Multidimensional Signal and Color Image Processing Using Lattices

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A QUICK OVERVIEW
What are Multidimensional Signals?

- Multidimensional signals are information bearing functions of two or more independent variables, such as:
  - Position
  - Time
  - Wavelength
  - Frequency

- These independent variables are called the domain $\mathcal{D}$ of the signal.

- The number of independent variables $D$ is the dimension of the domain and we also call it the dimension of the signal.

- The range $\mathcal{R}$ of the signal can be scalar valued (gray-scale images) or vector valued (color images).
Multidimensional Signals

Grayscale image

\[ D = 2 \]
\[ \mathcal{D} = \mathbb{R}^2 \]
\[ \mathcal{R} = \mathbb{R} \]
Multidimensional Signals

\[ D = 2 \]
\[ D = \mathbb{R}^2 \]
\[ \mathcal{R} = \mathbb{C} \]
Multidimensional Signals

Time-varying image

(Video)

\[ D = 3 \]

\[ D = \mathbb{R}^2 \times \mathbb{R} \]

\[ R = \mathbb{C} \]
Multidimensional Signals

Volumetric image

\[ D = 3 \]
\[ D = \mathbb{R}^3 \]
\[ R = \mathbb{R} \]

From GPU Gems, NVIDIA, 2004
Continuous Domain and Discrete Domain

- Real-world signals are functions of continuous space and time variables.
- Examples are the light arriving on a camera sensor or the human retina, or the light emitted by display screen.
- Digital transmission and processing require a discrete-domain or sampled representation.

This book is mainly concerned with the representation and processing of discrete-domain multidimensional signals.

- Conversions between discrete-domain and continuous-domain representations are also addressed.
Discrete-domain signals are sampled on a regular array of points.

In many real-world situations, the images are not sampled on a rectangular array of points.

- Virtually all digital cameras use a color filter array (CFA) where some color components are not sampled on a rectangular array.
- Virtually all color displays sample color components on non-rectangular arrays.
- In printing, non-rectangular halftones are used.
- In video, non-rectangular interlaced scanning has been widely used.

All of these widely-used non-rectangular sampling structures can be described using lattices (used in the same sense as crystal lattice).

This book uses the lattice as the generic domain for MD discrete-domain signals.
Periodic multidimensional signals arise in many imaging systems. Notable examples are camera and display mosaics like those already shown. As another example, consider the Super CCD image sensor from Fujifilm. The mask consists of an octagonal region replicated on the points of a non-rectangular lattice.

This book uses the lattice to specify the periodicity of MD periodic signals.
A lattice $\Lambda$ in $D$ dimensions is a discrete set of points that can be expressed as the set of all linear combinations with integer coefficients of $D$ linearly independent vectors in $\mathbb{R}^D$ (called basis vectors)

$$\Lambda = \{n_1 v_1 + \cdots + n_D v_D | n_i \in \mathbb{Z}\}$$

$$= \{Vn | n \in \mathbb{Z}^D\}$$

$$V = [v_1 \cdots v_D]$$

Sampling Matrix

$$V = \begin{bmatrix} 0.9X & 1.4X \\ 0.3X & -0.5X \end{bmatrix}$$
Properties of a lattice

- A lattice is closed under addition and subtraction. It forms an additive group.
- A lattice is invariant under a shift by any element of the lattice.

We can develop a theory of linear shift-invariant (LSI) signal processing for signals defined on a lattice - main theme of the book

- The basis and sampling matrix for a given lattice are not unique. 
  \( \text{LAT}(V) = \text{LAT}(VE) \) if \( V \) is an integer matrix with \( |\det E| = 1 \).

The theory should not depend on any arbitrary choice of lattice basis or sampling matrix. The book favors a basis-independent development of signal processing on lattices.
The book presents four categories of signal spaces with associated linear shift-invariant signal processing theory and Fourier analysis:

- Continuous-domain aperiodic
- Discrete-domain aperiodic, sampling lattice $\Lambda$
- Discrete-domain periodic, sampling lattice $\Lambda$, periodicity lattice $\Gamma$
- Continuous-domain periodic, periodicity lattice $\Gamma$

Chapters 2-5 present these four categories of signals and signal processing theory in this order.
We convert a continuous-domain signal to a discrete-domain signal by sampling.

We convert an aperiodic signal to a periodic one by replicating it on the points of a lattice, and adding. We call this periodization.

Sampling in the signal domain corresponds to periodization in the frequency domain, and vice versa.

We can convert between the four categories of signals by sampling and periodization. This is captured in the Fourier-Poisson cube.

Chapter 6 presents these sampling and periodization results. It also treats the reconstruction of sampled signals.
Colors perceived by a human viewer with normal trichromatic vision belong to the three-dimensional vector space $\mathcal{C}$.

Three linearly independent colors form a basis $\mathcal{B} = \{[P_1], [P_2], [P_3]\}$ for this color vector space.

Any color can be represented with three real numbers $C_1, C_2, C_3$ called tristimulus values $[C] = C_1[P_1] + C_2[P_2] + C_3[P_3]$.

Chapter 7 fully develops the theory of the color vector space which forms the basis for color image processing using a novel approach.
A color signal has the form $[C](x)$; the range of the signal is the color vector space $C$.

A color image is represented by three scalar images with respect to a specific basis $B$: $[C](x) = C_1(x)[P_1] + C_2(x)[P_2] + C_3(x)[P_3]$

Chapter 8 develops the theory of linear systems for color signals including associated Fourier analysis. A linear system for color signals is formed of nine scalar systems.
It is very common for different component images of a color image with respect to a specific basis to be sampled on different sampling structures. Examples are color filter arrays in digital cameras, color mosaic displays and component image standards.

Chapter 8 develops a novel theory for the sampling of color signals with different sampling structures for different components.
Other Topics

- Stationary random field models for color signals (chapter 9)
- Multidimensional filter design with Matlab software (chapter 10)
- Sampling structure conversion (chapter 11)
- Symmetry-invariant signal processing; the generalized discrete cosine transform on lattices (chapter 12)
The book develops a novel approach to the theory of multidimensional signal processing heavily based on lattices. I emphasize the use of basis-independent, non-normalized representations of signals and transforms. I use the vector-space representation for color and present color image processing in this context. Several special topics, namely random field models, filter design, sampling structure conversion and symmetry-invariant signal processing are developed in this framework.
Matlab software to reproduce all the figures in the book, as well as spectral estimation and filter design routines

- solutions manual for all the problems in the book
- errata
- additional material such as the printable table of Fourier transform properties and links to presentations such as this one

Book web site: http://www.site.uottawa.ca/~edubois/mdsp/
Thanks!
Rectangular sampling structure
Bayer Color Filter Array
Pentile RGBW Display Mosaic
A general lattice
Super CCD Image Sensor
Fourier-Poisson Cube