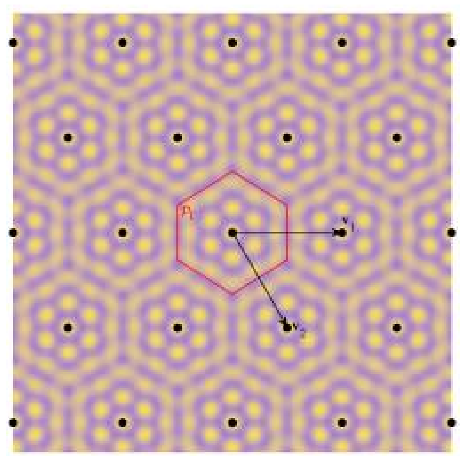
Multidimensional Signal and Color Image Processing Using Lattices

Problems and Solutions

©Eric Dubois 2019



Chapter 1

Introduction

This manual provides the author's solutions to the problems in Multidimensional Signal and Color Image Processing Using Lattices, Wiley, 2019. These solutions are made freely available on the book web site since they provide both complementary material and worked examples that can be of benefit to all users of the book.

Chapter 2

Continuous-Domain Signals and Systems

1. Consider a two-dimensional sinusoidal signal $f(x,y) = A\cos(2\pi(ux+vy)+\phi)$ where x and y are in ph and u and v are in c/ph. Form the one-dimensional signal $g(z)$ by tracing $f(x,y)$ along the line $y = cx$, where c is some real constant, as a function of distance along the line, $z = \sqrt{x^2+y^2}$.

(a) Show that $g(z)$ is a sinusoidal signal $g(z) = A\cos(2\pi wz + \phi)$ and determine the spatial frequency w in c/ph, as a function of u, v and c.

Solution: Evaluating f along the given line

$$\begin{aligned} f(x,cx) &= A\cos(2\pi(ux+vy)+\phi) \\ &= A\cos(2\pi(u+vc)x+\phi) \end{aligned}$$

Distance on the given line is $z = \sqrt{x^2+y^2} = \sqrt{x^2+c^2x^2} = \sqrt{1+c^2}x$. Thus

$$\begin{aligned} g(z) &= A\cos\left(2\pi\frac{(u+vc)z}{\sqrt{1+c^2}}+\phi\right) \\ &= A\cos(2\pi wz+\phi). \end{aligned}$$

Thus we identify the spatial frequency as

$$w = \frac{u+vc}{\sqrt{1+c^2}} \quad \text{c/ph.}$$

(b) Explain what happens when $c = 0$ and when $c \to \infty$.

Solution: If $c = 0$, then $w = u$. As $c \to \infty$, then $w = v$.

(c) Show that the spatial frequency w is greatest along the line $y = (v/u)x$, if $u \neq 0$. What is the value of this maximum spatial frequency? What happens if $u = 0$?

Solution: We want to maximize $(u+vc)/\sqrt{1+c^2}$ with respect to c. Setting the derivative with respect to c equal to 0,

$$\frac{v\sqrt{1+c^2} - (u+vc)(0.5)(1+c^2)^{-0.5}(2c)}{1+c^2} = 0.$$

Simplifying, $v(1+c^2) - (u+vc)c = 0$, or $v - uc = 0$. Thus, if $u \neq 0$, then $c = v/u$, and thus $w = \sqrt{u^2+v^2}$. If $u = 0$, the maximum frequency is v, along the line $x = 0$.

2. Show that for each of the following functions $\delta_\Delta(x, y)$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_\Delta(x, y)\, dx\, dy = 1$$

and

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_\Delta(x, y) f(x, y)\, dx\, dy = f(0, 0)$$

for any function $f(x, y)$ that is continuous at $(x, y) = (0, 0)$.

(a) $\delta_\Delta(x, y) = \frac{1}{\Delta^2} \operatorname{rect}(x/\Delta, y/\Delta)$.

Solution: For the first condition

$$\frac{1}{\Delta^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{rect}\left(\frac{x}{\Delta}, \frac{y}{\Delta}\right)\, dx\, dy = \frac{1}{\Delta^2} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} dx\, dy = 1.$$

For the second condition,

$$\frac{1}{\Delta^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{rect}\left(\frac{x}{\Delta}, \frac{y}{\Delta}\right) f(x, y)\, dx\, dy = \frac{1}{\Delta^2} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} f(x, y) dx\, dy.$$

Since $f(x, y)$ is continuous at (0,0), for any $\epsilon > 0$ there exists a sufficiently small Δ such that $|f(x, y) - f(0, 0)| < \epsilon$ for $-\Delta/2 < x, y < \Delta/2$. Then

$$\frac{1}{\Delta^2} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} (f(0,0) - \epsilon) dx\, dy < \frac{1}{\Delta^2} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} f(x, y) dx\, dy < \frac{1}{\Delta^2} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} (f(0,0) + \epsilon) dx\, dy,$$

and thus

$$f(0, 0) - \epsilon < \frac{1}{\Delta^2} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} f(x, y) dx\, dy < f(0, 0) + \epsilon.$$

Thus, by making Δ sufficiently small, we can make the integral of the second condition arbitrarily close to $f(0, 0)$, and so by definition, the limit as $\Delta \to 0$ is $f(0, 0)$.

(b) $\delta_\Delta(x,y) = \frac{1}{\Delta^2}\exp(-\pi(x^2+y^2)/\Delta^2)$.

Solution: For the first condition

$$\begin{aligned}\frac{1}{\Delta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-\pi(x^2+y^2)/\Delta^2)\,dx\,dy &= \frac{1}{\Delta^2}\int_{0}^{\infty}\int_{-\pi}^{\pi}\exp(-\pi r^2/\Delta^2)r\,dr\,d\theta\\ &= \frac{2\pi}{\Delta^2}\exp(-\pi r^2/\Delta^2)\left(-\frac{\Delta^2}{2\pi}\right)\Big|_0^\infty = 1.\end{aligned}$$

For the second condition to hold,we must impose additional constraints on $f(x,y)$ besides being continuous at the origin. For example, $f(x,y)$ cannot increase wildly, faster than the reciprocal of a Gaussian. For this problem, we will assume that $|f(x,y)|$ is bounded, with $\max|f(x,y)| = A < \infty$. We need to show that $|\int\int\delta_\Delta(x,y)f(x,y)\,dx\,dy - f(0,0)|$ can be made arbitrarily small by taking Δ sufficiently small.

Take any $\epsilon > 0$ choose $\delta > 0$ so that $|f(x,y) - f(0,0)| < \epsilon$ for $x^2+y^2 < \delta^2$ (we can do this since $f(x,y)$ is continuous at (0,0)). Then, choose M sufficiently large that

$$\frac{1}{2\pi}\int_{M}^{\infty}\int_{-\pi}^{\pi}\exp(-r^2/2)r\,dr\,d\theta < \epsilon,$$

i.e., $\exp(-M^2/2) < \epsilon$. Finally, take $\Delta = \sqrt{2\pi}\delta/M$.

