UNIFORM SCATTERING OF AUTONOMOUS MOBILE ROBOTS IN A GRID

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ABSTRACT

We consider the uniform scattering problem for a set of autonomous mobile robots deployed in a grid network: starting from an arbitrary placement in the grid, using purely localized computations, the robots must move so to reach in finite time a state of static equilibrium in which they cover uniformly the grid. The theoretical quest is on determining the minimal capabilities needed by the robots to solve the problem. We prove that uniform scattering is indeed possible even for very weak robots. The proof is constructive. We present a provably correct protocol for uniform self-deployment in a grid. The protocol is fully localized, collision-free, and it makes minimal assumptions: in particular: (1) it does not require any direct or explicit communication between robots; (2) it makes no assumption on robots synchronization or timing, hence the robots can be fully asynchronous in all their actions; (3) it requires only a limited visibility range; (4) it uses at each robot only a constant size memory, hence computationally the robots can be simple Finite-State Machines; (5) it does not need a global localization system but only orientation in the grid (e.g., a compass); (6) it does not require identifiers, hence the robots can be anonymous and totally identical.
1. Introduction

Consider a set of autonomous computational entities, called robots or agents, located in a grid network. A robot is capable of limited sensing and computational activities, and it can move in the grid from a node to a neighboring node. The initial positions of the robots in the grid is arbitrary; the goal is to reach a state of static equilibrium in which they are uniformly scattered in the grid. This problem, called uniform scattering (or covering, or self-deployment), occurs in practice, e.g. when robots are randomly deployed in a region but the requirement is for the region to be covered uniformly, so to maximize coverage. A scattering or self-deployment algorithm, the same for all robots, will specify which operations a robot must perform whenever it is active to achieve this goal. An essential requirement is clearly that the robots will reach a state of static equilibrium, that is the scattering will be completed within finite time.

How uniform scattering can be efficiently accomplished in a region is the subject of extensive research in several fields. In cooperative mobile robotics and swarm robotics, this question has been posed under the terms scattering, coverage, and as a special case of formation (e.g. [2, 3, 5, 11, 13, 17, 18]). Similar questions have been posed in terms of self-deployment in mobile sensor networks and in networks of robotic sensors (e.g., [7, 8, 9, 10, 15, 16]).

The existing protocols differ greatly from each other depending on the assumptions they make. The first and foremost difference is on the nature of the environment, depending on whether the environment is a subset of two-dimensional plane, a setting usually called continuous (e.g., [5, 14, 17, 18]), or it is a network or a graph, a setting usually called discrete or graph world (e.g., [1, 9, 11]). Other major differences exist depending on whether: the robots’ actions are synchronized (e.g., [2, 11, 17, 18]) or no timing assumptions exist (e.g., [4, 5, 13]); the robots have persistent memory (e.g., [2, 7, 18]) or are oblivious (e.g., [5, 10]); the robots have the computational power of Turing machines (e.g., [12, 17, 18]) or are simple Finite State machines (e.g., [1, 11]); the visibility/communication range is limited (e.g., [4, 7, 6, 11]) or extends to the entire region (e.g., [5]); exact or approximate uniform covering is reached within finite time (e.g., [4, 5, 11]) or the protocol converges without ever terminating (e.g., [2, 3]); the protocol is generic, i.e., it operates in any space/network (e.g., [5, 12, 18]) or only in specific for classes of regions/graphs (e.g., [2, 3, 6]); etc.

The robots we consider here are autonomous (i.e., without a central control), anonymous (i.e., identical), asynchronous (i.e., their actions take a finite but unpredictable amount of time), randomly dispersed in the environment, without a global localization system, with limited sensing range (called visibility radius), with no communication capabilities, and constant size memory.

Under these conditions, there is no generic protocol capable of a uniform coverage. This fact is hardly surprising considering both the weakness of the robots and the requirement that a generic protocol must work in any environment regardless of its topology or structure. This fact opens the natural question of whether it is possible for these weak robots to self-deploy achieving uniform coverage in spe-
specific environments (e.g., corridors, grids, rims). A solution algorithm has recently
been developed for the line (e.g., a rectilinear corridor) [2], and several have been
designed for the ring (e.g., the border of a convex region) [3, 4, 13].

We are interested in the uniform scattering of these weak robots in a grid:
starting from an initial random placement on the grid, the robots must position
themselves in the grid at equal distance, within finite time (see Figure 1). The
existing investigations on uniform covering of a grid by weak robots [1, 11] assume
that the robots enter the grid one at a time from fixed locations. However, nothing
is known, to date, on whether or not the problem is solvable when the robots are,
initially, arbitrarily dispersed in the grid.

In this paper we prove that indeed uniform coverage by very weak robots is
possible in a square grid starting from any arbitrary initial configuration and with-
out any collisions. The proof is constructive: we present a collision-free protocol
for uniform scattering in a grid and prove its correctness. The protocol is fully
localized and decentralized, and it makes minimal assumptions; in particular: (1) it
does not require any direct or explicit communication between robots; (2) it makes
no assumption on robots synchronization or timing, hence the robots can be fully
asynchronous in all their actions; (3) it requires only a limited visibility range; (4)
it uses at each robot only a constant persistent memory, hence computationally the
robots can be simple Finite-State Machines; (5) it does not need a global localiza-
tion system but only orientation in the grid (e.g., a compass); (6) it does not require
identifiers, hence the robots can be anonymous and totally identical.

