Multidimensional Signal and Color Image Processing using Lattices

A brief overview

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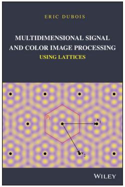
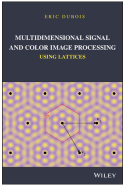


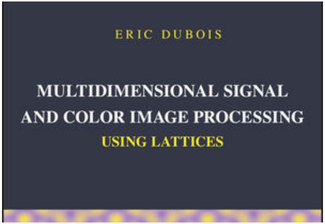
Abstract

- Overview of the book.
- Friendly book review by the author.
- What distinguishes the book from other works.
- Some contributions of the book.



Key themes of the book

- Processing of multidimensional (MD) signals (e.g., images, video, volumetric, etc.)
- Function of several independent variables (space, time, etc.)
- Scalar or vector valued signals.
- Continuous-domain and discrete-domain signals (where sampling structures are defined by lattices)
- Aperiodic and periodic signals (where periodicity is specified by a lattice)
- Color signals, where the signal range is a vector space.



Overview of the book contents

- MD signals and systems: four cases, continuous-domain aperiodic (CDA), discrete-domain aperiodic (DDA), discrete-domain periodic (DDP), continuous-domain periodic (CDP)
- Sampling and reconstruction - conversion between domains
- Color representation and color signals
- Random fields (wide-sense stationary case)
- Filter design
- Sampling structure conversion
- Symmetry invariant signal processing
- Separate chapter summarizing relevant facts about lattices

Continuous-domain MD signals

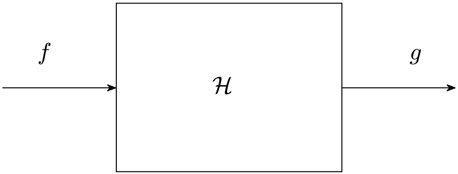
- A multidimensional signal f is a function of D independent variables

$$f(\mathrm{x}) = f(x_1, x_2, \ldots, x_D), \qquad \mathrm{x} \in \mathbb{R}^D.$$

- Although we are mainly concerned with $D > 1$, one-dimensional signals with $D = 1$ are simply a special case of the general theory.
- We assume that signal values belong to a vector space, in the simplest case the space of real numbers. Color images correspond to a three-dimensional vector space.
- We assume that signals belong to a vector space $\mathcal{S}$ over the complex numbers called the *signal space*. Thus signals can be added, scaled, negated, there is a zero signal, etc.

Continuous-domain MD systems

- A multidimensional system $\mathcal{H}$ acts on elements of a signal space $\mathcal{S}$. We consider deterministic systems where each input f results in a well-defined output. We write $\mathcal{H} : \mathcal{S} \to \mathcal{S} : g = \mathcal{H}f$.
- Example 1: $g(\mathrm{x}) = (f(\mathrm{x}))^2$ for all $\mathrm{x} \in \mathbb{R}^D$.
- Example 2: $\mathcal{T}_{\mathrm{d}} : g(\mathrm{x}) = f(\mathrm{x} - \mathrm{d})$ for some fixed $\mathrm{d} \in \mathbb{R}^D$. We call this the shift or translation system.
- We consider two main classes of systems:
- Linear systems: $\mathcal{H}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathcal{H} f_1 + \alpha_2 \mathcal{H} f_2$.
- Shift-invariant systems: $\mathcal{H}\mathcal{T}_{\mathrm{d}} = \mathcal{T}_{\mathrm{d}}\mathcal{H}$ for all d.
- A *linear shift-invariant* (LSI) system is both linear and shift invariant.



MD LSI systems: convolution

For an LSI system $\mathcal{H}$, we can show that if the input is f, then the output is $g = h * f$, where $h = \mathcal{H}\delta$ and δ is the Dirac delta.
Specifically,

$$\begin{aligned} g(\mathrm{x}) &= \int_{\mathbb{R}^D} h(\mathrm{s}) f(\mathrm{x}-\mathrm{s}) d\mathrm{s} \\ &= \int_{\mathbb{R}^D} f(\mathrm{s}) h(\mathrm{x}-\mathrm{s}) d\mathrm{s} \\ h * f &= f * h \end{aligned}$$

MD LSI systems: frequency response

An important class of signals are the complex exponentials ϕ_{u} given by

$$\phi_{\mathrm{u}}(\mathrm{x}) = \exp(j2\pi\mathrm{u}\cdot\mathrm{x}) = \cos(2\pi\mathrm{u}\cdot\mathrm{x}) + j\sin(2\pi\mathrm{u}\cdot\mathrm{x})$$
$$\phi_{\mathrm{u}}(x_1,\ldots,x_D) = \exp(j2\pi(u_1x_1+\cdots+u_Dx_D))$$

for some fixed $\mathrm{u}\in\mathbb{R}^D$, referred to as the frequency vector.
For an LSI system, applying the convolution formula, we find that

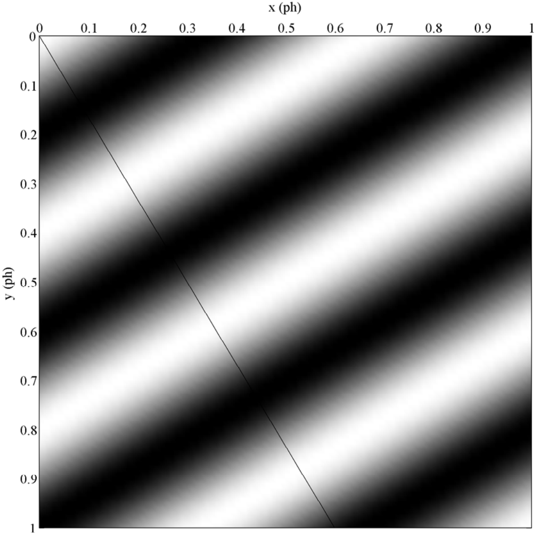
$$\mathcal{H}\phi_{\mathrm{u}} = H(\mathrm{u})\phi_{\mathrm{u}}$$

where for a given u, $H(\mathrm{u})$ is a complex number given by

$$H(\mathrm{u}) = \int_{\mathbb{R}^D} h(\mathrm{x})\exp(-j2\pi\mathrm{u}\cdot\mathrm{x})d\mathrm{x}.$$

Taken as a function of u, $H(\mathrm{u})$ is called the frequency response of the LSI system, and is the *Fourier Transform* of the impulse response.

Two-dimensional sinusoidal signal



$$f(x, y) = 0.5\cos(2\pi(1.5x + 2.5y)) + 0.5$$

Horizontal frequency $u = 1.5$ c/ph, vertical frequency $v = 2.5$ c/ph.