We can bound the second condition as follows:

$$\begin{aligned}&\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_\Delta(x,y)f(x,y)\,dx\,dy - f(0,0)\right| = \left|\frac{1}{\Delta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{\pi(x^2+y^2)}{\Delta^2}\right)f(x,y)\,dx\,dy - f(0,0)\right|\\ &\quad= \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{s_1^2+s_2^2}{2}\right)f\left(\frac{\Delta s_1}{\sqrt{2\pi}},\frac{\Delta s_2}{\sqrt{2\pi}}\right)ds_1\,ds_2 - f(0,0)\right|\\ &\quad= \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{s_1^2+s_2^2}{2}\right)\left(f\left(\frac{\Delta s_1}{\sqrt{2\pi}},\frac{\Delta s_2}{\sqrt{2\pi}}\right) - f(0,0)\right)ds_1\,ds_2\right|\\ &\quad< \left|\frac{1}{2\pi}\iint_{s_1^2+s_2^2<M^2}\exp\left(-\frac{s_1^2+s_2^2}{2}\right)\left(f\left(\frac{\Delta s_1}{\sqrt{2\pi}},\frac{\Delta s_2}{\sqrt{2\pi}}\right) - f(0,0)\right)ds_1\,ds_2\right| +\\ &\qquad\left|\frac{1}{2\pi}\iint_{s_1^2+s_2^2\geq M^2}\exp\left(-\frac{s_1^2+s_2^2}{2}\right)\left(f\left(\frac{\Delta s_1}{\sqrt{2\pi}},\frac{\Delta s_2}{\sqrt{2\pi}}\right) - f(0,0)\right)ds_1\,ds_2\right|\end{aligned}$$

The first term of the last expression above is dominated by

$$\frac{1}{2\pi}\iint_{s_1^2+s_2^2<M^2} \exp\left(-\frac{s_1^2+s_2^2}{2}\right)\left|f\left(\frac{\Delta s_1}{\sqrt{2\pi}},\frac{\Delta s_2}{\sqrt{2\pi}}\right)-f(0,0)\right| ds_1\,ds_2$$
$$< \epsilon \iint_{s_1^2+s_2^2<M^2} \exp\left(-\frac{s_1^2+s_2^2}{2}\right) ds_1\,ds_2 < \epsilon$$

since in this region

$$\frac{\Delta^2 s_1^2}{2\pi}+\frac{\Delta^2 s_2^2}{2\pi}<\frac{\Delta^2 M^2}{2\pi}=\delta^2.$$

The second term of that expression is dominated by

$$\frac{1}{2\pi}\iint_{s_1^2+s_2^2\geq M^2} \exp\left(-\frac{s_1^2+s_2^2}{2}\right) ds_1\,ds_2 \max_{(s_1,s_2)}\left|f\left(\frac{\Delta s_1}{\sqrt{2\pi}},\frac{\Delta s_2}{\sqrt{2\pi}}\right)-f(0,0)\right| < 2A\epsilon.$$

Thus, the total error is dominated by $\epsilon(1+2A)$ which we can make arbitrarily small, completing the proof.

(c) $\delta_\Delta(x, y) = \frac{1}{\pi\Delta^2} \operatorname{circ}(x/\Delta, y/\Delta)$.

Solution: For the first condition

$$\frac{1}{\pi\Delta^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{circ}\left(\frac{x}{\Delta}, \frac{y}{\Delta}\right) \, dx \, dy = \frac{1}{\pi\Delta^2} \int_0^{\Delta} \int_{-\pi}^{\pi} r \, dr \, d\theta = 1.$$

For the second condition,

$$\frac{1}{\Delta^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{circ}\left(\frac{x}{\Delta}, \frac{y}{\Delta}\right) f(x, y) \, dx \, dy = \frac{1}{\pi\Delta^2} \int_0^{\Delta} \int_{-\pi}^{\pi} f_R(r, \theta) \, r \, dr \, d\theta,$$

where $f_R(r, \theta) = f(r\cos\theta, r\sin\theta)$. Since $f(x, y)$ is continuous at (0,0), for any $\epsilon > 0$ there exists a sufficiently small Δ such that $|f_R(r, \theta) - f(0, 0)| < \epsilon$ for $r < \Delta$. Then

$$\frac{1}{\pi\Delta^2} \int_0^{\Delta} \int_{-\pi}^{\pi} (f(0,0) - \epsilon) \, r \, dr \, d\theta < \frac{1}{\pi\Delta^2} \int_0^{\Delta} \int_{-\pi}^{\pi} f_R(r, \theta) \, r \, dr \, d\theta < \frac{1}{\pi\Delta^2} \int_0^{\Delta} \int_{-\pi}^{\pi} (f(0,0) + \epsilon) \, r \, dr \, d\theta$$

and thus

$$f(0, 0) - \epsilon < \frac{1}{\pi\Delta^2} \int_0^{\Delta} \int_{-\pi}^{\pi} f_R(r, \theta) \, r \, dr \, d\theta < f(0, 0) + \epsilon.$$

Thus, by making Δ sufficiently small, we can make the integral of the second condition arbitrarily close to $f(0, 0)$, and so by definition, the limit as $\Delta \to 0$ is $f(0, 0)$.

3. Show that

$$\delta(ax, by) = \frac{1}{|ab|}\delta(x, y)$$

where $a, b \neq 0$.

Solution: Note that since the Dirac delta is not a regular function, the meaning of this scaling operation is not self evident. In fact, in distribution theory, this is the definition of the scaling operation, which is meant to be consistent with scaling of regular functions (see Richards and Youn for example). Here, we demonstrate it using the approximation $\delta_\Delta(x, y) = \frac{1}{\Delta^2}\operatorname{rect}(x/\Delta, y/\Delta)$. Since the Dirac delta and rect functions are separable, we can show it in the one-dimensional case.

$$\begin{aligned}
\delta_\Delta(ax) &= \frac{1}{\Delta}\operatorname{rect}\left(\frac{ax}{\Delta}\right) \\
&= \frac{1}{\Delta}\operatorname{rect}\left(\frac{x}{\Delta/|a|}\right) \\
&= \frac{1}{|a|}\frac{1}{\Delta/|a|}\operatorname{rect}\left(\frac{x}{\Delta/|a|}\right) \\
&= \frac{1}{|a|}\delta_{\Delta/|a|}(x),
\end{aligned}$$

where we use the fact that rect is symmetric about the origin in line 2. Thus, we see that as $\Delta \to 0$, then $\delta_\Delta(ax) \to \frac{1}{|a|}\delta(x)$, and so we conclude that $\delta(ax) = \frac{1}{|a|}\delta(x)$. Similarly, $\delta(by) = \frac{1}{|b|}\delta(y)$, and the result follows by separability.

This can be extended to a general linear transformation of the domain.

4. Prove that the following systems are linear systems.

(a) The shift system $\mathcal{T}_{\mathbf{d}}$ for any shift $\mathbf{d} \in \mathbb{R}^D$.

Solution: Let $\mathcal{S}$ be a signal space for which $\mathcal{T}_{\mathbf{d}}$ is well defined, i.e., if $f \in \mathcal{S}$ then $\mathcal{T}_{\mathbf{d}} f \in \mathcal{S}$. Let $f = \alpha_1 f_1 + \alpha_2 f_2$ for any $f_1, f_2 \in \mathcal{S}$ and any scalars α_1, α_2. This means that $f(\mathbf{x}) = \alpha_1 f_1(\mathbf{x}) + \alpha_2 f_2(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^D$. If $g = \mathcal{T}_{\mathbf{d}} f = \mathcal{T}_{\mathbf{d}}(\alpha_1 f_1 + \alpha_2 f_2)$, then $g(\mathbf{x}) = f(\mathbf{x} - \mathbf{d}) = \alpha_1 f_1(\mathbf{x} - \mathbf{d}) + \alpha_2 f_2(\mathbf{x} - \mathbf{d})$, and thus $g = \alpha_1 \mathcal{T}_{\mathbf{d}} f_1 + \alpha_2 \mathcal{T}_{\mathbf{d}} f_2$ and so $\mathcal{T}_{\mathbf{d}}$ is linear.