2. Model

The system is composed of a set $S$ of $K = (k + 1)^2$ identical computational
entities, called robots, dispersed in a a square grid $G$ of $N = (n + 1) \times (n + 1)$ nodes,
where $n = k \cdot d$, $d \geq 2$, and $k \geq 2$.

The robots in $S$ are autonomous, asynchronous, anonymous computational units
with sensory and locomotion capabilities. Each robot functions according to an
algorithm (the same for all robots) preprogrammed into it, and can move from a
node to any neighbouring node in the grid. They have a local sense of orientation (i.e., each robot has a consistent notion of "up-down" and "left-right", e.g., as provided by a compass) but no access to any global localization system, so in general they do not know their own position in the grid. The common orientation will be modeled by assuming that the edges of the grid are consistently labeled $Up$, $Down$, $Left$ and $Right$, and edge labels are visible to the robots. We will indicate by $(0,0)$ the left-most lower corner of $G$ and by $(i,j)$ the node belonging to column $i$ and row $j$.

The sensory devices on the robot allow it to have a vision of its immediate surrounding; we assume the robots to have restricted vision of radius $2d$ around it; i.e., a robot sees all nodes at distance (in the Manhattan or city block metric) at most $2d$ from it and the links between them. Even if two robots see each-other, they do not have any explicit means of communicating with each-other.

Each robot performs the following sequence of operations: $Look$: the robot obtains a snapshot of its surroundings within its visibility radius $2d$, to see which nodes are empty and which are occupied and on which links robots are moving; $Compute$: based on the obtained snapshot and the rules of the algorithm, the robot chooses as its next destination either one of the neighboring nodes or the node where it resides; $Move$: the robot moves to the computed destination node. The sequence $Look$-$Compute$-$Move$ constitutes the unit cycle, and each robot reiterates this cycle endlessly.

The robots are fully asynchronous; that is, between successive operations, the robot can be idle for a finite but unpredictable amount of time. While $Look$ is by definition instantaneous, $Compute$ and $Move$ may take an unpredictable (but finite) amount of time. Nevertheless, without any loss of generality, it is possible to assume that also these two operation are instantaneous. This is trivially true in the case of $Compute$ (e.g., by attributing its duration to idle time after the operation); in the case of $Move$, every protocol can be easily modified by adding rules to ensure atomicity of the operation (e.g., the current cycle will be ended if the snapshot shows some robots on the edges). Hence, in the following, we will assume that all operations are instantaneous.

The uniform scattering (or self-deployment) problem consists of starting from a random initial placement of the robots in $G$ and to reach an equilibrium configuration where the robots cover the grid evenly. Since $n = kd$ such an equilibrium configuration consists of nodes $(i \cdot d, j \cdot d)$, with $i, j \in [0, k]$ hosting exactly one robot each (see Fig 1).

In the description of the algorithm for robot $s$ we use the following notation: $\ell_t(s)$ denotes the location of $s$ at time $t$ (i.e., it is a pair $(i,j) \in [0,n]^2$), and $(x,y)_t^+$ denotes coordinates relative to $s$ at time $t$ (i.e. $(x,y)_t^+ = \ell_t(s) + (x,y)$). The algorithm will use only relative coordinates because the robots are not aware of their global coordinates; a robot, in fact, can only understand whether it is located inside the grid, on a corner or on a border. Note that, due to orientation it can also distinguish the four borders and corners, and due to visibility it can “see” a border or a corner if at distance smaller than $2d - 1$ from it: in particular, if it sees
a border it can also understand its global coordinates.

The visibility set of $s$ at time $t$ is the set of the nodes of the grid that are visible by $s$ at time $t$; that is $V_t(s) = \{\ell(s) + (x, y) : |x| + |y| \leq 2d\} \cap [0, n] \times [0, n]$.

Let $V_t^*(s) = \{\ell(s) + (x, y) : |x| + |y| \leq 2d, x > -2d\} \cap [1, n] \times [0, n]$ indicate the visibility set $V_t(s)$ of $s$ excluding any node that is, or might be, on the left border of the grid (we will need such a set for a phase of the algorithm). Notice that if robot $s$ does not “see” the border, the node $\ell(s) + (-2d, 0)$, which it perceives as $(-2d, 0)^+$, might belong to the border, so the need to exclude it as well.

In the following, when no ambiguity arises, we will omit the subscript $t$.

3. Uniform Scattering Protocol

The uniform scattering algorithm $\text{SCATTER}$ is a set of local rules for the robots. Each robot has a state variable (initialized to $-1$) which determines the set of rules to be followed, which solely depend on its current state, its position, and the positions of the robots within its visibility radius. Recall that if $0 \leq \delta < 2d$, a robot is able to determine whether it is at a distance $\delta$ from any of the borders of the grid.

Upon startup, the execution of the algorithm is logically divided in three phases.

- **Cleaning**: robots (if any) on the left and bottom borders move leaving those nodes empty (it is performed only by robots in state 0).
- **Collecting**: robots move towards the left-upper corner of the grid (it is performed by robots in states 1 and 2).
- **Deploying**: robots follow a distinguished path on the grid, called $\text{snake-path}$, eventually occupying their final positions (it is performed by robots in states 3 and 4). We refer to the first robot to enter this phase the head of the snake; however, such a robot does not need to be aware of being the head.

Waking up for the first time (in state $-1$), each robot executes the following set of rules:

<table>
<thead>
<tr>
<th>ROBOT $s$ IN STATE $-1$ (waking up for the first time):</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>LOOK</strong> $V(s)^*$</td>
</tr>
</tbody>
</table>

- If $s$ is at **left or bottom border**, but not on **upper corner** (i.e., $\ell(s) = (0, j)$, with $j < n$, or at $\ell(s) = (i, 0)$, with $i > 0$): $s$ enters state 0 (i.e., it performs **Cleaning**).
- If $s$ is **not in the left nor in the bottom border** (i.e., $\ell(s) = (i, j)$, with $i, j \geq 1$): $s$ enters state 1 (i.e., it performs **Collecting**).
- If $s$ is at **left upper corner**: (i.e., $\ell(s) = (0, n)$): $s$ enters state 3 (i.e., it starts **Deploying**).

Note that, due to asynchrony, at any point in time robots could be performing actions belonging to different phases. In particular, all robots that wake up for the first time on the left or bottom borders, enter the **Cleaning** phase; robots that are already inside the grid the first time they wake up enter the **Collecting** phase; if a
robot starts the algorithm in the upper left corner, it enters directly the Deploying phase. The asynchronous execution of the various phases requires special care in order to insure that robots continue to progress in their phase avoiding collisions and the creation of deadlocks.

3.1. Algorithm scatter: Cleaning

The Cleaning phase is performed only by robots in state 0. The goal is to have the robots move from the left and bottom borders. If moving from the bottom, a robot enters state 1 and starts the Collecting phase. The rules are described below (see also Figure 2).

<table>
<thead>
<tr>
<th>Robot s in state 0:</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOOK $V(s)^*$</td>
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<tr>
<td></td>
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<tr>
<td>• If $s$ is at left border but not on the corners (i.e., $\ell(s) = (0, j)$, with $0 &lt; j &lt; n$):</td>
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<tr>
<td></td>
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<tr>
<td></td>
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<tr>
<td>• If $s$ is at left bottom corner (i.e., $\ell(s) = (0, 0)$).</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>• If $s$ is at bottom border but not on the left corner (i.e., $\ell(s) = (j, 0)$, with $0 &lt; j \leq n$).</td>
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</tbody>
</table>
3.2. Algorithm SCATTER: Collecting

The Collecting phase is performed by robots in states 1 or 2. The goal of the robots is to arrive to the left-upper corner, i.e. node \((0, n)\). To avoid conflicts with robots possibly already in the Deploying phase, during the Collecting phase a robot should not consider robots that are on the left border; i.e., it limits its observation to \(V^*(s)\). With this aim, robots act according to the following rules\(^a\) (see also Figure 3).

**Robot s in state 1:**

**LOOK** \(V^*(s)\)

- If \(s\) is on the right neighbor of the upper left corner (i.e. \(\ell(s) = (1, n)\)):
  
  - If the upper left corner \(((0, n))\) is empty, \(s\) goes left and enters state 3 (i.e., starts the Deploying phase).
  
  - Otherwise, \(s\) waits and stays in state 1.

- Otherwise (i.e., \(\ell(s) = (i, j), i, j > 0; \ell(s) \neq (1, n)\)):
  
  - If the upper node \(((0, 1)^+)\) is empty, \(s\) goes up and stays in state 1.
  
  - If all the above nodes in its same column (i.e., \((0, y)^+, 1 \leq y \leq 2d\)), are occupied and the left and bottom-left nodes \((-1, 0)^+\) and \((-1, -1)^+\) are empty, \(s\) goes left and stays in state 1.\(^a\).
  
  - If within \(V(s)^*\) all the following conditions are satisfied, then \(s\) goes right and enters state 2.
    
    - All the nodes not below nor right (i.e., \((x, y)^+, x \leq 0 \text{ and } y \geq 0\), are occupied.
    
    - If the lower right node (i.e., \((1, -1)^+\)) is empty.
    
    - All the nodes in its same row on the right (i.e., \((x, 0)^+, x > 0\), are empty.
    
    - At least one of the the above and right nodes (i.e., \((x, y)^+, x > 0 \text{ and } y > 0\), is empty.
    
    - None of the robots in the above and right nodes at distance strictly less than \(2d\) (i.e., \((x, y)^+, x > 0, y > 0 \text{ and } x + y < 2d\), can go up.
  
  - Otherwise, \(s\) waits and stay in state 1.

\(^a\)Note that a robot in state 1 never moves to the left border unless it is at node \((1, n)\)

As we will show in Section 4, this set of rules allows the robots to be located in a monotone portion of the grid. These rules are shown in Figure 3.

\(^a\)Note that if \(s\) is in state 1 then it can not be on the left nor bottom border
Robot $s$ in state 2:

**LOOK $V^*(s)$**

- In all situations (i.e., $\ell(s) = (i, j)$, $i, j > 0$):

  - If the upper node ($(0, 1)^+$) is empty, $s$ goes up and enters again state 1.
  - If within $V(s)^*$ all the following conditions are satisfied, then $s$ goes right (staying in state 2).
    - All the nodes above and the ones not to the right (i.e., $(x, y)^+$, with $x \leq 0$ and $y > 0$), are occupied.
    - The lower right node (i.e., $(1, -1)^+$) is empty.
    - All the nodes in the same row on the right (i.e., $(x, 0)^+$, with $x > 0$), are empty.
    - At least one among the nodes above and the ones to the right (i.e., $(x, y)^+$, with $x > 0$ and $y > 0$), is empty.
    - None of the robots in the above and right nodes at distance strictly less than $2d$ (i.e., $(x, y)^+$, with $x > 0, y > 0$ and $x + y < 2d$), can go up.

  - Otherwise, $s$ enters state 1.

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*Notice that if $s$ is in state 2 then it can not be on the left nor bottom border.*

Notice that state 2 is dedicated to robots going right, which is a non-prioritary movement. As we will see, these movements allow the set of robots to adopt a shape in the grid that facilitates avoiding deadlocks. These rules are shown in Figure 4.

### 3.3. Algorithm scatter: Deploying

This phase is executed by robots in state 3 and 4. A robot starts this phase when it arrives to node $(0, n)$ and enters state 3. In this phase the robots move on the snake-path (see Figure 5) and eventually stop in their final positions, that is the nodes $(i \cdot d, j \cdot d)$, with $0 \leq i, j \leq k$, called final nodes.

Let $k$ be odd; the snake-path is the path $n_0, n_1, \ldots n_{(K-1)d}$, that starts at $n_0 = (0, n)$, ends at $n_{(K-1)d} = (d, n)$ and passes through every final node as shown in Figure 5. We will denote the nodes in the snake-path as $n_i$, for the $i$-th final node, and $n_{i \cdot d + j}$, $1 \leq j < d$, for the $j$-th node in the path between $n_i$ and $n_{i+1}$. By a
slight modification of this path and, consequently, of the algorithm, the snake-path can be defined for \( k \) even.

Figure 4: Rules for state 2 of the Collecting phase.

Figure 5: The snake-path for \( N = 21^2 \), \( K = 36, k = 5 \ d = 4 \).

Because of asynchrony, complications may arise. For example, if a robot in the Deploying phase enters in contact with robots that are still in the Collecting phase, the robot has to wait before continuing on the snake-path. Special care has to be taken to avoid deadlocks and guarantee progress.

**Rules for state 3.** A robot \( s \) enters state 3 when it reaches the left-upper corner, i.e. node \((0, n)\). In this state, \( s \) follows the left border moving only if it sees above another robot at distance lower than \( d \) and if the lower node is empty. If the robot is in the left upper corner it only moves if the right node is occupied, insuring in this way that there is at least one node in the Collecting phase. When the last node enters the Deploying phase it will stay at the upper left corner and eventually the other nodes will position themselves at distance \( d \) from each other. When in state
3, a robot \( s \) is following the left border of the grid, i.e., its location can be indicated by \( \ell(s) = n_i \), for \( 0 \leq i < kd \). It enters state 4 when it reaches the bottom-left corner. See Figure 6 for an illustration of the rules.

**Robot \( s \) in state 3:**

**LOOK \( V(s) \)**

- If \( s \) is at upper left corner (i.e., \( \ell(s) = n_0 = (0,n) \)):
  - If the right node ((1, \( n \))) is occupied and the lower node is empty then \( s \) moves down.
  - Otherwise \( s \) waits.
- If \( s \) is at left border but not on the corners (i.e., \( \ell(s) = n_j \), for \( 0 < j < kd \)):
  - If the next node on the snake-path (i.e., \( n_{j+1} \)) is occupied then \( s \) waits.
  - If all the \( d - 1 \) prior nodes on the snake-path (i.e., nodes from \( n_{j-(d-1)} \) to \( n_{j-1} \)) are empty, then \( s \) waits.
  - Otherwise, \( s \) goes down. If \( s \) reaches node \((0,0)\), it enters state 4.

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\( ^a \) If \( j < (d-1) \) robot \( s \) only sees the path from \( n_0 \) to \( n_{j-1} \)

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**Figure 6:** Rules for states 3 and 4. Gray nodes are final nodes; black nodes are hosting a robot. Robots in \( A \) are in state 3; robots in \( B \) are in state 4 and only look at the snake-path to decide whether to go on or wait; robots located in \( C \) and \( D \) are in state 4 and can continue only if they are alone in the marked areas.

**Rules for state 4.** A robot \( s \) in this state is following the snake-path, from the left-bottom corner. That is, its location is \( \ell(s) = n_i \), for \( kd \leq i \leq (K - 1)d \).

A robot \( s \) in the *Deploying* phase might see in the rows above some robots still performing the *Collecting* phase (the presence of these robots can be detected because of their “wrong” positions). In this case \( s \) waits. If \( s \) does not see any robot
out of the snake-path but there are no robots in the $d - 1$ preceding nodes or the next node is occupied, then it waits. Otherwise $s$ moves forward on the snake-path. When the last node enters the Deploying phase it will stay at the upper left corner and eventually all other nodes will stop at their final position.

<table>
<thead>
<tr>
<th>Robot $s$ in state 4: Look $V(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>• In all situations (i.e., $\ell(s) = n_j$, for $kd \leq j \leq (K - 1)d$):</td>
</tr>
<tr>
<td>• If all the $d - 1$ prior nodes in the snake-path (i.e., from node $n_{j-(d-1)}$ to $n_{j-1}$) are empty, then $s$ waits.</td>
</tr>
<tr>
<td>• If the next node in the snake-path ($n_{j+1}$) is occupied then $s$ waits.</td>
</tr>
<tr>
<td>• If $s$ is not on the two last row of the snake-path (i.e., $\ell(s) = n_j$, for $kd \leq j \leq (K - 2k - 1)d$):</td>
</tr>
<tr>
<td>• If $s$ detects a robot $z$ not in the snake-path, in the next $d + 1$ rows above and at most $d - 1$ columns farther, then $s$ waits (for example, robots in $C$ in Figure 6 apply this rule).</td>
</tr>
<tr>
<td>• If $s$ is in one of the last $kd - d + 1$ nodes of the snake-path (i.e., $\ell(s) = n_j$, for $(K - k)d \leq j &lt; (K - 1)d$):</td>
</tr>
<tr>
<td>• If $s$ detects a robot $z$ at left in the $d - 1$ rows below (i.e, at least one of the nodes $(-i, -j)^+$, with $1 \leq i, j \leq d - 1$ is occupied), then $s$ waits (for example, robots in $D$ in Figure 6 apply this rule).</td>
</tr>
<tr>
<td>• Otherwise, $s$ goes forward on the snake-path.</td>
</tr>
</tbody>
</table>

4. Correctness

In this Section we show the correctness of algorithm SCATTER: we prove that both collisions between robots and deadlocks are avoided, and also that any execution of the algorithm eventually ends in a uniform scattering configuration.

First we start with two observations.

Note that, according to the rules for state 4, a robot $s$ in state 4, which is not on the last two rows of the snake-path, never moves if it detects a robot $z$, not in the snake-path, in the $d + 1$ rows above it. Thus:

**Observation 4.1** *If $s$ is in state 4, and it is not on the $d - 1$ last rows of the snake-path, then every robot below $s$ is in the Deploying phase.*

Note also that, if $s$ is in one of the last $kd - d + 1$ nodes of the snake-path then, according to the rules for state 4, $s$ will not move if it detects a robot $z$ at left in the $d - 1$ rows below it. Thus, with this fact and observation 4.1 we have that:

**Observation 4.2** *If a robot $s$ in the Collecting phase is blocked by a robot in the Deploying phase, then it is at node $(1, 0)$.*

Using the above observations we can derive the following theorem.

**Theorem 1** *During the execution of algorithm SCATTER, no two robots can collide.*
We now define, for a given set of robots, a particular configuration in the grid which we call a stable configuration. This definition will be used later to prove that algorithm scatter cannot stop because of deadlock.

**Definition 1 (Stable configuration)** Let $S \subseteq S$ be a set of robots executing algorithm scatter. We say that the set $S$ is in an $m$-stable configuration with $t$ complete rows (possibly $t = 0$) if:

- all robots of $S$ are located in the upper $m$ rows;
- all robots located in the upper $m$ rows are in $S$;
- for every $j = 1, \ldots, m - t$, the robots of $S$ located in row $n - m + j$ occupy consecutive nodes $(1, n - m + j), \ldots, (\lambda_j, n - m + j)$;
- the $m-t$ positive integers $\lambda_1, \ldots, \lambda_{m-t}$ satisfy $\lambda_1 \geq 1$ and $kd > \lambda_j \geq \lambda_{j-1} + 2d - 1$, for every $j = 2, \ldots, m - t$;
- $\lambda_j = kd$, for every $j = m - t + 1, \ldots, m$.

We say that $S$ is in a stable configuration if it is a $m$-stable configuration, for some $m$. (See Figure 7.)

Notice that, for a set of robots $S$ to be in a stable configuration, we ask the robots to be located at the upper rows of the grid, occupying consecutive nodes and every row hosting at least $2d - 1$ nodes more than the previous one. In fact, this last condition cannot be fulfilled if the row is complete. This is reflected in the last $t$ rows, which host $kd$ robots each (i.e., every node is occupied).

The correctness of the algorithm follows from a sequence of lemmas.

In the following Lemma it is shown that, if no robot in the Collecting phase enters in the Deploying phase for a sufficient amount of time then they will eventually reach a stable configuration.

**Lemma 1** Let $S$ be the set of robots in the Collecting phase. Then eventually they will reach a stable configuration, provided that no robot in $S$ enters the Deploying phase.

**Proof.** Assume that for a large enough number of steps, robots in $S$ cannot enter the Deploying phase. Let $h$ in row $n - m - 1$ be the head of the snake. By observation 4.1 we know that every robot in rows below $n - m - 1$ is in the Deploying phase. Moreover, we know by observation 4.2 that no robot in the Collecting phase could
be blocked by a robot in the Deploying phase which is not at node (0, 0). Thus, eventually the set of robots $S$ will be isolated in the last $m$ upper rows.

By the rules for states 1 and 2, if a robot $s$ decides to go right and enter state 2, either it will go up in a few steps or a faster robot will come from below, occupying an empty node in the same row as $s$ or in a row above. After that, $s$ will go back to state 1. Since robots never go down, after a finite number of steps in the execution of algorithm SCATTER, no robot enters state 2 again.

We can therefore assume that we have reached a situation in which none of the robots enter state 2 again. By an analogous reasoning, if all robots can only go left or up, then, after some steps, they all will be blocked.

We now show that, if none of the robots can move, then they are in a stable configuration.

If none of the robots can go up, nor left, the robots in $S$ are located in the upper $m'$ rows, for some $m' < m$. In every row, they occupy consecutive nodes. Moreover, if $\lambda_j$ is the number of occupied nodes in row $n - m' + j$, then $\lambda_1 \geq 1$, and $\lambda_j \geq \lambda_{j-1}$ for $j = 2, \ldots, m'$.

Let us concentrate now in robot $s_j$, with $j < m'$, located at node $\ell(s_j) = (\lambda_j, n - m' + j)$. We know that $s_j$ cannot enter state 2 and go right. We can distinguish the following three cases:

- if $\lambda_j = kd$, then $\lambda_i = kd$, for $j \leq i \leq m'$;
- if $\lambda_j \leq kd - (2d - 1)$, then all $2d - 1$ nodes $(\lambda_j + 1, n - m' + j + 1), (\lambda_j + 2, n - m' + j + 1), \ldots, (\lambda_j + 2d - 1, n - m' + j + 1)$ are occupied (otherwise, $s$ could go right, by rules for states 1 and 2), and $\lambda_{j+1} \geq \lambda_j + 2d - 1$;
- if $\lambda_j > kd - (2d - 1)$, then all $kd - j$ nodes $(\lambda_j + 1, n - m' + j + 1), (\lambda_j + 2, n - m' + j + 1), \ldots, (kd, n - m' + j + 1)$ are occupied (otherwise, $s$ could go right, by rules for states 1 and 2), and $\lambda_{j+1} = kd$.

So, $S$ satisfies the conditions of Definition 1, and thus it is an $m'$-stable configuration. This completes the proof.

In the next Lemma we give a lower bound on the number of robots in a stable configuration. Notice that the condition on $m$ does not necessarily imply that there are no complete rows, since we are computing a lower bound.

**Lemma 2** Let $S$ be a set of robots in an $m$-stable configuration. If $1 + (2d - 1)(m - 1) \leq kd$, then

$$|S| \geq m + \frac{2d - 1}{2}(m^2 - m)$$

**Proof.** The number of robots in the $m$-stable configuration is, by definition, $|S| = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ with $\lambda_1 \geq 1$, $\lambda_j \geq \lambda_{j-1} + 2d - 1$, for $j = 2, \ldots, m - t$. If $1 + (2d - 1)(m - 1) \leq kd$, then $\lambda_j \geq 1 + (j - 1)(2d - 1)$, for every $j = 1, \ldots, m$. Therefore $|S| \geq \sum_{j=1}^{m} 1 + (j - 1)(2d - 1) = m + (2d - 1)\sum_{i=1}^{m-1} i = m + \frac{2d - 1}{2}(m^2 - m)$.

To prove that there is no deadlock, we first introduce some lemmas.

**Lemma 3** No robot can be blocked in the Cleaning phase.
Proof. Notice that robots in state 0 (i.e., in the Cleaning phase) could only wait for robots in state 1 located at some nodes \((i, 1)\), i.e., in row 1. Moreover, robots in state 1 are waiting for robots in state 0 only if there are robots at some nodes \((j, 0)\) (i.e., in row 0). By construction however, robots in row 1 will eventually move up, leaving room for robots in state 0 to go up as well and enter state 1.

Lemma 4 Let \(s\) be the first robot, on the snake-path from the node \(n_0\), which is in the Deploying phase and blocked by a robot in the Collecting phase. Let \(s\) be located in node \(n_i = (x, y)\). Eventually one of the two conditions will occur: 1) all the nodes on the snake-path in rows below \(s\) (i.e., nodes \(n_j = (x', y'); y' < y\) are occupied by robots in the Deploying phase or 2) all the robots are in the Deploying phase.

Proof. Suppose that there is a case when at least one node in the Deploying phase is blocked by a robot in the Collecting phase. By contradiction suppose that some nodes behind it, in the snake-path and in a lower row, remains empty. It is possible that for some steps none of the robots in the Collecting phase enter the Deploying phase; however, we know by lemma 1, that they eventually reach a stable configuration. Let \(s\), in node \(n_i\), be the first robot on the snake-path from node \(n_0\), which is blocked by a robot in the Collecting phase. Let \(n_j; 0 \leq j < i\) be the first empty node on the snake-path in a lower row. If \(j = 0\) then the robot in \((1, n)\) could move entering the Deploying phase thus reaching a contradiction.

On the other hand, if \(j > 0\), we can assure that eventually there will be at least one robot in the path \(\Gamma := \{n_0, n_1, \ldots n_{j-1}\}\), in the Deploying phase. Let \(s'\) be the robot in path \(\Gamma\) which is closest (in \(\Gamma\)) to the empty node \(n_j\). We next show that robot \(s'\) will eventually arrive to \(n_j\). We can distinguish two cases:

- If \(1 \leq j \leq kd\), then robot \(s'\) is in state 3. Rules for state 3 allow \(s\) to go down and to arrive to node \(n_j\).
- If \(j > kd\), then robot \(s'\) is in state 4.

Notice that robots in the Collecting phase never move down. Moreover, robot \(s\) has already passed through nodes \(n_0, n_1, \ldots n_{j-1}, n_j\). This allows us to assure that no robot in the Collecting phase is neither entering \(\Gamma\), nor in the \(d + 1\) rows above \(n_j\). Applying the rules for state 4, \(s'\) is able to follow the snake-path and to arrive to node \(n_j\). In both cases, robot \(s'\) could move to \(n_j\) leaving a new empty node in the same row or in the row below. So, the robots in \(\Gamma\) will move successively until \(n_0\) is left empty. At this moment, a robot in the Collecting phase will move to \(n_0\) entering the Deploying phase thus reaching a contradiction.

As a consequence of the previous lemmas we have that:

Lemma 5 If there is no robot in the Deploying phase blocked by a robot in the Collecting phase nor robot in the Collecting phase blocked by a robot in the Deploying phase then Deadlock is not possible.

We are ready to prove that algorithm \textsc{scatter} never stops in a deadlock situation.

Theorem 2 No deadlocks will occur during any execution of algorithm \textsc{scatter}.
Proof. By Lemma 3, after some steps in the execution of algorithm scatter, all robots are in the Collecting phase or in the Deploying phase. Assume by contradiction that there is a deadlock; then, by Lemma 5, we know that there is at least one robot in the Deploying phase blocked by a robot in the Collecting phase. Let $s$ be the first robot, in the Deploying phase, in the snake-path blocked by a robot in the Collecting phase.

By Lemma 4, after some steps all the nodes in the snake-path in a row lower than $s$ are occupied by robots in the Deploying phase. Robot $t$ in node $n_0$ will wait, provided that $s$ is waiting. Therefore, by Lemma 1, after some steps, the set $S$ of robots in the Collecting phase will be in a $m''$-stable configuration, for some $m''$ (see gray robots in Figure 8).

Let $m'' = m + m'$ such that $m$ satisfies $1 + (2d - 1)(m - 1) \leq kd < 1 + (2d - 1)m$.

By definition 1, we know that the upper $m'$ rows contain $kd$ robots each. Note that robots in the Deploying phase can only occupy nodes on the snake path. This means that, for a deadlock to occur when robots are in the Collecting phase we would need a higher number of robots if they are in an $m''$-stable configuration than if they are in an $m$-stable configuration. Thus we can assume that the robots in the Collecting phase are in an $m$-stable configuration with $1 + (2d - 1)(m - 1) \leq kd$. This will allow us to apply Lemma 2 for a lower bound on the number of robots in the stable configuration.

We now show that there is a contradiction, based on the fact that there are always at least $d + 1$ empty rows between robots in the Collecting phase (the gray robots in Figure 8) and robot $s$. First of all we know that:

- There are $(k + 1)d$ robots in the Deploying phase in the $d$ left-most columns. In fact, the snake path contains $kd + 1$ nodes on the left border and only one node, the bottom-most, on each of the next $d - 1$ columns, and by Lemma 4 all these nodes have been occupied.

- By Lemma 2, the number of robots $S(m)$ in the stable configuration satisfies $S(m) \geq m + \frac{2d-1}{2}(m^2 - m)$.

- The number of rows below $s$ in the Deploying phase is $kd - m - d$. By Lemma 4, the number of robots in the Deploying phase is thus $S(d) \geq (k + 1)d + \left\lceil \frac{kd - m - d}{d} \right\rceil kd$.

The total number of robots used is $S(t) = S(m) + S(d)$. To obtain a contradiction we have to prove that the number of used robots exceeds the number of robots in the space, that is, $S(t) \geq (k + 1)^2$. The lower bound on $S(m)$ implies that:

$$S(t) \geq m + \frac{2d-1}{2}(m^2 - m) + (k + 1)d + \left\lceil \frac{kd - m - d}{d} \right\rceil kd$$

$$\geq m + \frac{2d-1}{2}(m^2 - m) + (k + 1)d + k(kd - m - d)$$

$$= (k + 1)d + k^2d - kd + \left(1 - \frac{2d-1}{2} - k\right)m + \frac{2d-1}{2}m^2$$
Let us subtract \((k + 1)^2\) to this amount and let us denote the result by \(f(m)\):
\[
f(m) = (k^2 + 1)(d - 1) - 2k + \frac{-2d - 2k + 3}{2}m + \frac{2d - 1}{2}m^2
\]
We will show that \(f(m) > 0\).

First, we show that \(f(m)\) is increasing for \(m > m^* = \frac{2d + 2k - 3}{2(2d - 1)}\). In fact \(f'(m) = (2d - 1)m + \frac{-2d - 2k + 3}{2}\). Now, \(f''(m^*) = 0\) if and only if \(m^* = \frac{2d + 2k - 3}{2(2d - 1)}\). Since \(f''(m) = (2d - 1) > 0\) we have that for \(m > m^*\) the function \(f(m)\) is strictly increasing.

We know that \(m > d(k^2 - 1)\), otherwise the number of rows below \(s\) would be greater or equal than \(\frac{k}{d}\). The reason is that if we had \(s \geq \frac{k}{d}\), the snake path could room the \((k + 1)^2\) robots behind \(s\) because \(d \geq 2\), which is not possible if there is a deadlock.

Notice also that \((\frac{k}{2} - 1)d - \frac{2d + 2k - 3}{2(2d - 1)} = \frac{d^2(2k - 4) - k(d + 2) + 3}{2(2d - 1)} > 0\) if \(k \geq 4\). This is true because \(d^2 \geq (d + 2)\) for \(d \geq 2\), and \((2k - 4) \geq k\) for \(k \geq 4\). Furthermore, substituting \(k = 3\) we obtain \((d - 3)(2d + 3) + 6 > 0\) which is true for \(d \geq 3\). Therefore, \(f(m') \geq f((\frac{k}{2} - 1)d)\) for any \(m' \geq d(k^2 - 1)\) if \(k \geq 4\), or \(k = 3\) and \(d \geq 3\).

Now we show that \(f((\frac{k}{2} - 1)d) > 0\).
\[
\begin{align*}
f((\frac{k}{2} - 1)d) &= (k^2 + 1)(d - 1) - 2k + \frac{-2d - 2k + 3}{2}((k - 2)d) + \frac{2d - 1}{2}((k - 2)d)^2 \\
&= \frac{k^2(2d^3 - d^2 + 4d - 8) - 2k(4d^3 - 7d + 8) + 8d^3 + 4d^2 - 4d - 8}{8}
\end{align*}
\]

Let us denote the numerator as the function \(g(k) = k^2a - 2kb + c\), where \(a = 2d^3 - d^2 + 4d - 8\), \(b = (d - 2)(2d^2 + 3d + 10) + 12\), \(c = 4d^3 - 7d + 8\), \(d = ((d - 2)(4d^2 + 8d + 9) + 26\) and \(c = 8d^3 + 4d^2 - 4d - 8 = 8(d^3 - 1) + 4(d - 1)\). Notice that, since \(d \geq 2\), the three terms \(a\), \(b\) and \(c\) are positive. Thus, \(g(k)\) is a parabola opening up and its vertex is in \(k = \frac{b}{a}\).
\[
g\left(\frac{b}{a}\right) = \frac{ac - b^2}{a} = \frac{76d^4 - 124d^3 - 89d^2 + 112d}{a} = \frac{d((d - 2)((d - 1)(76d + 104) + 71) + 46)}{a}
\]

Since \(d \geq 2\), then \(g\left(\frac{b}{a}\right) > 0\) and thus \(f((\frac{k}{2} - 1)d) > 0\).

This implies that, when \(i)\) \(k = 3\) and \(d \geq 3\), or \(ii)\) \(k \geq 4\), for deadlock to exist there must be more than \((k + 1)^2\) robots. Since there are only \((k + 1)^2\) robots a deadlock is not possible in these cases. Finally, it is easy to verify (by simple but tedious calculations) that deadlock cannot occur in the special cases of \(i)\) \(k = 3\) and \(d = 2\), \(ii)\) \(k = 2\).

Therefore, no deadlocks can occur during any execution of algorithm scatter.

\[\square\]

Hence, by Theorems 1 and 2 the correctness of the possibility result follows.
Theorem 3 Any execution of algorithm SCATTER terminates within finite time with a uniform scattering.

Proof. By theorem 2, deadlock is avoided. Thus, eventually every robot will enter state 3. When the last robot \( s_0 \) enters state 3 reaching node \( n_0 = (0, n) \) by the rules for state 3 it will remain at node \( (0, n) \). The first robot \( s_1 \) in front of \( s_0 \) will move until be at distance \( d \) from \( s_0 \). At that moment \( s_1 \) will be at node \( n_d \). In this way the robot \( n_i \) will stop at distance \( d \) from \( n_{(i-1)} \) its successor reaching a final node. This process ends when the head reaches the node \( n_{(K-1)d} \) and every robot is in a final node.

5. Concluding Remarks and Open Problems

We have proved that the problem of uniform scattering in a grid is indeed solvable without collisions even for very weak robots. The results of this paper have been proven for square grids. They however hold also in rectangular grids, provided that the direction of the largest dimension is known.

Although the focus of this work has been on the computability rather than the complexity aspects of the problem, it is easy to verify that the proposed solution requires no more than \( O(N/d) \) time units in the worst case. An interesting open research question is whether this cost is optimal, or a more efficient solution strategy can be designed.

An important research direction involves the study of the problem in more complex orthogonal spaces (e.g., with holes). It appears doubtful that these cases could be solved without increasing the capabilities of the robots. In particular, the computational power and performance capabilities of weak robots that are however
provided with explicit communication seems to be a promising direction for future research on the uniform deployment problem.

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