Continuous-domain Fourier transform properties

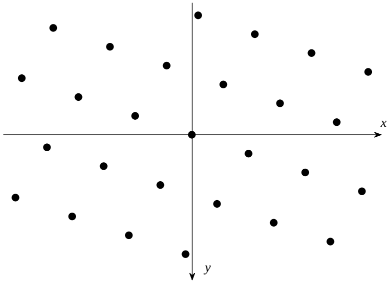
	$f(\mathbf{x}) = \int_{\mathbb{R}^D} F(\mathbf{u}) \exp(j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{u}$	$F(\mathbf{u}) = \int_{\mathbb{R}^D} f(\mathbf{x}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{x}$
(2.1)	$Af(\mathbf{x}) + Bg(\mathbf{x})$	$AF(\mathbf{u}) + BG(\mathbf{u})$
(2.2)	$f(\mathbf{x} - \mathbf{x}_0)$	$F(\mathbf{u}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x}_0)$
(2.3)	$f(\mathbf{x}) \exp(j2\pi\mathbf{u}_0\cdot\mathbf{x})$	$F(\mathbf{u} - \mathbf{u}_0)$
(2.4)	$f(\mathbf{x}) * g(\mathbf{x})$	$F(\mathbf{u})G(\mathbf{u})$
(2.5)	$f(\mathbf{x})g(\mathbf{x})$	$F(\mathbf{u}) * G(\mathbf{u})$
(2.6)	$f(\mathbf{A}\mathbf{x})$	$\frac{1}{\lvert\det \mathbf{A}\rvert} F(\mathbf{A}^{-T}\mathbf{u})$
(2.7)	$\nabla_{\mathbf{x}} f(\mathbf{x})$	$j2\pi\mathbf{u}F(\mathbf{u})$
(2.8)	$\mathbf{x}f(\mathbf{x})$	$\frac{j}{2\pi}\nabla_{\mathbf{u}} F(\mathbf{u})$
(2.9)	$f^*(\mathbf{x})$	$F^*(-\mathbf{u})$
(2.10)	$F(\mathbf{x})$	$f(-\mathbf{u})$
(2.11)	$f_1(x_1)\cdots f_D(x_D)$	$F_1(u_1)\cdots F_D(u_D)$
(2.12)	$\int_{\mathbb{R}^D} f(\mathbf{x})g^*(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^D} F(\mathbf{u})G^*(\mathbf{u})\, d\mathbf{u}$	

Discrete-domain MD signals

- An MD discrete-domain signal is defined at a discrete set of points Ψ in $\mathbb{R}^D$ called the *sampling structure*.

$$f[\mathbf{x}], \qquad \mathbf{x} \in \Psi.$$

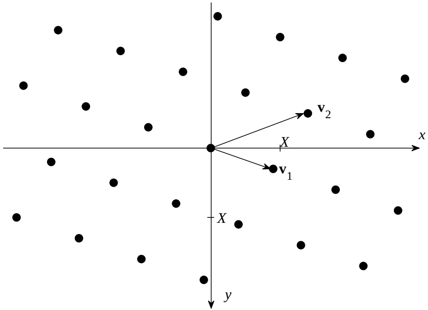
- The main tool used to describe and analyze sampling structures is the *lattice* (as in crystal lattice).
- A lattice is a uniform discrete set of points in $\mathbb{R}^D$; the neighborhood of a lattice point looks the same at every point



Lattices

A lattice Λ in D dimensions is a discrete set of points that can be expressed as the set of all linear combinations with *integer* coefficients of D linearly independent vectors in $\mathbb{R}^D$ (called basis vectors),

$$\begin{aligned}\Lambda &= \{n_1\mathsf{v}_1 + \cdots + n_D\mathsf{v}_D \mid n_i \in \mathbb{Z}\} \\ &= \{\mathsf{Vn} \mid \mathsf{n} \in \mathbb{Z}^D\} = \mathrm{LAT}(\mathsf{V}), \\ \mathsf{V} &= [\mathsf{v}_1, \cdots, \mathsf{v}_D].\end{aligned}$$

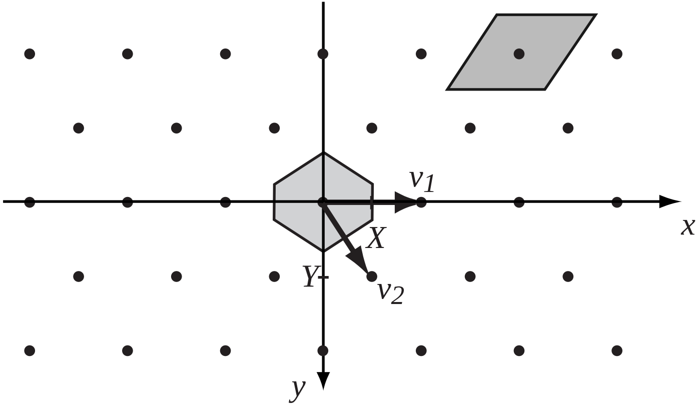


$$\mathsf{V} = \begin{bmatrix} 0.9X & 1.4X \\ 0.3X & -0.5X \end{bmatrix}$$

Some properties of a lattice Λ

- The basis and sampling matrix for a given lattice Λ are not unique.
- $\mathrm{LAT}(\mathrm{V}) = \mathrm{LAT}(\mathrm{VE})$ if E is an integer matrix with $|\det \mathrm{E}| = 1$.
- $0 \in \Lambda$
- $\mathrm{x} \pm \mathrm{y} \in \Lambda$ if $\mathrm{x}, \mathrm{y} \in \Lambda$.
- $\Lambda + \mathrm{d} = \Lambda$ if $\mathrm{d} \in \Lambda$.
- A unit cell of a lattice is a region $\mathcal{P} \subset \mathbb{R}^D$ such that $\cup_{\mathrm{s} \in \Lambda} \mathcal{P} + \mathrm{s} = \mathbb{R}^D$ while $(\mathcal{P} + \mathrm{s}_1) \cap (\mathcal{P} + \mathrm{s}_2) = \emptyset$ for and $\mathrm{s}_1, \mathrm{s}_2 \in \Lambda$ such that $\mathrm{s}_1 \neq \mathrm{s}_2$. It is not unique.
- The volume of a unit cell is $d(\Lambda) = |\det \mathrm{V}|$. $1/d(\Lambda)$ is the sampling density.