(b) The system induced by a nonsingular transformation of the domain, $\mathcal{M}_{\mathbf{A}} : g = \mathcal{M}_{\mathbf{A}} f : g(\mathbf{x}) = f(\mathbf{A}\mathbf{x})$, where $\mathbf{A}$ is any nonsingular $D \times D$ matrix.

Solution: Let $\mathcal{S}$ be a signal space for which $\mathcal{M}_{\mathbf{A}}$ is well defined, i.e., if $f \in \mathcal{S}$ then $\mathcal{M}_{\mathbf{A}} f \in \mathcal{S}$. As in (a), let $f = \alpha_1 f_1 + \alpha_2 f_2$ for any $f_1, f_2 \in \mathcal{S}$ and any scalars α_1, α_2. This means that $f(\mathbf{x}) = \alpha_1 f_1(\mathbf{x}) + \alpha_2 f_2(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^D$. If $g = \mathcal{M}_{\mathbf{A}} f = \mathcal{M}_{\mathbf{A}}(\alpha_1 f_1 + \alpha_2 f_2)$, then $g(\mathbf{x}) = f(\mathbf{A}\mathbf{x}) = \alpha_1 f_1(\mathbf{A}\mathbf{x}) + \alpha_2 f_2(\mathbf{A}\mathbf{x})$, and thus $g = \alpha_1 \mathcal{M}_{\mathbf{A}} f_1 + \alpha_2 \mathcal{M}_{\mathbf{A}} f_2$ and so $\mathcal{M}_{\mathbf{A}}$ is linear.

(c) The cascade of any two linear systems $\mathcal{H}_1$ and $\mathcal{H}_2$. Thus, the system induced by an affine transformation of the domain is a linear system.

Solution: Suppose that $\mathcal{H}_1 : \mathcal{S}_1 \to \mathcal{S}_2$ and $\mathcal{H}_2 : \mathcal{S}_2 \to \mathcal{S}_3$ are linear systems, and that $\mathcal{H} = \mathcal{H}_2 \mathcal{H}_1 : \mathcal{S}_1 \to \mathcal{S}_3$ is the cascade. This means $\mathcal{H} f = \mathcal{H}_2(\mathcal{H}_1 f)$. Let $f = \alpha_1 f_1 + \alpha_2 f_2$ for any $f_1, f_2 \in \mathcal{S}$ and any scalars α_1, α_2. Then

$$\begin{aligned}
\mathcal{H}(\alpha_1 f_1 + \alpha_2 f_2) &= \mathcal{H}_2(\mathcal{H}_1(\alpha_1 f_1 + \alpha_2 f_2)) \\
&= \mathcal{H}_2(\alpha_1 \mathcal{H}_1 f_1 + \alpha_2 \mathcal{H}_1 f_2) \\
&= \alpha_1 \mathcal{H}_2(\mathcal{H}_1 f_1) + \alpha_2 \mathcal{H}_2(\mathcal{H}_1 f_2) \\
&= \alpha_1 \mathcal{H} f_1 + \alpha_2 \mathcal{H} f_2
\end{aligned}$$

and so $\mathcal{H}$ is linear. From (a) and (b), it follows that $\mathcal{Q}_{\mathbf{A},\mathbf{d}} = \mathcal{T}_{\mathbf{d}} \mathcal{M}_{\mathbf{A}} = \mathcal{M}_{\mathbf{A}} \mathcal{T}_{\mathbf{Ad}}$ is linear.

(d) The parallel combination of two linear systems with the same domain and range, $\mathcal{H}_1 + \mathcal{H}_2$.

Solution: Let $\mathcal{H}_1 : \mathcal{S}_1 \to \mathcal{S}_2$ and $\mathcal{H}_2 : \mathcal{S}_1 \to \mathcal{S}_2$ be two linear systems with the same domain and range. The parallel combination of these systems is $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$, defined by

$\mathcal{H}f = \mathcal{H}_1 f + \mathcal{H}_2 f$. If $f = \alpha_1 f_1 + \alpha_2 f_2$ for any $f_1, f_2 \in \mathcal{S}_1$, then

$$\begin{aligned}\mathcal{H}f &= \mathcal{H}_1(\alpha_1 f_1 + \alpha_2 f_2) + \mathcal{H}_2(\alpha_1 f_1 + \alpha_2 f_2)\\ &= \alpha_1 \mathcal{H}_1 f_1 + \alpha_2 \mathcal{H}_1 f_2 + \alpha_1 \mathcal{H}_2 f_1 + \alpha_2 \mathcal{H}_2 f_2\\ &= \alpha_1(\mathcal{H}_1 f_1 + \mathcal{H}_2 f_1) + \alpha_2(\mathcal{H}_1 f_2 + \mathcal{H}_2 f_2)\\ &= \alpha_1 \mathcal{H} f_1 + \alpha_2 \mathcal{H} f_2.\end{aligned}$$

Thus, the system is linear.

(e) The partial derivative systems ∂_x and ∂_y defined in Section 2.5.2.

Solution: For this problem, we simply invoke the linearity of the derivative operation from its definition in calculus. Take for example ∂_x. By definition

$$(\partial_x f)(x, y) = \lim_{h \to 0} \frac{f(x, y) - f(x + h, y)}{h}.$$

For any given h, this can be thought of as the parallel combination of $\frac{1}{h}\mathcal{T}_{0,0}$ and $-\frac{1}{h}\mathcal{T}_{-h,0}$. Then, from parts (a) and (d) of this problem, the system is linear for any h and so will be linear in the limit.

5. Prove that the following systems are linear shift-invariant systems.

(a) The shift system $\mathcal{T}_{\mathbf{d}}$ for any shift $\mathbf{d} \in \mathbb{R}^D$.

Solution: From Problem 4(a), the shift system is linear. Let $\mathbf{s}$ be any element of $\mathbb{R}^D$ and $f \in \mathcal{S}$. Then $(\mathcal{T}_{\mathbf{s}}(\mathcal{T}_{\mathbf{d}}f))(\mathbf{x}) = (\mathcal{T}_{\mathbf{d}}f)(\mathbf{x}-\mathbf{s}) = f(\mathbf{x}-\mathbf{s}-\mathbf{d}) = (\mathcal{T}_{\mathbf{s}}f)(\mathbf{x}-\mathbf{d}) = (\mathcal{T}_{\mathbf{d}}(\mathcal{T}_{\mathbf{s}}f)(\mathbf{x})$. Thus $\mathcal{T}_{\mathbf{s}}\mathcal{T}_{\mathbf{d}} = \mathcal{T}_{\mathbf{d}}\mathcal{T}_{\mathbf{s}}$ for any $\mathbf{s} \in \mathbb{R}^D$ and so $\mathcal{T}_{\mathbf{d}}$ is an LSI system.

(b) The cascade of two LSI systems $\mathcal{H}_1$ and $\mathcal{H}_2$.

