Multidimensional Signal and Color Image Processing using Lattices

A brief overview

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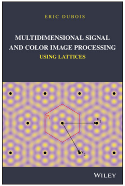
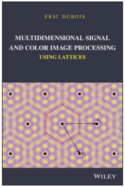


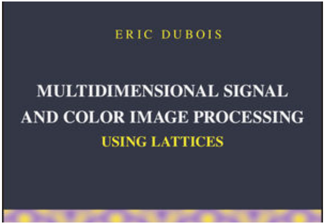
Abstract

- Overview of the book.
- Friendly book review by the author.
- What distinguishes the book from other works.
- Some contributions of the book.



Key themes of the book

- Processing of multidimensional (MD) signals (e.g., images, video, volumetric, etc.)
- Function of several independent variables (space, time, etc.)
- Scalar or vector valued signals.
- Continuous-domain and discrete-domain signals (where sampling structures are defined by lattices)
- Aperiodic and periodic signals (where periodicity is specified by a lattice)
- Color signals, where the signal range is a vector space.



Overview of the book contents

- MD signals and systems: four cases, continuous-domain aperiodic (CDA), discrete-domain aperiodic (DDA), discrete-domain periodic (DDP), continuous-domain periodic (CDP)
- Sampling and reconstruction - conversion between domains
- Color representation and color signals
- Random fields (wide-sense stationary case)
- Filter design
- Sampling structure conversion
- Symmetry invariant signal processing
- Separate chapter summarizing relevant facts about lattices

Continuous-Domain Signals and Systems

Continuous-domain MD signals

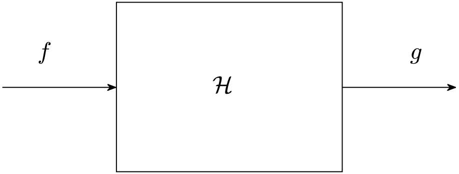
- A multidimensional signal f is a function of D independent variables

$$f(\mathrm{x}) = f(x_1, x_2, \ldots, x_D), \qquad \mathrm{x} \in \mathbb{R}^D.$$

- Although we are mainly concerned with $D > 1$, one-dimensional signals with $D = 1$ are simply a special case of the general theory.
- We assume that signal values belong to a vector space, in the simplest case the space of real numbers. Color images correspond to a three-dimensional vector space.
- We assume that signals belong to a vector space $\mathcal{S}$ over the complex numbers called the *signal space*. Thus signals can be added, scaled, negated, there is a zero signal, etc.

Continuous-domain MD systems

- A multidimensional system $\mathcal{H}$ acts on elements of a signal space $\mathcal{S}$. We consider deterministic systems where each input f results in a well-defined output. We write $\mathcal{H} : \mathcal{S} \to \mathcal{S} : g = \mathcal{H}f$.
- Example 1: $g(\mathrm{x}) = (f(\mathrm{x}))^2$ for all $\mathrm{x} \in \mathbb{R}^D$.
- Example 2: $\mathcal{T}_{\mathrm{d}} : g(\mathrm{x}) = f(\mathrm{x} - \mathrm{d})$ for some fixed $\mathrm{d} \in \mathbb{R}^D$. We call this the shift or translation system.
- We consider two main classes of systems:
- Linear systems: $\mathcal{H}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathcal{H} f_1 + \alpha_2 \mathcal{H} f_2$.
- Shift-invariant systems: $\mathcal{H}\mathcal{T}_{\mathrm{d}} = \mathcal{T}_{\mathrm{d}}\mathcal{H}$ for all d.
- A *linear shift-invariant* (LSI) system is both linear and shift invariant.



MD LSI systems: convolution

For an LSI system $\mathcal{H}$, we can show that if the input is f, then the output is $g = h * f$, where $h = \mathcal{H}\delta$ and δ is the Dirac delta.
Specifically,

$$\begin{aligned} g(\mathrm{x}) &= \int_{\mathbb{R}^D} h(\mathrm{s}) f(\mathrm{x} - \mathrm{s}) d\mathrm{s} \\ &= \int_{\mathbb{R}^D} f(\mathrm{s}) h(\mathrm{x} - \mathrm{s}) d\mathrm{s} \\ h * f &= f * h \end{aligned}$$

MD LSI systems: frequency response

An important class of signals are the complex exponentials ϕ_{u} given by

$$\phi_{\mathsf{u}}(\mathsf{x}) = \exp(j2\pi \mathsf{u} \cdot \mathsf{x}) = \cos(2\pi \mathsf{u} \cdot \mathsf{x}) + j \sin(2\pi \mathsf{u} \cdot \mathsf{x})$$

$$\phi_{\mathsf{u}}(x_1, \ldots, x_D) = \exp(j2\pi(u_1 x_1 + \cdots + u_D x_D))$$

for some fixed $\mathsf{u} \in \mathbb{R}^D$, referred to as the frequency vector.
For an LSI system, applying the convolution formula, we find that

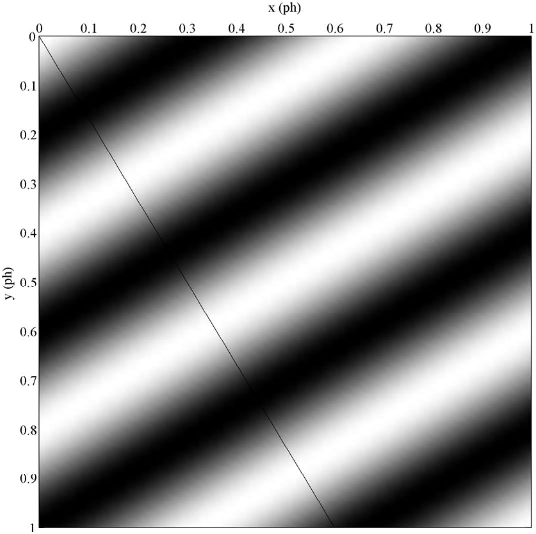
$$\mathcal{H}\phi_{\mathsf{u}} = H(\mathsf{u})\phi_{\mathsf{u}}$$

where for a given u, $H(\mathsf{u})$ is a complex number given by

$$H(\mathsf{u}) = \int_{\mathbb{R}^D} h(\mathsf{x}) \exp(-j2\pi \mathsf{u} \cdot \mathsf{x}) d\mathsf{x}.$$

Taken as a function of u, $H(\mathsf{u})$ is called the frequency response of the LSI system, and is the *Fourier Transform* of the impulse response.

Two-dimensional sinusoidal signal



$$f(x, y) = 0.5\cos(2\pi(1.5x + 2.5y)) + 0.5$$

Horizontal frequency $u = 1.5$ c/ph, vertical frequency $v = 2.5$ c/ph.

Continuous-domain Fourier transform properties

	$f(\mathbf{x}) = \int_{\mathbb{R}^D} F(\mathbf{u}) \exp(j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{u}$	$F(\mathbf{u}) = \int_{\mathbb{R}^D} f(\mathbf{x}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{x}$
(2.1)	$Af(\mathbf{x}) + Bg(\mathbf{x})$	$AF(\mathbf{u}) + BG(\mathbf{u})$
(2.2)	$f(\mathbf{x} - \mathbf{x}_0)$	$F(\mathbf{u}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x}_0)$
(2.3)	$f(\mathbf{x}) \exp(j2\pi\mathbf{u}_0\cdot\mathbf{x})$	$F(\mathbf{u} - \mathbf{u}_0)$
(2.4)	$f(\mathbf{x}) * g(\mathbf{x})$	$F(\mathbf{u})G(\mathbf{u})$
(2.5)	$f(\mathbf{x})g(\mathbf{x})$	$F(\mathbf{u}) * G(\mathbf{u})$
(2.6)	$f(\mathbf{A}\mathbf{x})$	$\frac{1}{\lvert\det \mathbf{A}\rvert} F(\mathbf{A}^{-T}\mathbf{u})$
(2.7)	$\nabla_{\mathbf{x}} f(\mathbf{x})$	$j2\pi\mathbf{u}F(\mathbf{u})$
(2.8)	$\mathbf{x}f(\mathbf{x})$	$\frac{j}{2\pi}\nabla_{\mathbf{u}} F(\mathbf{u})$
(2.9)	$f^*(\mathbf{x})$	$F^*(-\mathbf{u})$
(2.10)	$F(\mathbf{x})$	$f(-\mathbf{u})$
(2.11)	$f_1(x_1)\cdots f_D(x_D)$	$F_1(u_1)\cdots F_D(u_D)$
(2.12)	$\int_{\mathbb{R}^D} f(\mathbf{x})g^*(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^D} F(\mathbf{u})G^*(\mathbf{u})\, d\mathbf{u}$	

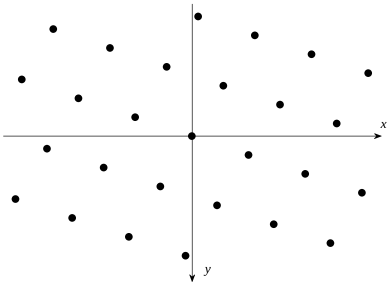
Discrete-Domain Signals and Systems

Discrete-domain MD signals

- An MD discrete-domain signal is defined at a discrete set of points Ψ in $\mathbb{R}^D$ called the *sampling structure*.

$$f[\mathrm{x}], \qquad \mathrm{x} \in \Psi.$$

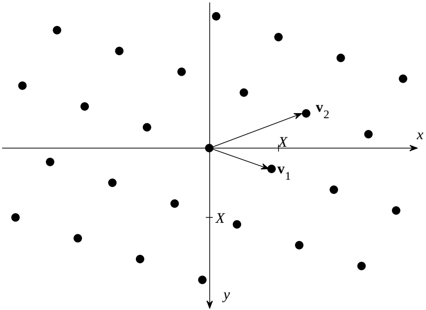
- The main tool used to describe and analyze sampling structures is the *lattice* (as in crystal lattice).
- A lattice is a uniform discrete set of points in $\mathbb{R}^D$; the neighborhood of a lattice point looks the same at every point



Lattices

A lattice Λ in D dimensions is a discrete set of points that can be expressed as the set of all linear combinations with *integer* coefficients of D linearly independent vectors in $\mathbb{R}^D$ (called basis vectors),

$$\begin{aligned}\Lambda &= \{n_1\mathsf{v}_1 + \cdots + n_D\mathsf{v}_D \mid n_i \in \mathbb{Z}\} \\ &= \{\mathsf{V}\mathsf{n} \mid \mathsf{n} \in \mathbb{Z}^D\} = \mathsf{LAT}(\mathsf{V}), \\ \mathsf{V} &= [\mathsf{v}_1, \cdots, \mathsf{v}_D].\end{aligned}$$

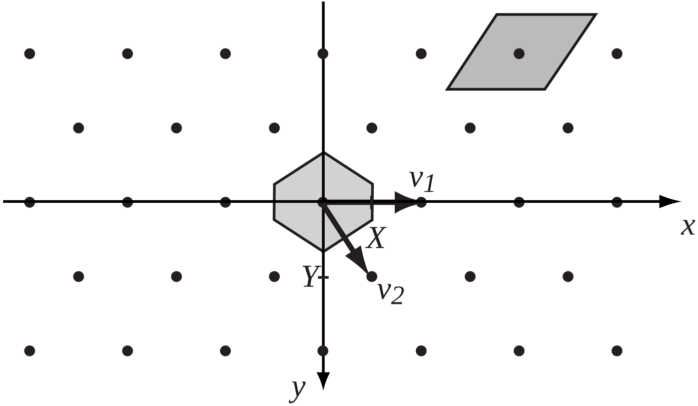


$$\mathsf{V} = \begin{bmatrix} 0.9X & 1.4X \\ 0.3X & -0.5X \end{bmatrix}$$

Some properties of a lattice Λ

- The basis and sampling matrix for a given lattice Λ are not unique.
- $\mathrm{LAT}(\mathrm{V}) = \mathrm{LAT}(\mathrm{VE})$ if E is an integer matrix with $|\det \mathrm{E}| = 1$.
- $\mathrm{x} \pm \mathrm{y} \in \Lambda$ if $\mathrm{x}, \mathrm{y} \in \Lambda$.
- $0 \in \Lambda$
- $\Lambda + \mathrm{d} = \Lambda$ if $\mathrm{d} \in \Lambda$.
- A unit cell of a lattice is a region $\mathcal{P} \subset \mathbb{R}^D$ such that $\cup_{\mathrm{s}\in\Lambda} \mathcal{P} + \mathrm{s} = \mathbb{R}^D$ while $(\mathcal{P} + \mathrm{s}_1) \cap (\mathcal{P} + \mathrm{s}_2) = \emptyset$ for and $\mathrm{s}_1, \mathrm{s}_2 \in \Lambda$ such that $\mathrm{s}_1 \neq \mathrm{s}_2$. It is not unique.
- The volume of a unit cell is $d(\Lambda) = |\det \mathrm{V}|$. $1/d(\Lambda)$ is the sampling density.

Illustration of unit cells



Discrete-domain MD systems

- For discrete-domain signals on a lattice Λ, the concepts of signal space $\mathcal{S}_\Lambda$, system $\mathcal{H}$, linear system, shift-invariant system, LSI system are all formally the same as for continuous-domain MD systems.
- The only proviso is that the shift $\mathbf{d}$ in a shift-system $\mathcal{T}_\mathbf{d}$ must be itself an element of the lattice Λ.
- The output of an LSI system is again given by a convolution formula

$$\begin{aligned} g[\mathbf{x}] &= \sum_{\mathbf{s}\in\Lambda} h[\mathbf{s}]f[\mathbf{x}-\mathbf{s}] \\ &= \sum_{\mathbf{s}\in\Lambda} f[\mathbf{s}]h[\mathbf{x}-\mathbf{s}] \qquad \mathbf{x}\in\Lambda \end{aligned}$$

- We write this as $g = h * f = f * h$.
- Here, h is the unit sample response $h = \mathcal{H}\delta_\Lambda$, where

$$\delta_\Lambda[\mathbf{x}] = \begin{cases} 1 & \mathbf{x} = \mathbf{0} \\ 0 & \mathbf{x} = \Lambda\backslash\mathbf{0} \end{cases}$$

MD discrete-domain LSI systems: frequency response

- Discrete-domain complex-exponential sinusoidal signals ϕ_{u} are defined in the same way

$$\phi_{\mathsf{u}}[\mathsf{x}] = \exp(j2\pi \mathsf{u} \cdot \mathsf{x}), \qquad \mathsf{x} \in \Lambda$$

  for some fixed frequency vector $\mathsf{u} \in \mathbb{R}^D$.

- The complex sinusoids are periodic in the frequency vector

$$\phi_{\mathsf{u}} = \phi_{\mathsf{u}+\mathsf{r}} \qquad \text{if } \mathsf{r} \in \Lambda^*$$

  where $\Lambda^* = \mathrm{LAT}(\mathsf{V}^{-T})$ is called the reciprocal (or dual) lattice.

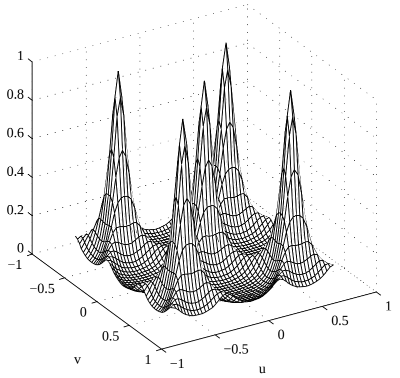
- The complex sinusoids are eigenfunctions of any LSI system, $\mathcal{H}\phi_{\mathsf{u}} = H(\mathsf{u})\phi_{\mathsf{u}}$, where

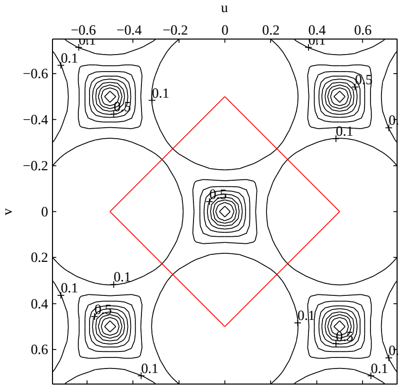
$$H(\mathsf{u}) = \sum_{\mathsf{x} \in \Lambda} h[\mathsf{x}] \exp(-j2\pi \mathsf{u} \cdot \mathsf{x})$$

Discrete-domain Fourier transform properties

	$f[\mathbf{x}] = d(\Lambda)\int_{\mathcal{P}^*} F(\mathbf{u}) \exp(j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{u}$	$F(\mathbf{u}) = \sum_{\mathbf{x}\in\Lambda} f[\mathbf{x}] \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})$
(3.1)	$Af[\mathbf{x}] + Bg[\mathbf{x}]$	$AF(\mathbf{u}) + BG(\mathbf{u})$
(3.2)	$f[\mathbf{x} - \mathbf{x}_0]$	$F(\mathbf{u}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x}_0)$
(3.3)	$f[\mathbf{x}] \exp(j2\pi\mathbf{u}_0\cdot\mathbf{x})$	$F(\mathbf{u} - \mathbf{u}_0)$
(3.4)	$f[\mathbf{x}] * g[\mathbf{x}]$	$F(\mathbf{u})G(\mathbf{u})$
(3.5)	$f[\mathbf{x}]g[\mathbf{x}]$	$d(\Lambda)\int_{\mathcal{P}^*} F(\mathbf{r})G(\mathbf{u}-\mathbf{r})\, d\mathbf{r}$
(3.6)	$f[\mathbf{A}\mathbf{x}]$	$F(\mathbf{A}^{-T}\mathbf{u})$
(3.7)	$\mathbf{x}f[\mathbf{x}]$	$\frac{j}{2\pi}\nabla_{\mathbf{u}} F(\mathbf{u})$
(3.8)	$f^*[\mathbf{x}]$	$F^*(-\mathbf{u})$
(3.9)	$\tilde{F}[\mathbf{x}]$	$d(\Gamma)\tilde{f}(-\mathbf{u})$
(3.10)	$\sum_{\mathbf{x}\in\Lambda} f[\mathbf{x}]g^*[\mathbf{x}] = d(\Lambda)\int_{\mathcal{P}^*} F(\mathbf{u})G^*(\mathbf{u})\, d\mathbf{u}$	

Example: Fourier transform of an exponential function on a hexagonal lattice





Comparison of approach in this book with the conventional approach in the literature

- The book presents discrete-domain signal processing on lattices, with rectangular sampling structures as a special case.
- This book presents a development of multidimensional discrete-domain signal processing that does not depend on arbitrarily chosen entities, particularly bases for lattices.
- In conventional presentations, a discrete-domain signal is defined on $\mathbb{Z}^D$,

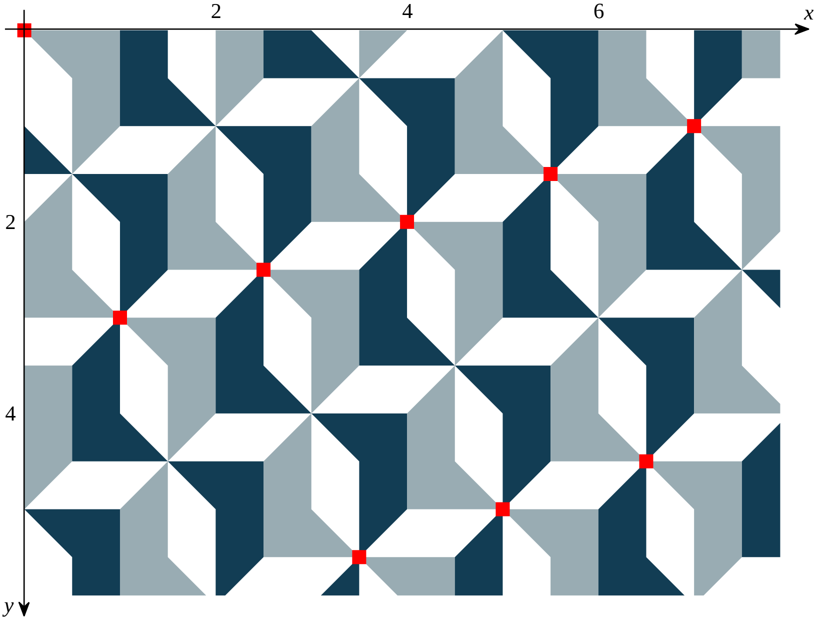
$$f[\mathrm{n}] = f[n_1, n_2, \ldots, n_D]$$

- Sampling is relative to an underlying continuous-domain signal

$$f_d[\mathrm{n}] = f_c(\mathrm{V}\mathrm{n}), \qquad \mathrm{n} \in \mathbb{Z}^D.$$

- This definition is dependent on the *non-unique* sampling matrix V.

Multidimensional periodic signals



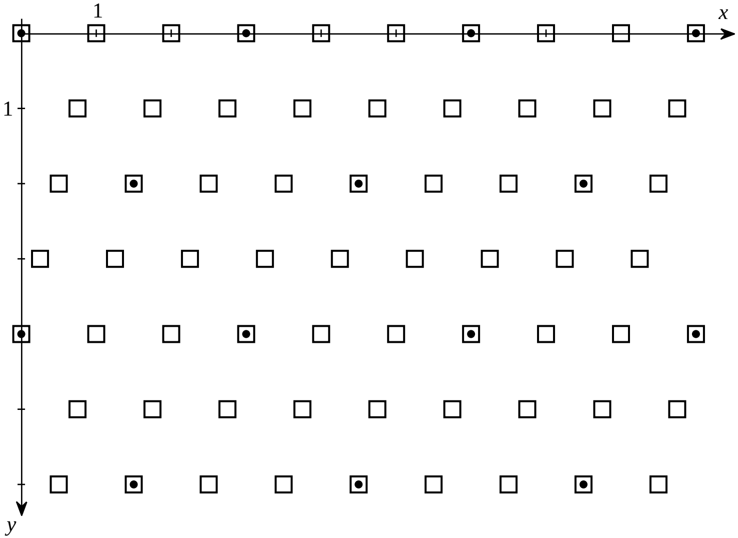
Multidimensional periodic signals

- Multidimensional periodic signals are covered in Chapter 4 (discrete domain) and Chapter 5 (continuous domain).
- For a periodic signal, periodicity is determined by a lattice, that we call the periodicity lattice.
- $\tilde{f}[\mathrm{x} + \mathrm{t}] = \tilde{f}[\mathrm{x}], \quad$ for all $\mathrm{t} \in \Gamma$, where Γ is the periodicity lattice.
- For continuous-domain signals, $\mathrm{x} \in \mathbb{R}^D$ and Γ is any lattice.
- For discrete-domain signals, Γ must be a sublattice of the sampling lattice Λ.
- One period of the signal consists of the signal restricted to any unit cell of the periodicity lattice Γ.

Discrete-domain MD periodic signals

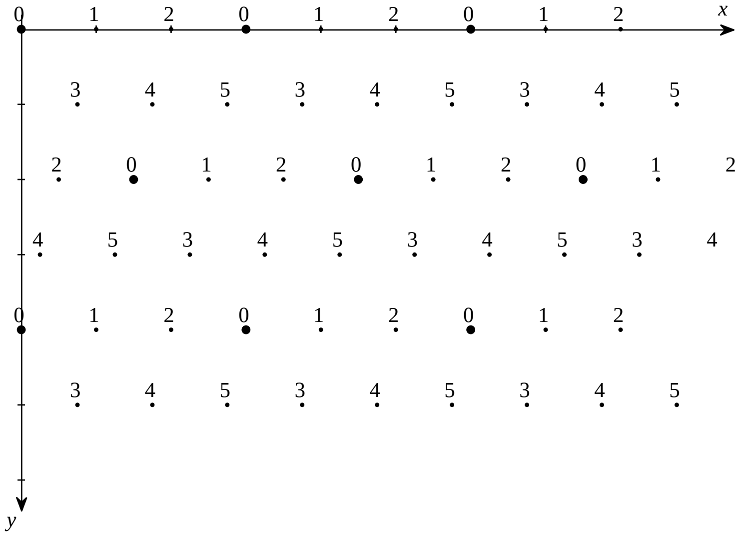
- The periodicity lattice is a sublattice of the sampling lattice, $\Gamma \subset \Lambda$.
- The number of sample points in any unit cell of Γ is the integer $K = d(\Gamma)/d(\Lambda)$.
- The set $\mathrm{b} + \Gamma = \{\mathrm{b} + \mathrm{x} \mid \mathrm{x} \in \Gamma\}$ for any $\mathrm{b} \in \Lambda$ is called a coset of Γ in Λ.
- There are K distinct cosets that partition Λ.
- Let $\mathrm{b}_0, \mathrm{b}_1, \ldots, \mathrm{b}_{K-1}$ be *arbitrary* elements chosen from each coset. Then
$$\Lambda = \bigcup_{k=0}^{K-1} (\mathrm{b}_k + \Gamma).$$
- A periodic signal is constant on cosets of Γ in Λ.

Lattice and sublattice



$\Lambda : \square \qquad \Gamma : \bullet$

Cosets of a sublattice in a lattice



MD discrete-domain periodic signals and systems

- The concepts of signal, signal space, system, linear systems, LSI systems have essentially the same definitions as in MD discrete-domain case.
- The output of an LSI system is given by a periodic convolution

$$\tilde{g}[\mathrm{x}] = \sum_{k=0}^{K-1} \tilde{f}[\mathrm{b}_k]\tilde{h}[\mathrm{x} - \mathrm{b}_k]$$

  which has K distinct values.
- The eigenfunctions of an LSI systems are complex exponential sinusoids

$$\phi_{\mathrm{u}}[\mathrm{x}] = \exp(j2\pi \mathrm{u} \cdot \mathrm{x}) \qquad \mathrm{x} \in \Lambda$$

  To be periodic, we must have $\mathrm{u} \in \Gamma^*$. There are only K distinct ϕ_{u}, for $\mathrm{u} = \mathrm{d}_0, \mathrm{d}_1, \ldots, \mathrm{d}_{K-1}$.

MD discrete-domain periodic Fourier transform

- The eigenvalues of the MD discrete-domain periodic LSI system are

$$\tilde{H}[\mathrm{d}_i] = \sum_{k=0}^{K-1} \tilde{h}[\mathrm{b}_k] \exp(-j2\pi \mathrm{d}_i \cdot \mathrm{b}_k), \quad i = 0, \ldots, K-1$$

- This is the multidimensional extension of the one-dimensional discrete Fourier transform (DFT), also referred to as the discrete-time Fourier series coefficients.
- We adopt the name *discrete-domain Fourier series*.
- We note that if $\Gamma \subset \Lambda$, then $\Lambda^* \subset \Gamma^*$.
- The d_i are (arbitrarily chosen) coset representatives of Λ^* in Γ^*.

Summary of multidimensional Fourier transforms

Signal domain		Frequency domain
continuous, aperiodic	$\overset{\text{CDFT}}{\longleftrightarrow}$	continuous, aperiodic
Λ discrete, aperiodic	$\overset{\text{DDFT}}{\longleftrightarrow}$	continuous, Λ^* periodic
Λ discrete, Γ periodic	$\overset{\text{DDFS}}{\longleftrightarrow}$	Γ^* discrete, Λ^* periodic
continuous, Γ periodic	$\overset{\text{CDFS}}{\longleftrightarrow}$	Γ^* discrete, aperiodic

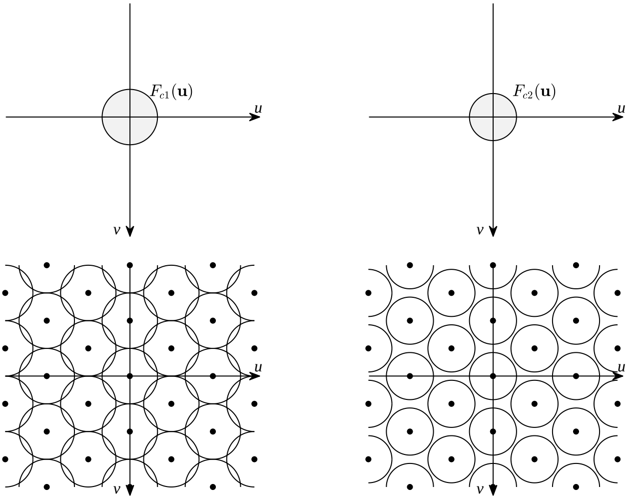
Fourier transform properties

Domain	**Continuous-domain, non-periodic**	**Discrete-domain** (Λ)**, non-periodic**	**Continuous-domain, periodic** (Γ)	**Discrete-domain** (Λ)**, periodic** $(\Gamma \subset \Lambda)$
Name of the transform	Continuous-domain Fourier transform (CDFT)	Discrete-domain Fourier transform (DDFT)	Continuous-domain Fourier series (CDFS)	Discrete-domain Fourier series (DDFS)
Signals and domains	$f_c(\mathbf{x}) \xleftrightarrow{\text{CDFT}} F_c(\mathbf{u}) \quad \mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^D$ $g_c(\mathbf{x}) \xleftrightarrow{\text{CDFT}} G_c(\mathbf{u}) \quad \mathbf{u}, \mathbf{u}_0 \in \mathbb{R}^D$	$f[\mathbf{x}] \xleftrightarrow{\text{DDFT}} F(\mathbf{u}) \quad \mathbf{x}, \mathbf{x}_0 \in \Lambda$ $g[\mathbf{x}] \xleftrightarrow{\text{DDFT}} G(\mathbf{u}) \quad \mathbf{u}, \mathbf{u}_0 \in \mathbb{R}^D$	$\tilde{f}_c(\mathbf{x}) \xleftrightarrow{\text{CDFS}} \tilde{F}_c[\mathbf{u}] \quad \mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^D$ $\tilde{g}_c(\mathbf{x}) \xleftrightarrow{\text{CDFS}} \tilde{G}_c[\mathbf{u}] \quad \mathbf{u}, \mathbf{u}_0 \in \Gamma^*$	$\tilde{f}[\mathbf{x}] \xleftrightarrow{\text{DDFS}} \tilde{F}[\mathbf{u}] \quad \mathbf{x}, \mathbf{x}_0 \in \Lambda$ $\tilde{g}[\mathbf{x}] \xleftrightarrow{\text{DDFS}} \tilde{G}[\mathbf{u}] \quad \mathbf{u}, \mathbf{u}_0 \in \Gamma^*$
Periodicity	none	$F(\mathbf{u}+\mathbf{r}) = F(\mathbf{u}), \quad \mathbf{r} \in \Lambda^*$	$\tilde{f}_c(\mathbf{x}+\mathbf{s}) = \tilde{f}_c(\mathbf{x}), \quad \mathbf{s} \in \Gamma$	$\tilde{f}[\mathbf{x}+\mathbf{s}] = \tilde{f}[\mathbf{x}], \quad \mathbf{s} \in \Gamma$ $\tilde{F}[\mathbf{u}+\mathbf{r}] = \tilde{F}[\mathbf{u}], \quad \mathbf{r} \in \Lambda^*$
Period	none	$\mathbf{u} : \mathcal{P}_{\Lambda^*},\ \lvert\mathcal{P}_{\Lambda^*}\rvert = 1/d(\Lambda)$	$\mathbf{x} : \mathcal{P}_\Gamma,\ \lvert\mathcal{P}_\Gamma\rvert = d(\Gamma)$	$\mathbf{x} : \mathcal{B},\ \mathbf{u} : \mathcal{D},\ \lvert\mathcal{B}\rvert = \lvert\mathcal{D}\rvert = K$
Analysis	$F_c(\mathbf{u}) = \int_{\mathbb{R}^D} f_c(\mathbf{x}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{x}$	$F(\mathbf{u}) = \sum_{\mathbf{x}\in\Lambda} f[\mathbf{x}] \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})$	$\tilde{F}_c[\mathbf{u}] = \int_{\mathcal{P}_\Gamma} \tilde{f}_c(\mathbf{x}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{x}$	$\tilde{F}[\mathbf{u}] = \sum_{\mathbf{x}\in\mathcal{B}} \tilde{f}[\mathbf{x}] \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})$
Synthesis	$f_c(\mathbf{x}) = \int_{\mathbb{R}^D} F_c(\mathbf{u}) \exp(j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{u}$	$f[\mathbf{x}] = d(\Lambda) \int_{\mathcal{P}_{\Lambda^*}} F(\mathbf{u}) \exp(j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{u}$	$\tilde{f}_c(\mathbf{x}) = \frac{1}{d(\Gamma)} \sum_{\mathbf{u}\in\Gamma^*} \tilde{F}_c[\mathbf{u}] \exp(j2\pi\mathbf{u}\cdot\mathbf{x})$	$\tilde{f}[\mathbf{x}] = \frac{1}{K} \sum_{\mathbf{u}\in\mathcal{D}} \tilde{F}[\mathbf{u}] \exp(j2\pi\mathbf{u}\cdot\mathbf{x})$
Linearity	$Af_c(\mathbf{x}) + Bg_c(\mathbf{x}) \xleftrightarrow{\text{CDFT}} AF_c(\mathbf{u}) + BG_c(\mathbf{u})$	$Af[\mathbf{x}] + Bg[\mathbf{x}] \xleftrightarrow{\text{DDFT}} AF(\mathbf{u}) + BG(\mathbf{u})$	$A\tilde{f}_c(\mathbf{x}) + B\tilde{g}_c(\mathbf{x}) \xleftrightarrow{\text{CDFS}} A\tilde{F}_c[\mathbf{u}] + B\tilde{G}_c[\mathbf{u}]$	$A\tilde{f}[\mathbf{x}] + B\tilde{g}[\mathbf{x}] \xleftrightarrow{\text{DDFS}} A\tilde{F}[\mathbf{u}] + B\tilde{G}[\mathbf{u}]$
Shift	$f_c(\mathbf{x}-\mathbf{x}_0) \xleftrightarrow{\text{CDFT}} F_c(\mathbf{u}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x}_0)$	$f[\mathbf{x}-\mathbf{x}_0] \xleftrightarrow{\text{DDFT}} F(\mathbf{u}) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x}_0)$	$\tilde{f}_c(\mathbf{x}-\mathbf{x}_0) \xleftrightarrow{\text{CDFS}} \tilde{F}_c[\mathbf{u}] \exp(-j2\pi\mathbf{u}\cdot\mathbf{x}_0)$	$\tilde{f}[\mathbf{x}-\mathbf{x}_0] \xleftrightarrow{\text{DDFS}} \tilde{F}[\mathbf{u}] \exp(-j2\pi\mathbf{u}\cdot\mathbf{x}_0)$
Modulation	$f_c(\mathbf{x}) \exp(j2\pi\mathbf{u}_0\cdot\mathbf{x}) \xleftrightarrow{\text{CDFT}} F_c(\mathbf{u}-\mathbf{u}_0)$	$f[\mathbf{x}] \exp(j2\pi\mathbf{u}_0\cdot\mathbf{x}) \xleftrightarrow{\text{DDFT}} F(\mathbf{u}-\mathbf{u}_0)$	$\tilde{f}_c(\mathbf{x}) \exp(j2\pi\mathbf{u}_0\cdot\mathbf{x}) \xleftrightarrow{\text{CDFS}} \tilde{F}_c[\mathbf{u}-\mathbf{u}_0]$	$\tilde{f}[\mathbf{x}] \exp(j2\pi\mathbf{u}_0\cdot\mathbf{x}) \xleftrightarrow{\text{DDFS}} \tilde{F}[\mathbf{u}-\mathbf{u}_0]$
Convolution	$\int_{\mathbb{R}^D} f_c(\mathbf{s}) g_c(\mathbf{x}-\mathbf{s})\, d\mathbf{s} \xleftrightarrow{\text{CDFT}} F_c(\mathbf{u}) G_c(\mathbf{u})$	$\sum_{\mathbf{s}\in\Lambda} f[\mathbf{s}] g[\mathbf{x}-\mathbf{s}] \xleftrightarrow{\text{DDFT}} F(\mathbf{u}) G(\mathbf{u})$	$\int_{\mathcal{P}_\Gamma} \tilde{f}_c(\mathbf{s}) \tilde{g}_c(\mathbf{x}-\mathbf{s})\, d\mathbf{s} \xleftrightarrow{\text{CDFS}} \tilde{F}_c[\mathbf{u}] \tilde{G}_c[\mathbf{u}]$	$\sum_{\mathbf{s}\in\mathcal{B}} \tilde{f}[\mathbf{s}] \tilde{g}[\mathbf{x}-\mathbf{s}] \xleftrightarrow{\text{DDFS}} \tilde{F}[\mathbf{u}] \tilde{G}[\mathbf{u}]$
Multiplication	$f_c(\mathbf{x}) g_c(\mathbf{x}) \xleftrightarrow{\text{CDFT}} \int_{\mathbb{R}^D} F_c(\mathbf{w}) G_c(\mathbf{u}-\mathbf{w})\, d\mathbf{w}$	$f[\mathbf{x}] g[\mathbf{x}] \xleftrightarrow{\text{DDFT}} d(\Lambda) \int_{\mathcal{P}_{\Lambda^*}} F(\mathbf{w}) G(\mathbf{u}-\mathbf{w})\, d\mathbf{w}$	$\tilde{f}_c(\mathbf{x}) \tilde{g}_c(\mathbf{x}) \xleftrightarrow{\text{CDFS}} \frac{1}{d(\Gamma)} \sum_{\mathbf{w}\in\Gamma^*} \tilde{F}_c[\mathbf{w}] \tilde{G}_c[\mathbf{u}-\mathbf{w}]$	$\tilde{f}[\mathbf{x}] \tilde{g}[\mathbf{x}] \xleftrightarrow{\text{DDFS}} \frac{1}{K} \sum_{\mathbf{w}\in\mathcal{D}} \tilde{F}[\mathbf{w}] \tilde{G}[\mathbf{u}-\mathbf{w}]$
Automorphism of domain	$f_c(\mathbf{A}\mathbf{x}) \xleftrightarrow{\text{CDFT}} \frac{1}{\lvert\det\mathbf{A}\rvert} F_c(\mathbf{A}^{-T}\mathbf{u})$	$f[\mathbf{A}\mathbf{x}] \xleftrightarrow{\text{DDFT}} F(\mathbf{A}^{-T}\mathbf{u})$	$\tilde{f}_c(\mathbf{A}\mathbf{x}) \xleftrightarrow{\text{CDFS}} \tilde{F}_c[\mathbf{A}^{-T}\mathbf{u}]$	$\tilde{f}[\mathbf{A}\mathbf{x}] \xleftrightarrow{\text{DDFS}} \tilde{F}[\mathbf{A}^{-T}\mathbf{u}]$
Differentiation	$\nabla_\mathbf{x} f_c(\mathbf{x}) \xleftrightarrow{\text{CDFT}} j2\pi\mathbf{u} F_c(\mathbf{u})$	N/A	$\nabla_\mathbf{x} \tilde{f}_c(\mathbf{x}) \xleftrightarrow{\text{CDFS}} j2\pi\mathbf{u} \tilde{F}_c[\mathbf{u}]$	N/A
Differentiation in frequency	$\mathbf{x} f_c(\mathbf{x}) \xleftrightarrow{\text{CDFT}} \frac{j}{2\pi} \nabla_\mathbf{u} F_c(\mathbf{u})$	$\mathbf{x} f[\mathbf{x}] \xleftrightarrow{\text{DDFT}} \frac{j}{2\pi} \nabla_\mathbf{u} F(\mathbf{u})$	N/A	N/A
Complex conjugation	$f_c^*(\mathbf{x}) \xleftrightarrow{\text{CDFT}} F_c^*(-\mathbf{u})$	$f^*[\mathbf{x}] \xleftrightarrow{\text{DDFT}} F^*(-\mathbf{u})$	$\tilde{f}_c^*(\mathbf{x}) \xleftrightarrow{\text{CDFS}} \tilde{F}_c^*[-\mathbf{u}]$	$\tilde{f}^*[\mathbf{x}] \xleftrightarrow{\text{DDFS}} \tilde{F}^*[-\mathbf{u}]$
Parseval	$\int_{\mathbb{R}^D} f_c(\mathbf{x}) g_c^*(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^D} F_c(\mathbf{u}) G_c^*(\mathbf{u})\, d\mathbf{u}$	$\sum_{\mathbf{x}\in\Lambda} f[\mathbf{x}] g^*[\mathbf{x}] = d(\Lambda) \int_{\mathcal{P}_{\Lambda^*}} F(\mathbf{u}) G^*(\mathbf{u})\, d\mathbf{u}$	$\int_{\mathcal{P}_\Gamma} \tilde{f}_c(\mathbf{x}) \tilde{g}_c^*(\mathbf{x})\, d\mathbf{x} = \frac{1}{d(\Gamma)} \sum_{\mathbf{u}\in\Gamma^*} \tilde{F}_c[\mathbf{u}] \tilde{G}_c^*[\mathbf{u}]$	$\sum_{\mathbf{x}\in\mathcal{B}} \tilde{f}[\mathbf{x}] \tilde{g}^*[\mathbf{x}] = \frac{1}{K} \sum_{\mathbf{w}\in\mathcal{D}} \tilde{F}[\mathbf{u}] \tilde{G}^*[\mathbf{u}]$
Duality	$F_c(\mathbf{x}) \xleftrightarrow{\text{CDFT}} f_c(-\mathbf{u})$	$\tilde{F}_c[\mathbf{x}] \xleftrightarrow{\text{DDFT}} d(\Gamma) \tilde{f}_c(-\mathbf{u})$	$F(\mathbf{x}) \xleftrightarrow{\text{CDFS}} \frac{1}{d(\Lambda)} f[-\mathbf{u}]$	$\tilde{F}[\mathbf{C}\mathbf{x}] \xleftrightarrow{\text{DDFS}} K \tilde{f}[-\mathbf{C}^{-1}\mathbf{u}]$

Sampling and reconstruction

Sampling of an MD continuous-domain signal

$$f[\mathsf{x}] = f_c(\mathsf{x}), \qquad \mathsf{x} \in \Lambda$$



Sampling of an MD continuous-domain signal

If a continuous-domain signal f_c with Fourier transform F_c is sampled on the lattice Λ

$$f[\mathrm{x}] = f_c(\mathrm{x}), \qquad \mathrm{x} \in \Lambda$$

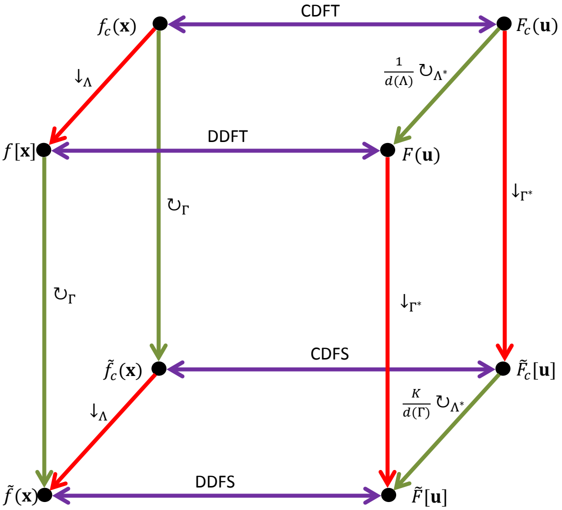
the Fourier transform of the sampled signal is given by

$$F(\mathrm{u}) = \frac{1}{d(\Lambda)} \sum_{\mathrm{r} \in \Lambda^*} F_c(\mathrm{u} - \mathrm{r})$$

If the support of F_c is limited to one unit cell of Λ^*, the replicated versions do not overlap and the continuous-domain signal can be reconstructed from the samples.
We call the sum on the right above the Λ^*-periodization of F_c, denoted $\circlearrowright_{\Lambda^*} F_c$.

$$f_c(\mathrm{x}) \xleftrightarrow{\text{CDFT}} F_c(\mathrm{u}) \quad \Longrightarrow \quad (\downarrow_\Lambda f_c)(\mathrm{x}) \xleftrightarrow{\text{DDFT}} \frac{1}{d(\Lambda)} (\circlearrowright_{\Lambda^*} F_c)(\mathrm{u})$$

Fourier-Poisson cube



Color signal processing

Color spaces

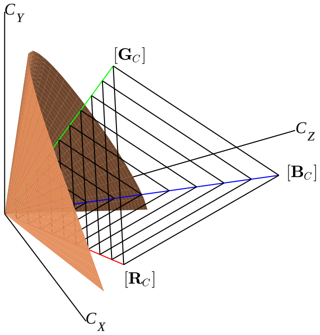
- The phenomenon of color perception is caused by visual light falling on the retina of the human eye.
- Colors perceived by a human viewer with normal trichromatic vision belong to the three-dimensional vector space $\mathcal{C}$.
- Three linearly independent colors form a basis $\mathcal{B} = \{[\mathrm{P}_1], [\mathrm{P}_2], [\mathrm{P}_3]\}$ for the color space.
- Any color can be represented with three real numbers C_1, C_2, C_3 called tristimulus values

$$[\mathrm{C}] = C_1[\mathrm{P}_1] + C_2[\mathrm{P}_2] + C_3[\mathrm{P}_3]$$

- A color space is associated with a specific viewer or device. Different bases just provide for different representations, or coordinates, for elements of this vector space.
- The Commission international de l'éclairage (CIE) has established various standard observers to represent typical human viewers.

Cone of realizable colors

- The set of physically realizable colors lie in a convex cone with a curved boundary formed by the spectral (monochromatic) colors and closed by the plane of purples.



- A subset of these colors with triangular cross-section can be synthesized with a positive linear combination of three display primaries.

Color coordinate systems

- A color coordinate system identifies colors within a given color space.
- The coefficients with respect to a given basis or set of primaries is the basic representation.
- There are many different bases for a given color space, such as RGB, XYZ, etc. These provide different coordinate systems.
- The change of representation between different sets of primaries is simply a change of basis operation from linear algebra.
- The chapter provides several bases and the transformation method between them.
- The color space is perceptually non-uniform. Equal Euclidean distance between color coordinates can correspond to widely different perceptual differences at different positions in the color space.
- Numerous perceptually-uniform color coordinate systems have been proposed and adopted, such as CIELAB.
- Many authors refer to different color coordinate systems as different color spaces, but I have avoided this practice.

Color signal processing

Color signal processing

- A color signal has the form $[\mathrm{C}](\mathrm{x})$, where typically $\mathrm{x} = (x, y)$ or $\mathrm{x} = (x, y, t)$, i.e., $D = 2$ or $D = 3$.
- In terms of a basis $\mathcal{B}$

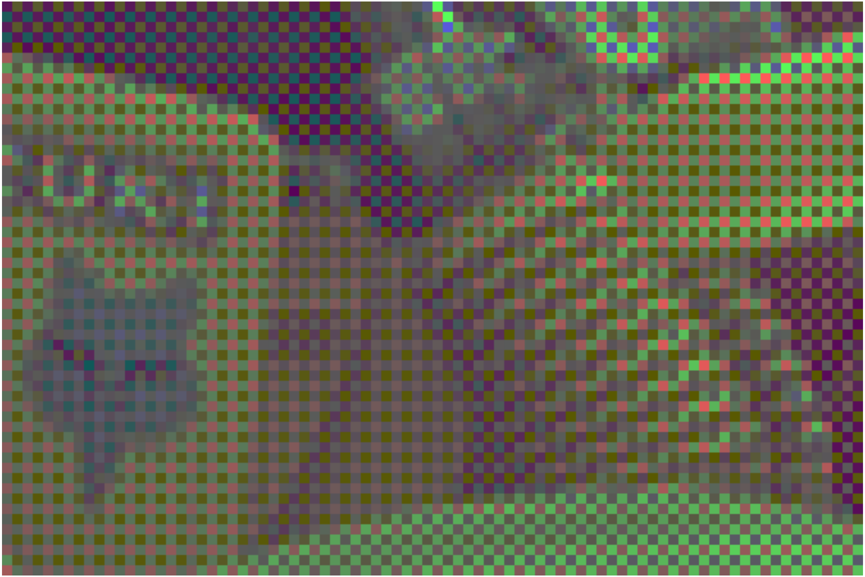
$$[\mathrm{C}](\mathrm{x}) = C_1(\mathrm{x})[\mathrm{P}_1] + C_2(\mathrm{x})[\mathrm{P}_2] + C_3(\mathrm{x})[\mathrm{P}_3]$$

- A color image is represented by three scalar images with respect to a specific basis $\mathcal{B}$.
- A linear system for color signals is formed of nine scalar systems $\mathcal{H}_{ki}$

$$\mathcal{H}([\mathrm{C}](\mathrm{x})) = \sum_{k=1}^{3} \left(\sum_{i=1}^{3} (H_{ki} * C_i)(\mathrm{x}) \right) [\mathrm{P}_k].$$

- Frequency response using Fourier transform can be defined for color signals.
- It is very common for different color components to be sampled on different sampling structures.

Bayer color filter array (CFA)



Random field models

Random field models

- Many image sources can be usefully modelled as wide-sense-stationary (WSS) random fields
- These WSS processes can be characterized by the autocorrelation function, or its Fourier transform, the power density spectrum. For discrete domain WSS random fields

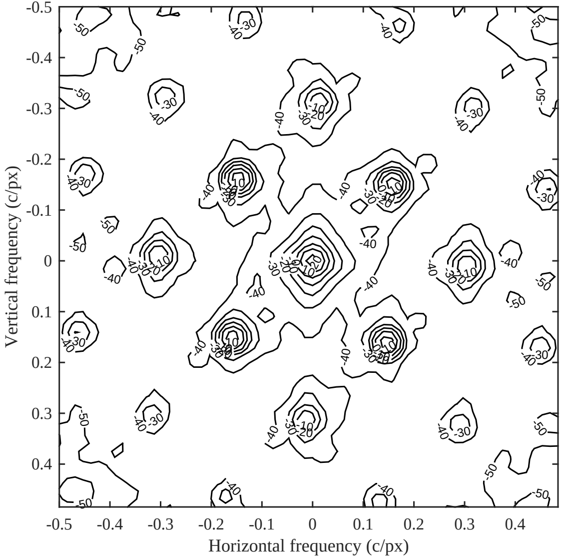
$$R_f[\mathrm{x}] = E[f(\mathrm{w}+\mathrm{x})f(\mathrm{w})]$$

$$S_f(\mathrm{u}) = \sum_{\mathrm{s}\in\Lambda} R_f[\mathrm{x}] \exp(-j2\pi\mathrm{u}\cdot\mathrm{x})$$

- The power density spectrum can be helpful in filter design.
- The chapter also covers WSS color random fields, which are characterized by a 3×3 spectral density matrix with respect to a given basis for the color space.
- The chapter presents the effect of sampling and filtering on the spectral density, and presents basic methods for spectral density estimation.

A halftone image

Power density spectrum of a halftone image

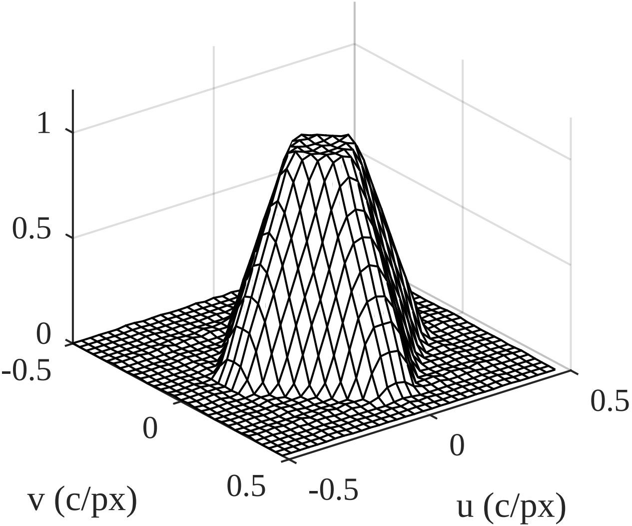


Multidimensional filter design

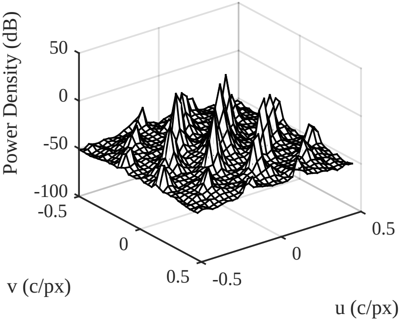
Multidimensional filter design

- Analysis of commonly used ad-hoc filter types: moving average and Gaussian filters.
- Design of band-pass and band-stop filters.
- Frequency-domain design using the window method.
- Frequency-domain design using least-pth optimization, including equality constraints.
- Software is provided on the book website.

Multidimensional digital filter design



Spectral density before and after filtering



Original

Filtered

Halftone image before and after filtering

Original

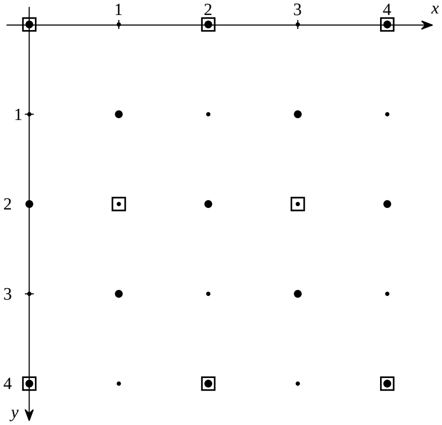
Filtered

Changing the sampling structure

Sampling structure conversion

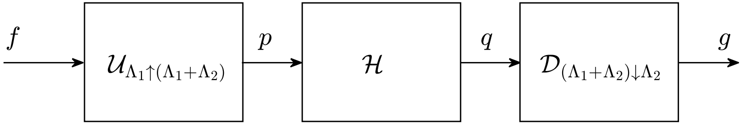
- The problem addressed is to convert a MD signal defined on one lattice to be defined on a different lattice.
- This can involve an increase or a decrease in the sampling density, or even no change.
- A general approach is presented based on upsampling and downsampling operations.
- This general approach involves upsampling to a common superlattice, followed by downsampling to the output lattice.
- Suitable filter design methodologies are presented.

Changing the sampling structure

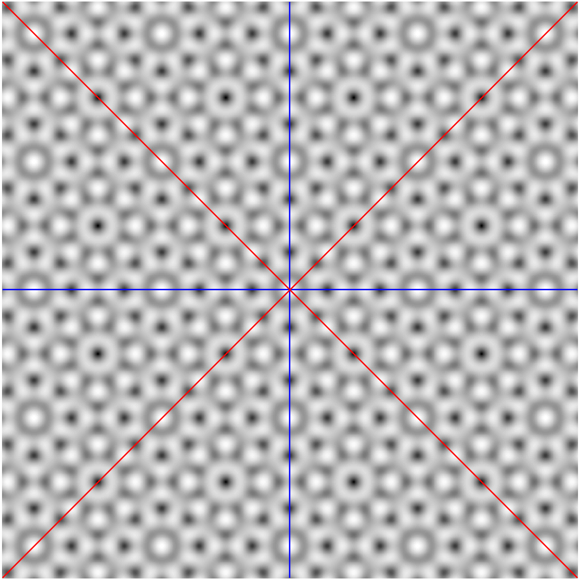


$\Lambda_1 : \bullet$ $\Lambda_2 : \square$ $\Lambda_1 + \Lambda_2 : \cdot$

Changing the sampling structure using a common superlattice



Symmetry invariant signals and systems



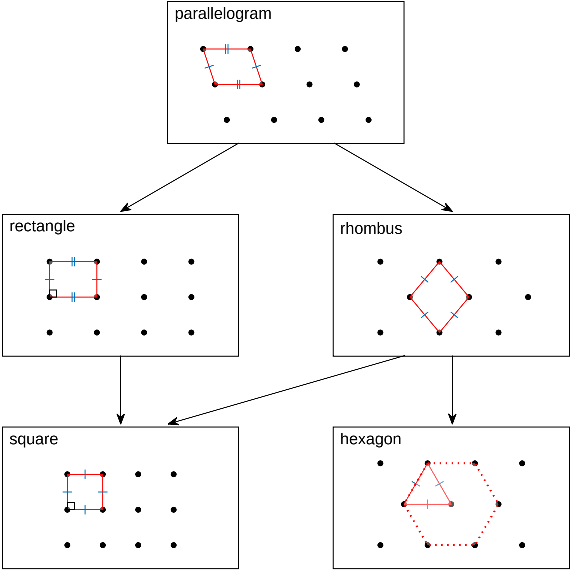
Symmetry invariant signals and systems

- A symmetry is a transformation that leaves a certain entity unchanged.
- For example, a shift leaves a lattice unchanged.
- Other symmetries of a lattice include rotations and reflections.
- Chapter 12 extends the notion of shift invariant systems to more general symmetry invariant systems.
- Finding the eigenfunctions and eigenvalues of such systems leads to a general class of Fourier-type transforms.
- An example is the multidimensional discrete cosine transform (DCT) defined on general lattices.

Symmetries of a lattice

- The basic concept in this chapter is the notion of a group of symmetries of a lattice, denoted $\mathcal{G}(\Lambda)$.
- The chapter depends to a large extent on the mathematical subject of group theory. Some basic concepts are summarized in Appendix B.
- These symmetry groups have been studied for a long time in fields such as crystallography.
- Different lattices allow different sets of symmetries.
- The following slide shows a classification of two-dimensional lattices according to how much symmetry they have.

Symmetries of a lattice



Symmetry-invariant systems

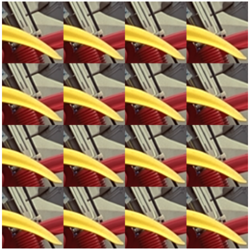
- The notion of systems invariant to a group of symmetries is developed for both aperiodic and periodic signal spaces.
- For aperiodic signal spaces, the symmetries leave the lattice unchanged.
- For periodic signal spaces, related symmetries must also leave the periodicity lattice unchanged.
- We are interested in the eigenfunctions of symmetry-invariant systems to perform Fourier-type analysis.
- These eigenfunctions are obtained by applying all the symmetries of the symmetry group to complex exponentials and adding.

$$\phi_{\mathcal{G}\mathrm{u}} = \frac{1}{L} \sum_{k=1}^{L} \mathcal{H}_{\mathrm{g}_k} \phi_{\mathrm{u}}$$

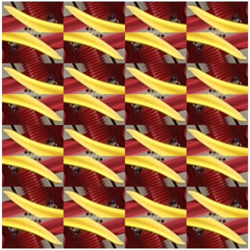
- Then, any symmetry invariant signal can expanded as a linear combination of these eigenfunctions, giving a generalized Fourier analysis.

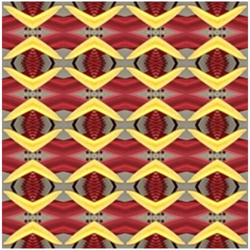
Periodic extension of image blocks

no symmetry

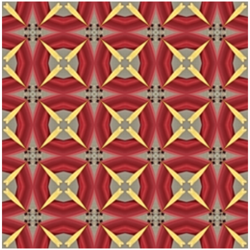


inversion symmetry





quadrantal symmetry



eight-fold symmetry

Summary

Summary

- In this book, I have presented the main elements of MD signal processing, mainly emphasizing image and video processing as examples.
- The theory of continuous-domain and discrete-domain signals and systems, both aperiodic and periodic, are presented in a common framework.
- Lattices are the basic tool to represent discrete-domain sampling structures and signal periodicities
- I emphasize the use of basis-independent, non-normalized representations.
- I use a vector-space representation for color and present color image processing in this context.
- I conclude the book with several special topics: random fields, filter design, sampling structure conversion and symmetry-invariant signal processing.
- A final chapter gathers all the relevant material on lattices needed for the book.

Other resources

Book web site:

http://www.site.uottawa.ca/~edubois/mdsp/

There are complementary resources available on the book web site, in ongoing development, including:

- Matlab software to reproduce all the figures in the book, as well as spectral estimation and filter design routines;
- solutions manual for all the problems in the book;
- errata;
- additional material such as the printable table of Fourier transform properties and links to presentations such as this one.

The End