Illustration of unit cells



Discrete-domain MD systems

- For discrete-domain signals on a lattice Λ, the concepts of signal space $\mathcal{S}_\Lambda$, system $\mathcal{H}$, linear system, shift-invariant system, LSI system are all formally the same as for continuous-domain MD systems.
- The only proviso is that the shift d in a shift-system $\mathcal{T}_\mathrm{d}$ must be itself an element of the lattice Λ.
- The output of an LSI system is again given by a convolution formula

$$\begin{aligned} g[\mathrm{x}] &= \sum_{\mathrm{s}\in\Lambda} h[\mathrm{s}] f[\mathrm{x}-\mathrm{s}] \\ &= \sum_{\mathrm{s}\in\Lambda} f[\mathrm{s}] h[\mathrm{x}-\mathrm{s}] \qquad \mathrm{x}\in\Lambda \end{aligned}$$

- We write this as $g = h * f = f * h$.
- Here, h is the unit sample response $h = \mathcal{H}\delta_\Lambda$, where

$$\delta_\Lambda[\mathrm{x}] = \begin{cases} 1 & \mathrm{x} = 0 \\ 0 & \mathrm{x} = \Lambda\backslash 0 \end{cases}$$

MD discrete-domain LSI systems: frequency response

- Discrete-domain complex-exponential sinusoidal signals ϕ_{u} are defined in the same way

$$\phi_{\mathrm{u}}[\mathrm{x}] = \exp(j2\pi\mathrm{u}\cdot\mathrm{x}), \qquad \mathrm{x}\in\Lambda$$

  for some fixed frequency vector $\mathrm{u}\in\mathbb{R}^D$.

- The complex sinusoids are periodic in the frequency vector

$$\phi_{\mathrm{u}} = \phi_{\mathrm{u}+\mathrm{r}} \qquad \text{if } \mathrm{r}\in\Lambda^*$$

  where $\Lambda^* = \mathrm{LAT}(\mathrm{V}^{-T})$ is called the reciprocal (or dual) lattice.

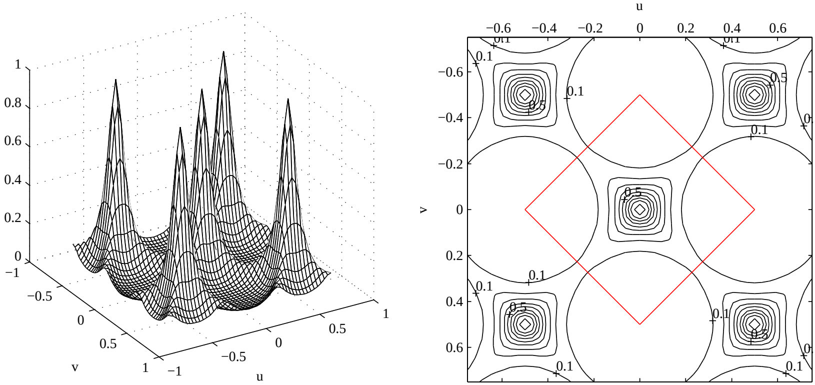
- The complex sinusoids are eigenfunctions of any LSI system, $\mathcal{H}\phi_{\mathrm{u}} = H(\mathrm{u})\phi_{\mathrm{u}}$, where

$$H(\mathrm{u}) = \sum_{\mathrm{x}\in\Lambda} h[\mathrm{x}]\exp(-j2\pi\mathrm{u}\cdot\mathrm{x})$$

Discrete-domain Fourier transform properties

	$f[\mathbf{x}] = d(\Lambda)\int_{\mathcal{P}^*} F(\mathbf{u}) \exp(j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{u}$	$F(\mathbf{u}) = \sum_{\mathbf{x}\in\Lambda} f[\mathbf{x}] \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})$
(3.1)	$Af[\mathbf{x}] + Bg[\mathbf{x}]$	$AF(\mathbf{u}) + BG(\mathbf{u})$
(3.2)	$f[\mathbf{x} - \mathbf{x}_0]$	$F(\mathbf{u}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x}_0)$
(3.3)	$f[\mathbf{x}] \exp(j2\pi\mathbf{u}_0\cdot\mathbf{x})$	$F(\mathbf{u} - \mathbf{u}_0)$
(3.4)	$f[\mathbf{x}] * g[\mathbf{x}]$	$F(\mathbf{u})G(\mathbf{u})$
(3.5)	$f[\mathbf{x}]g[\mathbf{x}]$	$d(\Lambda)\int_{\mathcal{P}^*} F(\mathbf{r})G(\mathbf{u} - \mathbf{r})\, d\mathbf{r}$
(3.6)	$f[\mathbf{A}\mathbf{x}]$	$F(\mathbf{A}^{-T}\mathbf{u})$
(3.7)	$\mathbf{x}f[\mathbf{x}]$	$\frac{j}{2\pi}\nabla_{\mathbf{u}} F(\mathbf{u})$
(3.8)	$f^*[\mathbf{x}]$	$F^*(-\mathbf{u})$
(3.9)	$\tilde{F}[\mathbf{x}]$	$d(\Gamma)\tilde{f}(-\mathbf{u})$
(3.10)	$\sum_{\mathbf{x}\in\Lambda} f[\mathbf{x}]g^*[\mathbf{x}] = d(\Lambda)\int_{\mathcal{P}^*} F(\mathbf{u})G^*(\mathbf{u})\, d\mathbf{u}$	

Example: Fourier transform of an exponential function on a hexagonal lattice



Comparison of approach in this book with the conventional approach in the literature

- This book presents a development of multidimensional discrete-domain signal processing that does not depend on arbitrarily chosen entities, particularly bases for lattices.
- This approach preserves the geometric characteristics of signals and Fourier transforms.
- In conventional presentations, a discrete-domain signal is defined on $\mathbb{Z}^D$,

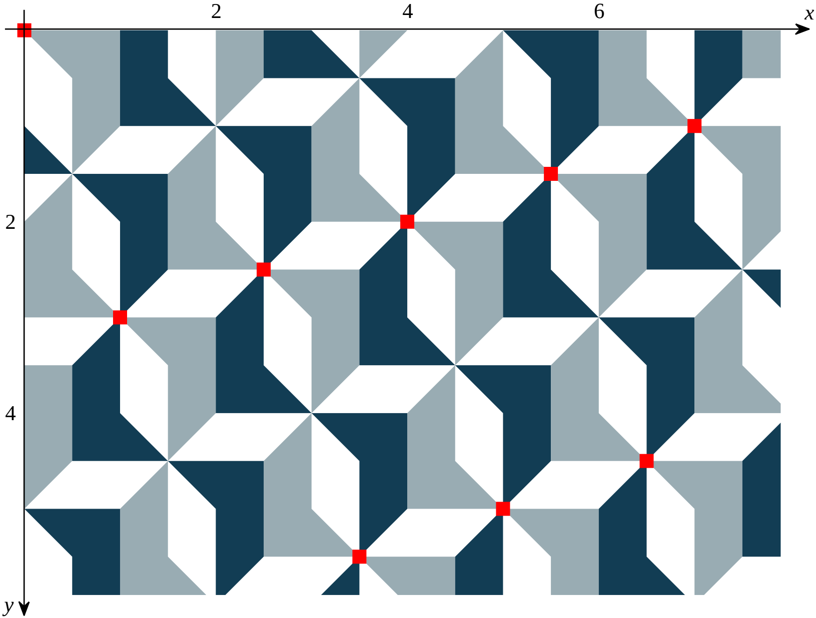
$$f[\mathrm{n}] = f[n_1, n_2, \ldots, n_D]$$

- Sampling is relative to an underlying continuous-domain signal

$$f_d[\mathrm{n}] = f_c(\mathrm{Vn}), \qquad \mathrm{n} \in \mathbb{Z}^D.$$

- This definition is dependent on the *non-unique* sampling matrix V.

Multidimensional periodic signals



Multidimensional periodic signals

- Multidimensional periodic signals are covered in Chapter 4 (discrete domain) and Chapter 5 (continuous domain).
- For a periodic signal, periodicity is determined by a lattice, that we call the periodicity lattice.
- $\tilde{f}[\mathrm{x}+\mathrm{t}] = \tilde{f}[\mathrm{x}], \quad$ for all $\mathrm{t} \in \Gamma$, where Γ is the periodicity lattice.
- For continuous-domain signals, $\mathrm{x} \in \mathbb{R}^D$ and Γ is any lattice.
- For discrete-domain signals, Γ must be a sublattice of the sampling lattice Λ.
- One period of the signal consists of the signal restricted to any unit cell of the periodicity lattice Γ.

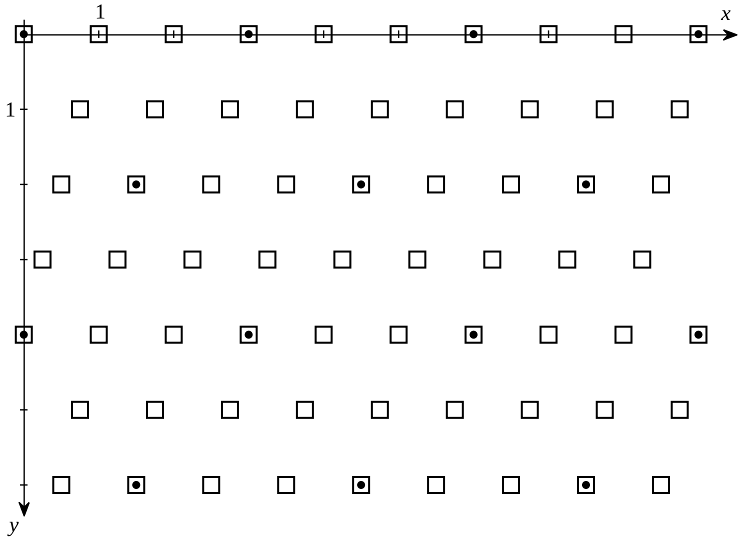
Discrete-domain MD periodic signals

- The periodicity lattice is a sublattice of the sampling lattice, $\Gamma \subset \Lambda$.
- The number of sample points in any unit cell of Γ is the integer $K = d(\Gamma)/d(\Lambda)$.
- The set $\mathbf{b} + \Gamma = \{\mathbf{b} + \mathbf{x} \mid \mathbf{x} \in \Gamma\}$ for any $\mathbf{b} \in \Lambda$ is called a coset of Γ in Λ.
- There are K distinct cosets that partition Λ.
- Let $\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_{K-1}$ be *arbitrary* elements chosen from each coset. Then

$$\Lambda = \bigcup_{k=0}^{K-1} (\mathbf{b}_k + \Gamma).$$

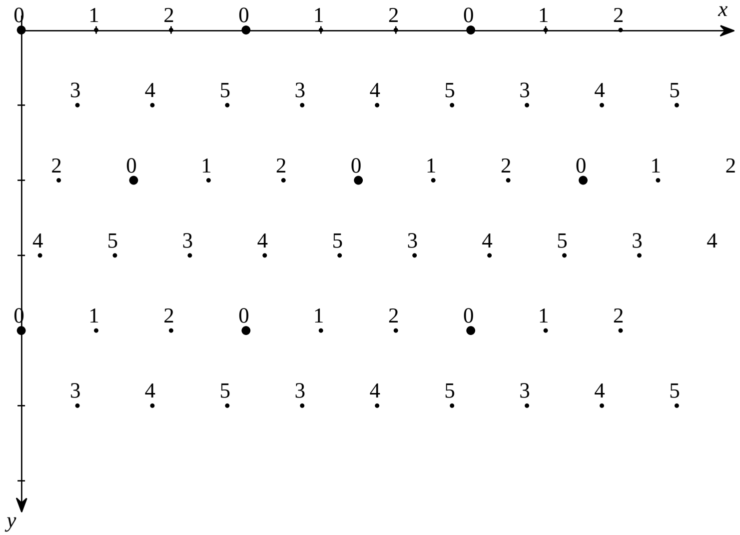
- A periodic signal is constant on cosets of Γ in Λ.

Lattice and sublattice



$\Lambda : \square \qquad \Gamma : \bullet$

Cosets of a sublattice in a lattice



MD discrete-domain periodic signals and systems

- The concepts of signal, signal space, system, linear systems, LSI systems have essentially the same definitions as in MD discrete-domain case.
- The output of an LSI system is given by a periodic convolution

$$\tilde{g}[\mathrm{x}] = \sum_{k=0}^{K-1} \tilde{f}[\mathrm{b}_k]\tilde{h}[\mathrm{x} - \mathrm{b}_k]$$

  which has K distinct values.
- The eigenfunctions of an LSI systems are complex exponential sinusoids

$$\phi_{\mathrm{u}}[\mathrm{x}] = \exp(j2\pi\mathrm{u}\cdot\mathrm{x}) \qquad \mathrm{x} \in \Lambda$$

  To be periodic, we must have $\mathrm{u} \in \Gamma^*$. There are only K distinct ϕ_{u}, for $\mathrm{u} = \mathrm{d}_0, \mathrm{d}_1, \ldots, \mathrm{d}_{K-1}$.

MD discrete-domain periodic Fourier transform

- The eigenvalues of the MD discrete-domain periodic LSI system are

$$\tilde{H}[\mathrm{d}_i] = \sum_{k=0}^{K-1} \tilde{h}[\mathrm{b}_k] \exp(-j2\pi \mathrm{d}_i \cdot \mathrm{b}_k), \quad i = 0, \ldots, K-1$$

- This is the multidimensional extension of the one-dimensional discrete Fourier transform (DFT), also referred to as the discrete-time Fourier series coefficients.
- We adopt the name *discrete-domain Fourier series*.
- We note that if $\Gamma \subset \Lambda$, then $\Lambda^* \subset \Gamma^*$.
- The d_i are (arbitrarily chosen) coset representatives of Λ^* in Γ^*.

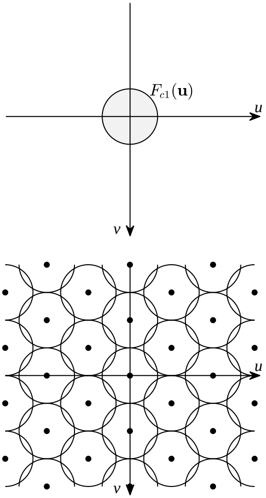
Summary of multidimensional Fourier transforms

Signal domain		Frequency domain
continuous, aperiodic	$\overset{\text{CDFT}}{\longleftrightarrow}$	continuous, aperiodic
Λ discrete, aperiodic	$\overset{\text{DDFT}}{\longleftrightarrow}$	continuous, Λ^* periodic
Λ discrete, Γ periodic	$\overset{\text{DDFS}}{\longleftrightarrow}$	Γ^* discrete, Λ^* periodic
continuous, Γ periodic	$\overset{\text{CDFS}}{\longleftrightarrow}$	Γ^* discrete, aperiodic

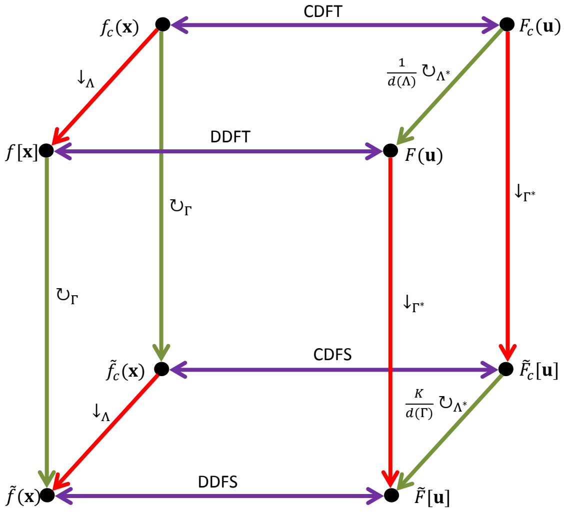
Fourier transform properties

Domain	**Continuous-domain, non-periodic**	**Discrete-domain** (Λ), **non-periodic**	**Continuous-domain, periodic** (Γ)	**Discrete-domain** (Λ), **periodic** ($\Gamma \subset \Lambda$)
Name of the transform	Continuous-domain Fourier transform (CDFT)	Discrete-domain Fourier transform (DDFT)	Continuous-domain Fourier series (CDFS)	Discrete-domain Fourier series (DDFS)
Signals and domains	$f_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} F_c(\mathbf{u}) \quad \mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^D$ $g_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} G_c(\mathbf{u}) \quad \mathbf{u}, \mathbf{u}_0 \in \mathbb{R}^D$	$f[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} F(\mathbf{u}) \quad \mathbf{x}, \mathbf{x}_0 \in \Lambda$ $g[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} G(\mathbf{u}) \quad \mathbf{u}, \mathbf{u}_0 \in \mathbb{R}^D$	$\tilde{f}_c(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{F}_c[\mathbf{u}] \quad \mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^D$ $\tilde{g}_c(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{G}_c[\mathbf{u}] \quad \mathbf{u}, \mathbf{u}_0 \in \Gamma^*$	$\tilde{f}[\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{F}[\mathbf{u}] \quad \mathbf{x}, \mathbf{x}_0 \in \Lambda$ $\tilde{g}[\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{G}[\mathbf{u}] \quad \mathbf{u}, \mathbf{u}_0 \in \Gamma^*$
Periodicity	none	$F(\mathbf{u}+\mathbf{r}) = F(\mathbf{u}), \quad \mathbf{r} \in \Lambda^*$	$\tilde{f}_c(\mathbf{x}+\mathbf{s}) = \tilde{f}_c(\mathbf{x}), \quad \mathbf{s} \in \Gamma$	$\tilde{f}[\mathbf{x}+\mathbf{s}] = \tilde{f}[\mathbf{x}], \quad \mathbf{s} \in \Gamma$ $\tilde{F}[\mathbf{u}+\mathbf{r}] = \tilde{F}[\mathbf{u}], \quad \mathbf{r} \in \Lambda^*$
Period	none	$\mathbf{u} : \mathcal{P}_{\Lambda^*}, \lvert\mathcal{P}_{\Lambda^*}\rvert = 1/d(\Lambda)$	$\mathbf{x} : \mathcal{P}_\Gamma, \lvert\mathcal{P}_\Gamma\rvert = d(\Gamma)$	$\mathbf{x} : \mathcal{B}, \mathbf{u} : \mathcal{D}, \lvert\mathcal{B}\rvert = \lvert\mathcal{D}\rvert = K$
Analysis	$F_c(\mathbf{u}) = \int_{\mathbb{R}^D} f_c(\mathbf{x}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{x}$	$F(\mathbf{u}) = \sum_{\mathbf{x}\in\Lambda} f[\mathbf{x}] \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})$	$\tilde{F}_c[\mathbf{u}] = \int_{\mathcal{P}_\Gamma} \tilde{f}_c(\mathbf{x}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{x}$	$\tilde{F}[\mathbf{u}] = \sum_{\mathbf{x}\in\mathcal{B}} \tilde{f}[\mathbf{x}] \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})$
Synthesis	$f_c(\mathbf{x}) = \int_{\mathbb{R}^D} F_c(\mathbf{u}) \exp(j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{u}$	$f[\mathbf{x}] = d(\Lambda) \int_{\mathcal{P}_{\Lambda^*}} F(\mathbf{u}) \exp(j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{u}$	$\tilde{f}_c(\mathbf{x}) = \frac{1}{d(\Gamma)} \sum_{\mathbf{u}\in\Gamma^*} \tilde{F}_c[\mathbf{u}] \exp(j2\pi\mathbf{u}\cdot\mathbf{x})$	$\tilde{f}[\mathbf{x}] = \frac{1}{K} \sum_{\mathbf{u}\in\mathcal{D}} \tilde{F}[\mathbf{u}] \exp(j2\pi\mathbf{u}\cdot\mathbf{x})$
Linearity	$Af_c(\mathbf{x}) + Bg_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} AF_c(\mathbf{u}) + BG_c(\mathbf{u})$	$Af[\mathbf{x}] + Bg[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} AF(\mathbf{u}) + BG(\mathbf{u})$	$A\tilde{f}_c(\mathbf{x}) + B\tilde{g}_c(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} A\tilde{F}_c[\mathbf{u}] + B\tilde{G}_c[\mathbf{u}]$	$A\tilde{f}[\mathbf{x}] + B\tilde{g}[\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} A\tilde{F}[\mathbf{u}] + B\tilde{G}[\mathbf{u}]$
Shift	$f_c(\mathbf{x}-\mathbf{x}_0) \stackrel{\text{CDFT}}{\longleftrightarrow} F_c(\mathbf{u}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x}_0)$	$f[\mathbf{x}-\mathbf{x}_0] \stackrel{\text{DDFT}}{\longleftrightarrow} F(\mathbf{u}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x}_0)$	$\tilde{f}_c(\mathbf{x}-\mathbf{x}_0) \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{F}_c[\mathbf{u}] \exp(-j2\pi\mathbf{u}\cdot\mathbf{x}_0)$	$\tilde{f}[\mathbf{x}-\mathbf{x}_0] \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{F}[\mathbf{u}] \exp(-j2\pi\mathbf{u}\cdot\mathbf{x}_0)$
Modulation	$f_c(\mathbf{x}) \exp(j2\pi\mathbf{u}_0\cdot\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} F_c(\mathbf{u}-\mathbf{u}_0)$	$f[\mathbf{x}] \exp(j2\pi\mathbf{u}_0\cdot\mathbf{x}) \stackrel{\text{DDFT}}{\longleftrightarrow} F(\mathbf{u}-\mathbf{u}_0)$	$\tilde{f}_c(\mathbf{x}) \exp(j2\pi\mathbf{u}_0\cdot\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{F}_c[\mathbf{u}-\mathbf{u}_0]$	$\tilde{f}[\mathbf{x}] \exp(j2\pi\mathbf{u}_0\cdot\mathbf{x}) \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{F}[\mathbf{u}-\mathbf{u}_0]$
Convolution	$\int_{\mathbb{R}^D} f_c(\mathbf{s}) g_c(\mathbf{x}-\mathbf{s})\, d\mathbf{s} \stackrel{\text{CDFT}}{\longleftrightarrow} F_c(\mathbf{u}) G_c(\mathbf{u})$	$\sum_{\mathbf{s}\in\Lambda} f[\mathbf{s}] g[\mathbf{x}-\mathbf{s}] \stackrel{\text{DDFT}}{\longleftrightarrow} F(\mathbf{u}) G(\mathbf{u})$	$\int_{\mathcal{P}_\Gamma} \tilde{f}_c(\mathbf{s}) \tilde{g}_c(\mathbf{x}-\mathbf{s})\, d\mathbf{s} \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{F}_c[\mathbf{u}] \tilde{G}_c[\mathbf{u}]$	$\sum_{\mathbf{s}\in\mathcal{B}} \tilde{f}[\mathbf{s}] \tilde{g}[\mathbf{x}-\mathbf{s}] \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{F}[\mathbf{u}] \tilde{G}[\mathbf{u}]$
Multiplication	$f_c(\mathbf{x}) g_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} \int_{\mathbb{R}^D} F_c(\mathbf{w}) G_c(\mathbf{u}-\mathbf{w})\, d\mathbf{w}$	$f[\mathbf{x}] g[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} d(\Lambda) \int_{\mathcal{P}_{\Lambda^*}} F(\mathbf{w}) G(\mathbf{u}-\mathbf{w})\, d\mathbf{w}$	$\tilde{f}_c(\mathbf{x}) \tilde{g}_c(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \frac{1}{d(\Gamma)} \sum_{\mathbf{w}\in\Gamma^*} \tilde{F}_c[\mathbf{w}] \tilde{G}_c[\mathbf{u}-\mathbf{w}]$	$\tilde{f}[\mathbf{x}] \tilde{g}[\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} \frac{1}{K} \sum_{\mathbf{w}\in\mathcal{D}} \tilde{F}[\mathbf{w}] \tilde{G}[\mathbf{u}-\mathbf{w}]$
Automorphism of domain	$f_c(\mathbf{A}\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} \frac{1}{\lvert\det \mathbf{A}\rvert} F_c(\mathbf{A}^{-T}\mathbf{u})$	$f[\mathbf{A}\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} F(\mathbf{A}^{-T}\mathbf{u})$	$\tilde{f}_c(\mathbf{A}\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{F}_c[\mathbf{A}^{-T}\mathbf{u}]$	$\tilde{f}[\mathbf{A}\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{F}[\mathbf{A}^{-T}\mathbf{u}]$
Differentiation	$\nabla_{\mathbf{x}} f_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} j2\pi\mathbf{u} F_c(\mathbf{u})$	N/A	$\nabla_{\mathbf{x}} \tilde{f}_c(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} j2\pi\mathbf{u} \tilde{F}_c[\mathbf{u}]$	N/A
Differentiation in frequency	$\mathbf{x} f_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} \frac{j}{2\pi} \nabla_{\mathbf{u}} F_c(\mathbf{u})$	$\mathbf{x} f[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} \frac{j}{2\pi} \nabla_{\mathbf{u}} F(\mathbf{u})$	N/A	N/A
Complex conjugation	$f_c^*(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} F_c^*(-\mathbf{u})$	$f^*[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} F^*(-\mathbf{u})$	$\tilde{f}_c^*(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{F}_c^*[-\mathbf{u}]$	$\tilde{f}^*[\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{F}^*[-\mathbf{u}]$
Parseval	$\int_{\mathbb{R}^D} f_c(\mathbf{x}) g_c^*(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^D} F_c(\mathbf{u}) G_c^*(\mathbf{u})\, d\mathbf{u}$	$\sum_{\mathbf{x}\in\Lambda} f[\mathbf{x}] g^*[\mathbf{x}] = d(\Lambda) \int_{\mathcal{P}_{\Lambda^*}} F(\mathbf{u}) G^*(\mathbf{u})\, d\mathbf{u}$	$\int_{\mathcal{P}_\Gamma} \tilde{f}_c(\mathbf{x}) \tilde{g}_c^*(\mathbf{x})\, d\mathbf{x} = \frac{1}{d(\Gamma)} \sum_{\mathbf{u}\in\Gamma^*} \tilde{F}_c[\mathbf{u}] \tilde{G}_c^*[\mathbf{u}]$	$\sum_{\mathbf{x}\in\mathcal{B}} \tilde{f}[\mathbf{x}] \tilde{g}^*[\mathbf{x}] = \frac{1}{K} \sum_{\mathbf{w}\in\mathcal{D}} \tilde{F}[\mathbf{u}] \tilde{G}^*[\mathbf{u}]$
Duality	$F_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} f_c(-\mathbf{u})$	$\tilde{F}_c[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} d(\Gamma) \tilde{f}_c(-\mathbf{u})$	$F(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \frac{1}{d(\Lambda)} f[-\mathbf{u}]$	$\tilde{F}[\mathbf{C}\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} K \tilde{f}[-\mathbf{C}^{-1}\mathbf{u}]$

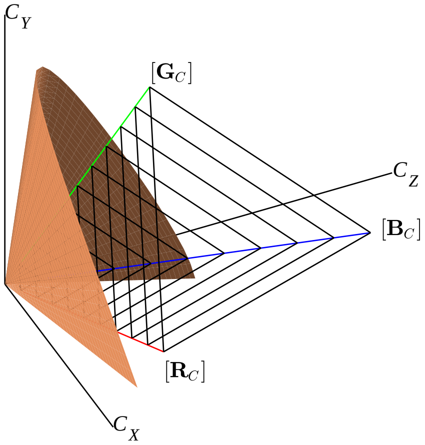
Sampling of an MD continuous-domain signal



Fourier-Poisson cube



Color vector space



Color spaces

- Colors perceived by a human viewer with normal trichromatic vision belong to the three-dimensional vector space $\mathcal{C}$.
- Three linearly independent colors form a basis $\mathcal{B} = \{[\mathrm{P}_1], [\mathrm{P}_2], [\mathrm{P}_3]\}$ for the color space.
- Any color can be represented with three real numbers C_1, C_2, C_3 called tristimulus values

$$[\mathrm{C}] = C_1[\mathrm{P}_1] + C_2[\mathrm{P}_2] + C_3[\mathrm{P}_3]$$

- The set of physically realizable colors lie in a convex cone with a curved boundary formed by the spectral (monochromatic) colors and closed by the plane of purples.
- A subset of these colors with triangular cross-section can be synthesized with a positive linear combination of three display primaries.

Color signal processing

Color signal processing

- A color signal has the form $[\mathrm{C}](\mathrm{x})$, where typically $\mathrm{x} = (x, y)$ or $\mathrm{x} = (x, y, t)$, i.e., $D = 2$ or $D = 3$.
- In terms of a basis $\mathcal{B}$

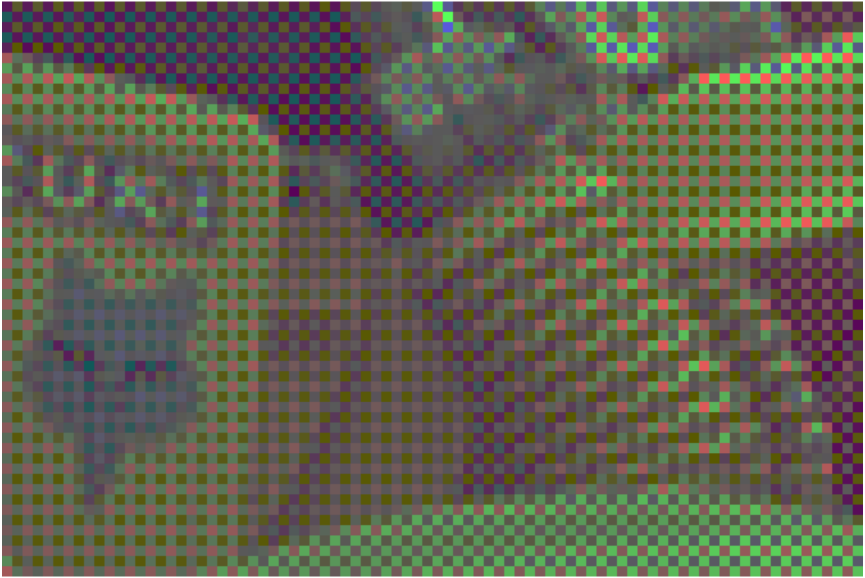
$$[\mathrm{C}](\mathrm{x}) = C_1(\mathrm{x})[\mathrm{P}_1] + C_2(\mathrm{x})[\mathrm{P}_2] + C_3(\mathrm{x})[\mathrm{P}_3]$$

- A color image is represented by three scalar images with respect to a specific basis $\mathcal{B}$.
- A linear system for color signals is formed of nine scalar systems $\mathcal{H}_{ki}$

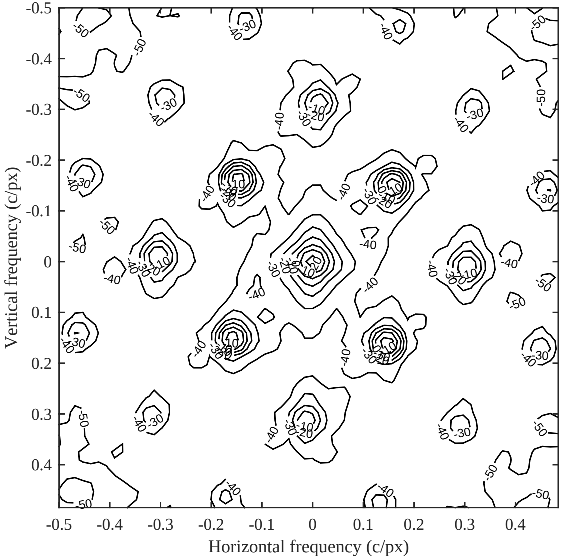
$$\mathcal{H}([\mathrm{C}](\mathrm{x})) = \sum_{k=1}^{3} \left(\sum_{i=1}^{3} (H_{ki} * C_i)(\mathrm{x}) \right) [\mathrm{P}_k].$$

- Frequency response and Fourier transform can be defined for color signals.
- It is very common for different color components to be sampled on different sampling structures.

Bayer color filter array (CFA)



Random field models

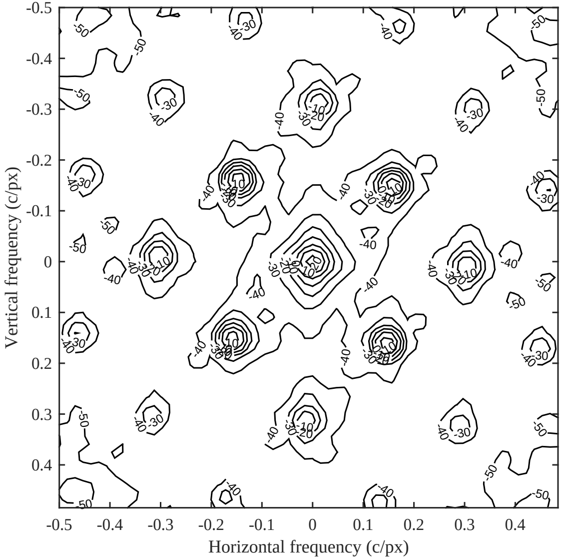


Random field models

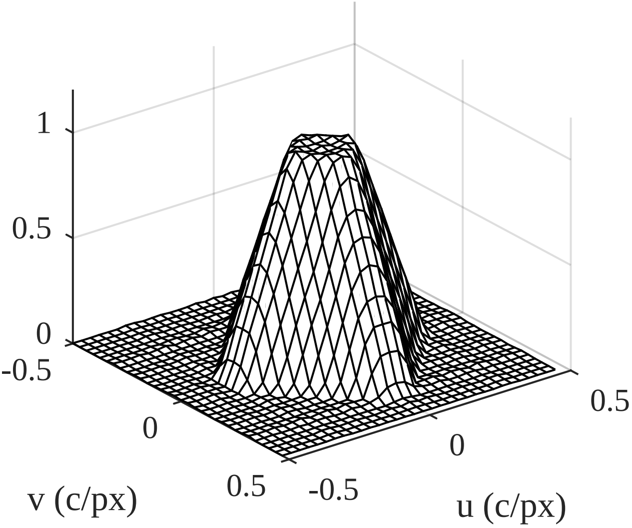
- Many image sources can be usefully modelled as wide-sense-stationary (WSS) random fields
- These WSS processes can be characterized by the autocorrelation function, or its Fourier transform, the power density spectrum.
- The power density spectrum can be helpful in filter design.
- The chapter also covers WSS color random fields, which are characterized by a 3×3 spectral density matrix with respect to a given basis for the color space.
- The chapter presents the effect of sampling and filtering on the spectral density.

A halftone image

Power density spectrum of a halftone image



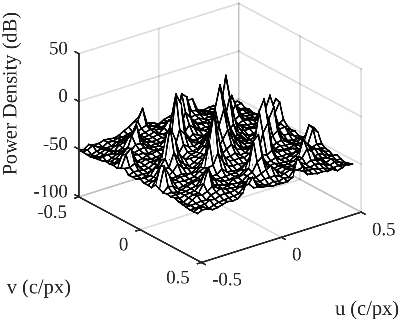
Multidimensional digital filter design



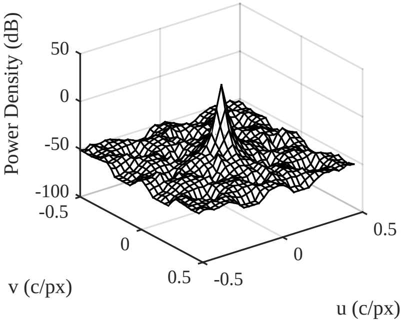
Multidimensional filter design

- Analysis of commonly used ad-hoc filter types: moving average and Gaussian filters.
- Design of band-pass and band-stop filters.
- Frequency-domain design using the window method.
- Frequency-domain design using least-pth optimization, including equality constraints.
- Software is provided on the book website.

Spectral density before and after filtering



Original



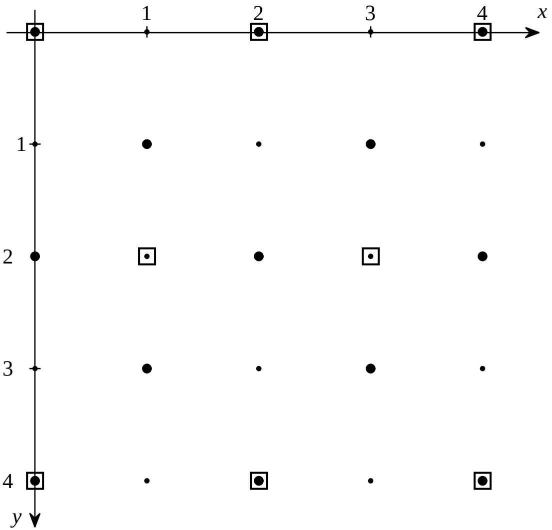
Filtered

Halftone image before and after filtering

Original

Filtered

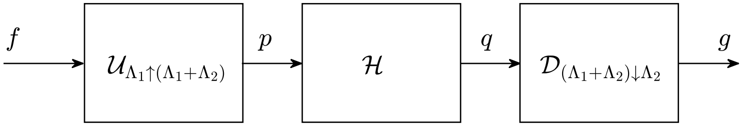
Changing the sampling structure



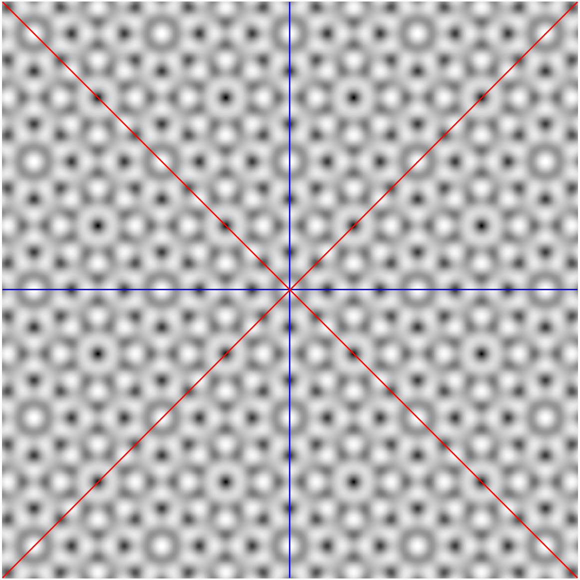
Sampling structure conversion

- The problem addressed is to convert a MD signal defined on one lattice to be defined on a different lattice.
- This can involve an increase or a decrease in the sampling density, or even no change.
- A general approach is presented based on upsampling and downsampling operations.
- The general approach involves upsampling to a common superlattice, followed by downsampling to the output lattice.
- Suitable filter design methodologies are presented.

Changing the sampling structure using a common superlattice



Symmetry invariant signals and systems



Symmetry invariant signals and systems

- A symmetry is a transformation that leaves a certain entity unchanged.
- For example, a shift leaves a lattice unchanged.
- Other symmetries of a lattice include rotations and reflections.
- Chapter 12 extends the notion of shift invariant systems to more general symmetry invariant systems.
- Finding the eigenfunctions and eigenvalues of such systems leads to a general class of Fourier-type transforms.
- An example is the multidimensional discrete cosine transform (DCT) defined on general lattices.

What is not in the book?

- polyphase structures for sampling structure conversion
- fast Fourier transforms
- Markov random fields
- motion analysis and estimation
- tempered distributions
- adaptive and non-linear filtering
- image compression

Thank you!