Solution: Let $\mathbf{d}$ be any element of $\mathbb{R}^D$. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are LSI systems, $\mathcal{T}_{\mathbf{d}}\mathcal{H}_1 = \mathcal{H}_1\mathcal{T}_{\mathbf{d}}$ and $\mathcal{T}_{\mathbf{d}}\mathcal{H}_2 = \mathcal{H}_2\mathcal{T}_{\mathbf{d}}$. It follows that

$$\mathcal{T}_{\mathbf{d}}\mathcal{H}_2\mathcal{H}_1 = \mathcal{H}_2\mathcal{T}_{\mathbf{d}}\mathcal{H}_1 = \mathcal{H}_2\mathcal{H}_1\mathcal{T}_{\mathbf{d}}$$

and so $\mathcal{H}_2\mathcal{H}_1$ is an LSI system.

(c) The parallel combination of two LSI systems with the same domain and range, $\mathcal{H}_1 + \mathcal{H}_2$.

Solution: Let $\mathbf{d}$ be any element of $\mathbb{R}^D$. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are LSI systems, $\mathcal{T}_{\mathbf{d}}\mathcal{H}_1 = \mathcal{H}_1\mathcal{T}_{\mathbf{d}}$ and $\mathcal{T}_{\mathbf{d}}\mathcal{H}_2 = \mathcal{H}_2\mathcal{T}_{\mathbf{d}}$. It follows that

$$\mathcal{T}_{\mathbf{d}}(\mathcal{H}_1 + \mathcal{H}_2) = \mathcal{T}_{\mathbf{d}}\mathcal{H}_1 + \mathcal{T}_{\mathbf{d}}\mathcal{H}_2 = \mathcal{H}_1\mathcal{T}_{\mathbf{d}} + \mathcal{H}_2\mathcal{T}_{\mathbf{d}} = (\mathcal{H}_1 + \mathcal{H}_2)\mathcal{T}_{\mathbf{d}}$$

and so $\mathcal{H}_1 + \mathcal{H}_2$ is an LSI system.

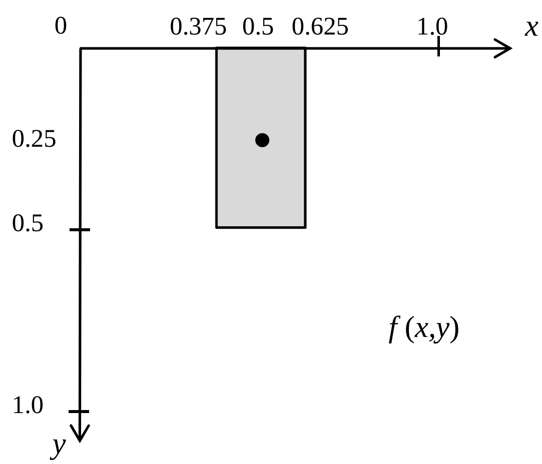
(d) The partial derivative systems ∂_x and ∂_y defined in Section 2.5.2.

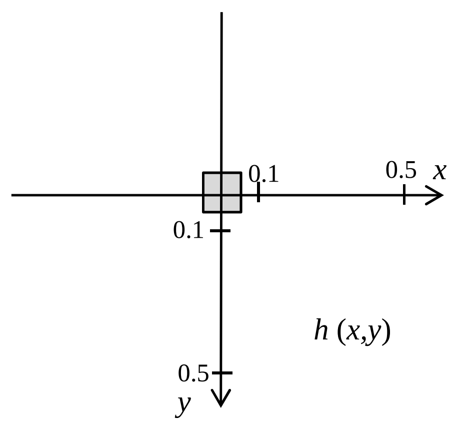
Solution: As in Problem 4(e), $\partial_x = \lim_{h\to 0}(\frac{1}{h}\mathcal{T}_{0,0} - \frac{1}{h}\mathcal{T}_{-h,0})$. This system is LSI for any fixed h using the results of parts (a) and (c) of this problem, and so will be LSI in the limit.

6. Let $f(x,y) = 0.5\,\mathrm{rect}(4(x-0.5), 2(y-0.25))$ and $h(x,y) = \mathrm{rect}(10x, 10y)$, where x and y are in ph.

(a) Sketch the region of support of $f(x,y)$ and $h(x,y)$ in the XY-plane (i.e., the area where these two signals are nonzero).

Solution: $f(x,y)$ has a rectangular region of support centered at (0,5, 0.25), of width $\frac{1}{4}$ and of height $\frac{1}{2}$. $h(x,y)$ has a square region of support cenetered at the origin, of width and height $\frac{1}{10}$. These regions are shown in the figures below.





(b) Compute the two-dimensional convolution $f(x,y) * h(x,y)$ from the definition using integration in the spatial domain.

Solution: From the definition of convolution (Equation (2.46)) and the definition of $f(x,y)$, the convolution $g = f * h$ is given by

$$\begin{aligned}
g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(w,z)h(x-w,y-z)\,dw\,dz \\
&= 0.5\int_{0}^{0.5}\int_{0.375}^{0.625} h(x-w,y-z)\,dw\,dz \\
&= 0.5\int_{0}^{0.5}\int_{0.375}^{0.625} h(w-x,z-y)\,dw\,dz,
\end{aligned}$$

where the last line follows from the fact that h has quadrantal symmetry. For any (x, y), the output $g(x, y)$ is equal to the area of overlap between the 0.1×0.1 square centered at (x, y) and the nonzero portion of f, multiplied by 0.5 (the value of f in the nonzero region). There are ten possible situations: no overlap, full overlap, overlap on one of the four sides, overlap on one of the four corners. The following figure gives examples of these ten cases, that applies in regions labelled A to J. We show $h(w-x, z-y)$ for sample values of (x, y) in each of the ten regions, and the overlap is the shaded portion.

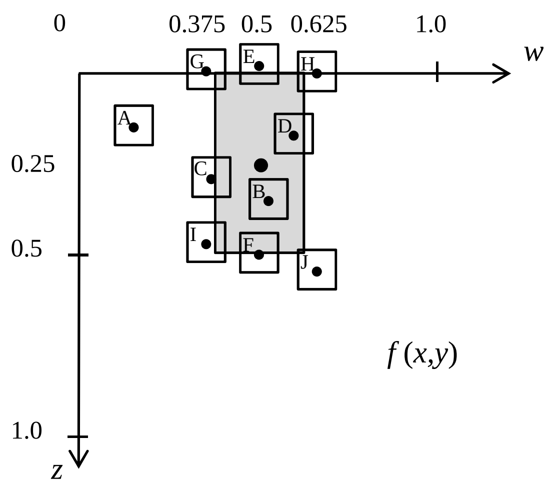
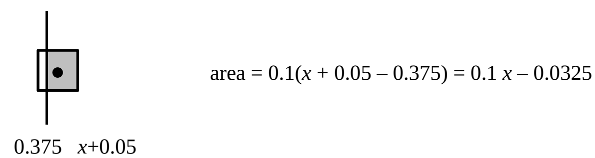
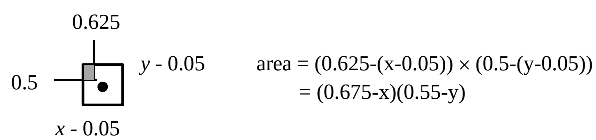


Figure P2.6b Illustration of the overlap between $h(w-x, z-y)$ and $f(w, z)$ for (x, y) in each of the ten regions A-J.

By inspecting Figure P2.6b, we can easily identify the ten regions for which the expression for $g(x, y)$ will have a different form. These regions are enumerated in the following table. In region A (no overlap), clearly $g(x, y) = 0$. In region B (full overlap), $g(x, y) = \frac{1}{2}(0.1)^2 = 0.005$. In the side regions (C, D, E, F), the overlap area is 0.1 times the overlap length. For example, in region C, this would be



For the corner regions (G, H, I, J) the area is the product of the two overlap lengths, as in the following illustration for region J.



A no overlap	$x < 0.325$ OR $x > 0.675$ OR $y < -0.05$ OR $y > 0.55$
B full overlap	$0.425 < x < 0.575$ AND $0.05 < y < 0.45$
C left side	$0.325 \leq x \leq 0.425$ AND $0.05 < y < 0.45$
D right side	$0.575 \leq x \leq 0.675$ AND $0.05 \leq y \leq 0.45$
E top side	$0.425 \leq x \leq 0.575$ AND $-0.05 \leq y \leq 0.05$
F bottom side	$0.425 < x < 0.575$ AND $0.45 \leq y \leq 0.55$
G top left	$0.325 \leq x \leq 0.425$ AND $-0.05 \leq y \leq 0.05$
H top right	$0.575 \leq x \leq 0.675$ AND $0.05 \leq y \leq 0.05$
I bottom left	$0.325 \leq x \leq 0.425$ AND $0.45 \leq y \leq 0.55$
J bottom right	$0.575 \leq x \leq 0.675$ AND $0.45 \leq y \leq 0.55$

Carrying out this approach for all the regions, and multiplying by 0.5, (the value of f), the overall output is

$$g(x,y) = \begin{cases} 0 & (x,y) \in A \\ 0.005 & (x,y) \in B \\ 0.05 - 0.01625 & (x,y) \in C \\ 0.03375 - 0.05x & (x,y) \in D \\ 0.05y + 0.0025 & (x,y) \in E \\ 0.0275 - 0.05y & (x,y) \in F \\ 0.5xy + 0.025x - 0.1625y - 0.008125 & (x,y) \in G \\ -0.5xy - 0.025x + 0.3375y + 0.016875 & (x,y) \in H \\ -0.5xy + 0.275x + 0.1625y - 0.089375 & (x,y) \in I \\ 0.5xy - 0.275x - 0.3375y + 0.185625 & (x,y) \in J \end{cases}$$

Note that since both f and h are separable, a simpler solution could be obtained using this separability.

(c) Suppose that $f(x,y)$ is the input to a two-dimensional system, and the output of this system is computed as in (b). What can we say about this system?

Solution: Since the output is given by the convolution of the input and h, we can conclude that the system is a linear, shift-invariant system and that its impulse response is h.

(d) Determine the continuous-space Fourier transforms $F(u,v)$, $H(u,v)$ and $G(u,v)$ of the above three signals. Make liberal use of Fourier transform properties. What are the units of u and v?

Solution: $H(u,v)$ is the Fourier transform of $h(x,y)$ which is a simple scaling of the standard rect function, as already done in Example 2.3, with $a = b = 10$ and $c = 1$. Thus, from that solution

$$H(u,v) = \frac{\sin(\pi u/10)\sin(\pi v/10)}{\pi^2 uv}.$$

Let $f_1(x,y) = 0.5\,\mathrm{rect}(4x,2y)$. Then $f(x,y) = f_1(x-0.5, y-0.25)$. Again, using the result of Example 2.3 with $a=4$, $b=2$, $c=0.5$, we obtain

$$F_1(u,v) = 0.5\frac{\sin(\pi u/4)\sin(\pi v/2)}{\pi^2 uv}.$$

Then, applying Property 2.2 with $\mathbf{x}_0 = (0.5, 0.25)$

$$\begin{aligned} F(u,v) &= F_1(u,v)\exp(-j2\pi(0.5u+0.25v)) \\ &= 0.5\frac{\sin(\pi u/4)\sin(\pi v/2)}{\pi^2 uv}\exp(-j2\pi(0.5u+0.25v)). \end{aligned}$$

Finally, since $g = f * h$, $G(u,v) = F(u,v)H(u,v)$ with $F(u,v)$ and $H(u,v)$ given as above.

Since x and y are in units of ph, u and v are in units of c/ph.

(e) Continuing with question (c), what is the interpretation of $H(u,v)$?

Solution: Since $H(u,v)$ is the Fourier transform of the impulse response $h(x,y)$, it is the frequency response of the linear shift-invariant system.

7. Determine the response of an LSI system $\mathcal{H}$ with impulse response $h(\mathbf{x})$ to a real sinusoidal signal $f(\mathbf{x}) = A + B\cos(2\pi\mathbf{u}\cdot\mathbf{x} + \phi)$ where $A > 0$ and $0 < B < A$.

   Solution: Note that this signal is positive everywhere and so it can be displayed as an image.

   We express the given signal as a sum of complex exponentials, and use the result from Equation (2.48) along with linearity of the system.

$$
\begin{aligned}
f(\mathbf{x}) &= A\exp(j2\pi\mathbf{0}\cdot\mathbf{x}) + \frac{B}{2}\exp(j(2\pi\mathbf{u}\cdot\mathbf{x}+\phi)) + \frac{B}{2}\exp(-j(2\pi\mathbf{u}\cdot\mathbf{x}+\phi)) \\
&= A\exp(j2\pi\mathbf{0}\cdot\mathbf{x}) + \frac{B}{2}\exp(j\phi)\exp(j2\pi\mathbf{u}\cdot\mathbf{x}) + \frac{B}{2}\exp(-j\phi)\exp(-j2\pi\mathbf{u}\cdot\mathbf{x}).
\end{aligned}
$$

   Now applying linearity and the result from Section 2.5.5, we obtain

$$
(\mathcal{H}f)(\mathbf{x}) = AH(0) + \frac{B}{2}\exp(j\phi)H(\mathbf{u})\exp(j2\pi\mathbf{u}\cdot\mathbf{x}) + \frac{B}{2}\exp(-j\phi)H(-\mathbf{u})\exp(-j2\pi\mathbf{u}\cdot\mathbf{x}),
$$

   where $H(\mathbf{u})$ is the continuous-domain Fourier transform of $h(\mathbf{x})$.

   The solution can be simplified if the impulse response $h(\mathbf{x})$ is real. In this case, from Property 2.9, $H(\mathbf{u}) = H^*(-\mathbf{u})$. If we express $H(\mathbf{u}) = |H(\mathbf{u})|\exp(j\angle H(\mathbf{u}))$, then $H(-\mathbf{u}) = H^*(\mathbf{u}) = |H(\mathbf{u})|\exp(-j\angle H(\mathbf{u}))$. Then,

$$
(\mathcal{H}f)(\mathbf{x}) = AH(0) + B|H(\mathbf{u})|\cos(2\pi\mathbf{u}\cdot\mathbf{x} + \phi + \angle H(\mathbf{u})).
$$

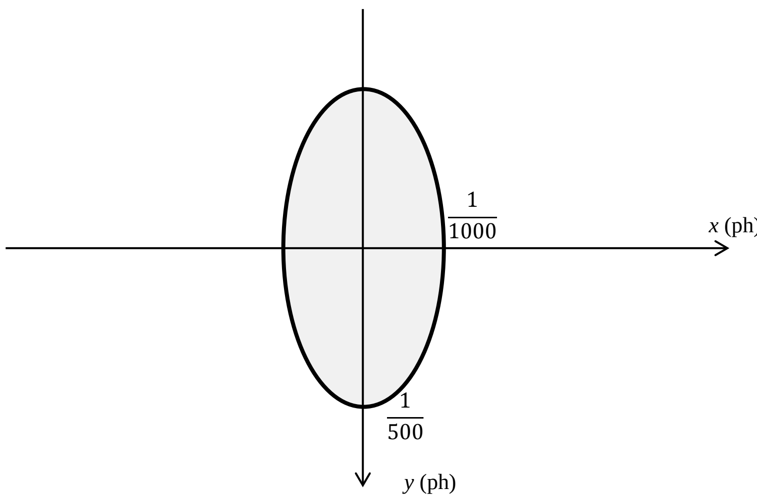
8. A two-dimensional continuous-space linear shift-invariant system has impulse response

$$h(x,y) = \begin{cases} \frac{1}{2\pi R_1 R_2}, & \left(\frac{x}{R_1}\right)^2 + \left(\frac{y}{R_2}\right)^2 \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $R_1 = 1/1000$ ph and $R_2 = 1/500$ ph.

(a) Sketch the region of support of the impulse response in the XY-plane, following the conventions used in the course for the labelling of axes. Express $h(x, y)$ in terms of the circ function.

Solution: The region of support is an elliptical region.



Written in terms of the circ function,

$$h(x,y) = \frac{1}{2\pi R_1 R_2} \operatorname{circ}\left(\frac{x}{R_1}, \frac{y}{R_2}\right) = \frac{250,000}{\pi} \operatorname{circ}(1000x, 500y).$$

(b) Find the frequency response $H(u, v)$ of this system, where u and v are in c/ph.

Solution: The frequency response is given by the Fourier transform of the impulse response. We use the Fourier transform of the circ function given in Table 2.2, along with

Fourier transform properties 2.1 and 2.6 from Table 2.1. For Property 2.6, we use the transformation matrix with detminant and inverse transform

$$\mathbf{A} = \begin{bmatrix} 1000 & 0 \\ 0 & 500 \end{bmatrix}, \quad \det \mathbf{A} = 500,000, \quad \mathbf{A}^{-T} = \begin{bmatrix} \frac{1}{1000} & 0 \\ 0 & \frac{1}{500} \end{bmatrix}.$$

Applying the properties and simplifying, we obtain

$$\begin{aligned} H(u,v) &= \frac{250,000}{\pi} \frac{1}{500,000} \frac{1}{\sqrt{(u/1000)^2 + (v/500)^2}} J_1 \left(2\pi \sqrt{\left(\frac{u}{1000}\right)^2 + \left(\frac{v}{500}\right)^2} \right) \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{(u/1000)^2 + (v/500)^2}} J_1 \left(2\pi \sqrt{\left(\frac{u}{1000}\right)^2 + \left(\frac{v}{500}\right)^2} \right). \end{aligned}$$

(c) The image $f(x,y) = \text{rect}(5(x - .5), 2(y - .5))$ is filtered with this system to produce the output $g(x,y) = f(x,y) * h(x,y)$. Determine the Fourier transform of the output, $G(u,v)$.

Solution: From Property 2.4, $G(u,v) = F(u,v)H(u,v)$ where $H(u,v)$ was found in part (b). To find $F(u,v)$, we use the Fourier transform of the rect function from Table 2.2 along with Fourier transform Property 2.2 with $\mathbf{x}_0 = [0.5, 0.5]^T$ and Property 2.6 with

$$\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, \quad \det \mathbf{A} = 10, \quad \mathbf{A}^{-T} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

It follows that

$$\begin{aligned} F(u,v) &= \exp(-j2\pi(0.5u + 0.5v)) \cdot \frac{1}{10} \frac{\sin\left(\frac{\pi u}{5}\right) \sin\left(\frac{\pi v}{2}\right)}{\frac{\pi u}{5} \cdot \frac{\pi v}{2}} \\ &= \exp(-j\pi(u+v)) \frac{\sin\left(\frac{\pi u}{5}\right) \sin\left(\frac{\pi v}{2}\right)}{\pi^2 uv}. \end{aligned}$$

9. Compute the two-dimensional continuous-space Fourier transform of the following signals:

(a) The separable signal $f(x,y) = h_X^{(1)}(x)h_Y^{(1)}(y)$ where

$$h_T^{(1)}(t) = \begin{cases} 1 - \frac{|t|}{T} & |t| \leq T, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: Using Property 2.11 in Table 2.1, $H(u,v) = H_X^{(1)}(u)H_Y^{(1)}(v)$. Thus, we need to find the one-dimensional Fourier transform of $h_T^{(1)}(t)$. This is straightforward if we note that

$$h_T^{(1)}(t) = \frac{1}{T}\,\mathrm{rect}(t/T) * \mathrm{rect}(t/T).$$

The Fourier transform of $\mathrm{rect}(t/T)$ is easily seen to be $\sin(\pi f T)/\pi f$ (see Examples 2.2 and 2.3). Thus, using Property 2.4,

$$H_T^{(1)}(f) = \frac{1}{T}\left(\frac{\sin(\pi f T)}{\pi f}\right)^2 = \frac{\sin^2(\pi f T)}{\pi^2 f^2 T}.$$

It follows directly that

$$H(u,v) = \frac{\sin^2(\pi u X)\sin^2(\pi v Y)}{\pi^4 u^2 v^2 XY}.$$

9. Compute the two-dimensional continuous-space Fourier transform of the following signals:

(b) A Gaussian function

$$f(x, y) = \frac{1}{2\pi r_0^2} \exp -(x^2 + y^2)/2r_0^2.$$

(i) Obtain the result from the entry in Table 2.2 (with $r_0 = 1$). (ii) Prove the result in Table 2.2. Extra question: For what value of r_0 is $F(u, v) = f(u, v)$?

Solution: (i) To use the result in Table 2.2, we use properties 2.1 and 2.6. As in Table 2.2, let

$$f_0(x, y) = \exp(-(x^2 + y^2)/2), \qquad F_0(u, v) = 2\pi \exp(-2\pi^2(u^2 + v^2)).$$

Then

$$f(x, y) = \frac{1}{2\pi r_0^2} f_0(x/r_0, y/r_0).$$

We use Property 2.6 with $\mathbf{A} = \text{diag}(1/r_0, 1/r_0)$, $\det \mathbf{A} = 1/r_0^2$, and $\mathbf{A}^{-T} = \text{diag}(r_0, r_0)$. This gives

$$F(u, v) = \frac{1}{2\pi r_0^2} r_0^2 F_0(r_0 u, r_0 v) = \exp(-2\pi^2 r_0^2(u^2 + v^2)).$$

(ii) Since $f(x, y)$ is separable, we know $F(u, v)$ will also be separable. Thus, we can find the one-dimensional Fourier transform of $f_0(x) = \exp(-x^2)$ and then apply Fourier transform properties to get the desired result.

$$\begin{aligned}
F_0(u) &= \int_{-\infty}^{\infty} \exp(-x^2) \exp(-j2\pi ux)\, dx \\
&= \int_{-\infty}^{\infty} \exp(-x^2 - j2\pi ux)\, dx \\
&= \int_{-\infty}^{\infty} \exp(-(x + j\pi u)^2) \exp(-\pi^2 u^2)\, dx \quad \text{completing the square} \\
&= \exp(-\pi^2 u^2) \int_{-\infty}^{\infty} \exp(-(x + j\pi u)^2)\, dx
\end{aligned}$$

The integral on the right is a contour integral of the complex function $\exp(-z^2)$ on the horizontal contour $z = x + j\pi u$. Since $\exp(-z^2)$ is an analytic function, its integral along any closed contour is zero (Cauchy's integral theorem). Take a contour from $-R$ to R to $R+j\pi u$ to $-R+j\pi u$ to $-R$. For R sufficiently large, integrals along the vertical segments at $x = \pm R$ can be made arbitrarily small and thus the integral on the contour $z = x + j\pi u$ is equal to the integral on the contour $z = x$. Thus,

$$I = \int_{-\infty}^{\infty} \exp(-(x + j\pi u)^2)\, dx = \int_{-\infty}^{\infty} \exp(-x^2)\, dx.$$

This integral cannot be directly evaluated, so we use a trick. (This result is well-known to anyone who uses the Gaussian distribution.

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} \exp(-x^2)\, dx \int_{-\infty}^{\infty} \exp(-y^2)\, dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-(x^2 + y^2))\, dxdy \\
&= \int_{0}^{\infty} \int_{0}^{2\pi} \exp(-r^2) r\, d\theta dr \\
&= 2\pi \int_{0}^{\infty} \exp(-r^2) r\, dr \\
&= -\pi \exp(-r^2)\big|_0^{\infty} \\
&= \pi
\end{aligned}$$

and thus $I = \sqrt{\pi}$ and

$$F_0(u) = \sqrt{\pi} \exp(-\pi^2 u^2).$$

If we define $f_1(x, y) = f_0(x) f_0(y) = \exp(-(x^2 + y^2))$, then $F_1(u, v) = \pi \exp(-\pi^2(u^2 + v^2))$. The desired function is

$$f(x, y) = \frac{1}{2\pi r_0^2} f_1\left(\frac{x}{\sqrt{2} r_0}, \frac{y}{\sqrt{2} r_0}\right).$$

Applying the linearity and scaling properties

$$\begin{aligned}
F(u, v) &= \frac{1}{2\pi r_0^2} 2 r_0^2 F_1(\sqrt{2} r_0 u, \sqrt{2} r_0 v) \\
&= \exp(-2\pi^2 r_0^2 (u^2 + v^2))
\end{aligned}$$

Note that if $r_0 = 1/\sqrt{2\pi}$, then $F(u, v) = f(u, v) = \exp(-\pi(u^2 + v^2))$. Also, the entry in Table 2.2 corresponds to $r_0 = 1$, with f scaled by 2π.

9. Compute the two-dimensional continuous-space Fourier transform of the following signals:

(c) A real zone plate, $f(x,y) = \cos(\pi(x^2+y^2)/r_0^2)$. (Hint: Find the Fourier transform of the complex zone plate $\exp(j\pi(x^2+y^2)/r_0^2)$ and use linearity. You can use $\int_{-\infty}^{\infty} e^{jy^2}\,dy = \sqrt{\pi}e^{j\pi/4}$.)

Solution: As suggested, we express the real zone plate in terms of complex zone plates,

$$\cos(\pi(x^2+y^2)/r_0^2) = \tfrac{1}{2}(\exp(j\pi(x^2+y^2)/r_0^2) + \exp(-j\pi(x^2+y^2)/r_0^2)).$$

Thus, we will compute the Fourier transform of a complex zone plate $f_C(x,y) = \exp(j\pi\alpha(x^2+y^2))$ and use linearity. Furthermore, the complex zone plate is separable, so we can find the one-dimensional Fourier transform of the function $f_1(x) = \exp(j\pi\alpha x^2)$ and use Property 2.11.

Computing this one-dimensional Fourier transform

$$\begin{aligned}
F_1(u) &= \int_{-\infty}^{\infty} \exp(j\pi\alpha x^2)\exp(-j2\pi ux)\,dx \\
&= \int_{-\infty}^{\infty} \exp(j2\pi(\tfrac{\alpha}{2}x^2 - ux))\,dx \\
&= \exp(-j\pi u^2/\alpha)\int_{-\infty}^{\infty} \exp(j\pi\alpha(x-\tfrac{u}{\alpha})^2)\,dx \\
&= \exp(-j\pi u^2/\alpha)\frac{1}{\sqrt{\pi\alpha}}\int_{-\infty}^{\infty} \exp(jy^2)\,dy \qquad [y = \sqrt{\pi\alpha}(x-\tfrac{u}{\alpha})] \\
&= \frac{1}{\sqrt{\alpha}}\exp(-j\pi u^2/\alpha)\exp(j\pi/4) \qquad \text{[applying the given hint]}.
\end{aligned}$$

Using separability, we find the Fourier transform of the complex zone plate

$$F_C(u,v) = \frac{1}{\alpha} j \exp(-j\pi(u^2+v^2)/\alpha).$$

Finally, using linearity with $\alpha = 1/r_0^2$ and $\alpha = -1/r_0^2$ respectively, we find

$$\begin{aligned}
F(u,v) &= \frac{j}{2}(r_0^2\exp(-j\pi(u^2+v^2)r_0^2) - r_0^2\exp(j\pi(u^2+v^2)r_0^2)) \\
&= r_0^2\sin(\pi(u^2+v^2)r_0^2).
\end{aligned}$$

These results are given in Table 2.2 for the real and complex case with $r_0 = 1$ and $\alpha = 1$ respectively.

To compute the integral given in the hint,

$$\begin{aligned}
\int_{-\infty}^{\infty} \exp(jy^2)\,dy &= 2\int_0^{\infty} \exp(jy^2)\,dy \\
&= \int_0^{\infty} \frac{1}{\sqrt{x}} \exp(jx)\,dx \qquad [x = y^2; dy = \frac{1}{2\sqrt{x}} dx] \\
&= \frac{\Gamma(\frac{1}{2})}{(-j)^{1/2}} \\
&= \sqrt{\pi}\exp(j\pi/4) \qquad [\Gamma(\tfrac{1}{2}) = \sqrt{\pi}].
\end{aligned}$$

In this expression, $\Gamma(x)$ is the gamma function which satisfies

$$\int_0^{\infty} t^{\beta} \exp(-pt)\,dt = \frac{\Gamma(\beta+1)}{p^{\beta+1}}, p = \sigma + j\omega, \sigma \geq 0.$$

We have used $p = -j$ and $\beta = -1/2$.

9. Compute the two-dimensional continuous-space Fourier transform of the following signals:

(d) Diamond-shaped pulse

$$f(x,y) = \begin{cases} 1 & |x|+|y| \le a, \\ 0 & |x|+|y| > a. \end{cases}$$

(Hint: obtain this function from a rect function using a rotation transformation.)

Solution: The diamond-shaped pulse can be obtained by a rotating a rect function by 45° and scaling (in either order). The given diamond region is in fact a square of side $\sqrt{2}a$ rotated 45° from the horizontal. Such a square oriented horizontally is given by

$$f_1(x,y) = \mathrm{rect}\left(\frac{x}{\sqrt{2}a}, \frac{y}{\sqrt{2}a}\right).$$

Thus, $f(\mathbf{x}) = f_1(\mathbf{A}\mathbf{x})$ where

$$\mathbf{A} = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

From Example 2.3,

$$F_1(u,v) = \frac{\sin(\pi u\sqrt{2}a)\sin(\pi v\sqrt{2}a)}{\pi^2 uv}.$$

Using Property 2.6 from Table 2.1,

$$F(u,v) = \frac{1}{|\det \mathbf{A}|} F_1(\mathbf{A}^{-T}\mathbf{u})$$

where here $\det \mathbf{A} = 1$ and $\mathbf{A}^{-T} = \mathbf{A}$. Thus

$$\begin{aligned} F(u,v) &= F_1\left(\frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}, -\frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}\right) \\ &= \frac{\sin(\pi(u+v)a)\sin(\pi(-u+v)a)}{\pi^2 \frac{(u+v)}{\sqrt{2}} \frac{(-u+v)}{\sqrt{2}}} \\ &= \frac{2\sin(\pi(u+v)a)\sin(\pi(u-v)a)}{\pi^2(u^2-v^2)} \end{aligned}$$

9. Compute the two-dimensional continuous-space Fourier transform of the following signals:

(e) Gabor function

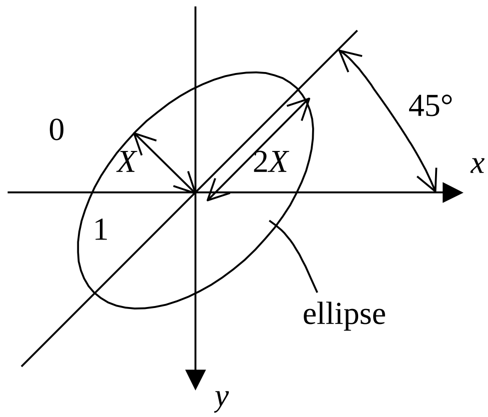
$$f(x,y) = \cos(2\pi(u_0 x + v_0 y)) \exp\left(-\frac{(x-x_0)^2 + (y-y_0)^2}{2r_0^2}\right)$$

Solution: This is a sinusoidal signal of frequency (u_0, v_0) multiplied by a Gaussian of spread r_0 centered at (x_0, y_0),

$$\begin{aligned}
\text{Let } f_1(x,y) &= \exp\left(-\frac{x^2+y^2}{2r_0^2}\right), \text{ with Fourier transform} \\
F_1(u,v) &= 2\pi r_0^2 \exp(-2\pi^2(u^2+v^2)r_0^2) \quad \text{[Table 2.2 and Property 2.6]} \\
\text{Let } f_2(x,y) &= f_1(x-x_0, y-y_0), \text{ with} \\
F_2(u,v) &= F_1(u,v)\exp(-j2\pi(ux_0+vy_0)) \quad \text{[Property 2.2)]} \\
\text{Then } f(x,y) &= \frac{1}{2}(\exp(j2\pi(u_0x+v_0y)) + \exp(-j2\pi(u_0x+v_0y)))f_2(x,y) \\
\text{and so } F(u,v) &= \frac{1}{2}(F_2(u-u_0, v-v_0) + F_2(u+u_0, v+v_0)) \quad \text{[Property 2.3]} \\
&= \pi r_0^2 \exp(-2\pi^2((u-u_0)^2 + (v-v_0)^2)r_0^2)\exp(-j2\pi(u-u_0)x_0 + (v-v_0)y_0) \\
&+ \pi r_0^2 \exp(-2\pi^2((u+u_0)^2 + (v+v_0)^2)r_0^2)\exp(-j2\pi(u+u_0)x_0 + (v+v_0)y_0)
\end{aligned}$$

9. Compute the two-dimensional continuous-space Fourier transform of the following signals:

(f) The two-dimensional zero-one function $p_{\mathcal{A}}(x, y)$ where $\mathcal{A}$ is an elliptical region, with semi-minor axis X and semi-major axis 2X, oriented at 45° as shown in the figure



Elliptical region of support of two-dimensional zero-one function.

Solution: Let$f(x, y) = \mathrm{circ}(x, y)$, $f_1(x, y) = \mathrm{circ}(x/2X, y/X)$. f_1 has an elliptical region of support with semi-minor axis X in the vertical direction and semi-major axis $2X$ in the horizontal direction. We can write $f_1(\mathbf{x}) = f(\mathbf{A}_1\mathbf{x})$, where

$$\mathbf{A}_1 = \begin{bmatrix} \frac{1}{2X} & 0 \\ 0 & \frac{1}{X} \end{bmatrix}.$$

Then $p_{\mathcal{A}}(\mathbf{x}) = f_1(\mathbf{A}_2\mathbf{x})$, where $\mathbf{A}_2$ causes a 45° counterclockwise rotation (or equivalently, a −45° clockwise rotation. From page 17 in the notes, we can achieve this with

$$\mathbf{A}_2 = \begin{bmatrix} \cos(-45^\circ) & \sin(-45^\circ) \\ -\sin(-45^\circ) & \cos(-45^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Combining the two, we obtain that $p_{\mathcal{A}}(\mathbf{x}) = f_1(\mathbf{A}_2\mathbf{x}) = f(\mathbf{A}_1\mathbf{A}_2\mathbf{x}) = f(\mathbf{A}\mathbf{x})$, where

$$\mathbf{A} = \mathbf{A}_1\mathbf{A}_2 = \begin{bmatrix} \frac{1}{2\sqrt{2}X} & -\frac{1}{2\sqrt{2}X} \\ \frac{1}{\sqrt{2}X} & \frac{1}{\sqrt{2}X} \end{bmatrix}.$$

Applying property 2.6 from Table 2.1, $P_{\mathcal{A}}(\mathbf{u}) = F(\mathbf{A}^{-T}\mathbf{u})/|\det(\mathbf{A})|$, where

$$\mathbf{A}^{-T} = \begin{bmatrix} \sqrt{2}X & -\sqrt{2}X \\ \frac{X}{\sqrt{2}} & \frac{X}{\sqrt{2}} \end{bmatrix}, \qquad \det(\mathbf{A}) = \frac{1}{2X^2}, \qquad \mathbf{A}^{-T}\mathbf{u} = \begin{bmatrix} \sqrt{2}X(u-v) \\ \frac{X(u+v)}{\sqrt{2}} \end{bmatrix}.$$

From Table 2.2, $F(u,v) = \frac{1}{\sqrt{u^2+v^2}} J_1(2\pi\sqrt{u^2+v^2})$. Thus,

$$\begin{aligned} P_{\mathcal{A}}(u,v) &= 2X^2 F(\sqrt{2}X(u-v), X(u+v)/\sqrt{2}) \\ &= \frac{2X}{\sqrt{2(u-v)^2 + (u+v)^2/2}} J_1(2\pi X\sqrt{2(u-v)^2 + (u+v)^2/2}) \end{aligned}$$

10. Derive the expression for the Fourier transform of a zero-one function on a polygon symmetric about the origin, as given in Equation (2.80).

    Solution: For a polygon symmetric about the origin, the number of vertices K must be even, and $\mathbf{a}_{k+K/2} = -\mathbf{a}_k$ for $k = 1, \ldots, K/2$. Furthermore, we see from the definitions that $\mathbf{n}_{k+\frac{K}{2}} = -\mathbf{n}_k$, $\mathbf{c}_{k+\frac{K}{2}} = -\mathbf{c}_k$ and $\mathbf{d}_{k+\frac{K}{2}} = \mathbf{d}_k$. