_1 x_0 + \bar{x}_1 \bar{x}_2), (x_3 \bar{x}_2 + \bar{x}_3 x_4), \cdots) \\
 & \times & ((x_2 x_1 + \bar{x}_2 \bar{x}_3), (x_4 \bar{x}_3 + \bar{x}_4 x_5), \cdots).
\end{array}$$

Similarly, $h \circ G = G' \circ h'$.

Finally, we have,

$$\begin{array}{rcl}
G' \circ (G' \circ h) & = & (G' \circ h') \circ G \\
 & = & h \circ G \circ G,
\end{array}$$

as required.

Since both systems in the cross product of B' are GWCA with γ as in the proof of Theorem 2, they will both converge to fixed points by Theorem 1. If $(p, \cdots, p) \times (q, \cdots, q)$ is the point of convergence of B', then $h^{-1}((p, \cdots, p) \times (q, \cdots, q)) = (p, q, 1-p, 1-q, p, q, 1-p, 1-q \cdots, p, q, 1-p, 1-q)$ is the point of convergence of B.

Since A is the right shift of B:

$$\begin{aligned}
\mathbf{X}^{4t} &\rightarrow (p, q, 1-p, 1-q, p, q, \cdots, 1-p, 1-q), \\
\mathbf{X}^{4t+1} &\rightarrow (q, 1-p, 1-q, p, \cdots, q, 1-p, 1-q, p), \\
\mathbf{X}^{4t+2} &\rightarrow (1-p, 1-q, p, q, \cdots, 1-p, 1-q, p, q), \\
\mathbf{X}^{4t+3} &\rightarrow (1-q, p, q, 1-p, \cdots, 1-q, p, q, 1-p).
\end{aligned}$$

When n is even but not divisible by 4, let B' be a cross product of 2 GWCA of length m, and let h map B to B' as follows:

$$\begin{aligned}
h(x_0, x_1, \cdots, x_{n-1}) &= (x_0, \bar{x}_2, x_4, \cdots, x_{n-2}, \bar{x}_0, x_2, \cdots, \bar{x}_{n-2}) \\
&\times (x_1, \bar{x_3}, \cdots, x_{n-1}, \bar{x}_1, x_2, \cdots, \bar{x}_{n-1}).
\end{aligned}$$

Let $h' = h \circ \mathrm{inv}$. Let G' be as in the proof of the previous theorem.

As before, we can show that $h \circ G \circ G = G' \circ G' \circ h$ by showing the intermediary steps using h'. This time, however, we see that if $x_i \rightarrow p$ then $1 - x_i \rightarrow p$ also and so $p = \frac{1}{2}$.

When n is odd, we consider one $GWCA$ of length $2n$, which alternates original values and negations. So for n modulo 4 equal to 1, we will have:

$$(x_0, \bar{x}_2, x_4, \cdots, x_{n-1}, \bar{x}_1, x_3, \cdots, x_{n-2}, \bar{x}_0, x_2, \cdots, \bar{x}_{n-1}, x_1, \bar{x}_3, \cdots, \bar{x}_{n-2}),$$

and for n modulo 4 equal to 3 we will have:

$$(x_0, \bar{x}_2, x_4, \cdots, \bar{x}_{n-1}, x_1, \bar{x}_3, \cdots, x_{n-2}, \bar{x}_0, x_2, \cdots, x_{n-1}, \bar{x}_1, x_3, \cdots, \bar{x}_{n-2}).$$

The rule G' uses appropriate weights to average neighbours in this array. Clearly, due to its structure, B' must converge to $\frac{1}{2}^n$. ∎

Summary of results for $t \rightarrow \infty$

Rule f_{46}	$x_i^{t+1} = (1 - x_i^t)x_{i+1}^t + x_i^t \bar{x}_{i-1}^t$
Spatial:	$n = 4k$: $\rightarrow (pq(1-p)(1-q))^{\frac{n}{4}}$, for some p, q $n \neq 4k$: $\rightarrow \frac{1}{2}^n$
Temporal:	shift

4 CONCLUDING REMARKS AND OPEN PROBLEMS

In this paper, we have focused on the asymptotic behaviour of circular elementary FCA beginning the analytical explanation of the empirical classification suggested in [9]. We started with weighted average rules: a class of rules that contains most of the observed behaviours and we studied their dynamics confirming the observations.

Based on the insights obtained by this analysis and on careful study of other common properties, we are are now studying the other elementary CA with the goal of completing the analysis of all elementary rules.

Assuming that the empirical classification is indeed accurate, an interesting open problem is to understand the reasons for these particular period lengths: why are all elementary circular FCA of size n eventually converging to temporal periodicity of length 1,2,4, or n? The periodicity is clearly connected to circularity and to the neighbourhood's size, but the explanation for these particular lengths is not known.

The asymptotic analyses in this paper rely heavily on the circularity of the lattice; quiescent backgrounds obviously do provide different types of convergence [16]. It would also be interesting to understand the behaviour for infinite configurations.

Finally, it has been proven in [2] that the fixed points of a FCA based on DNF-fuzzification are the stable densities of the mean field approximation for the corresponding Boolean rule. The most obvious consequence of this result is that when a FCA converges to a homogeneous configuration, the point of convergence is a rough estimate of the asymptotic density of the Boolean rule (this is the case, for example, of weighted average rule 172). What is more interesting, and is now under investigation, is what the FCA indicate about the density of the Boolean rule when they do not converge homogeneously.

5 ACKNOWLEDGEMENTS

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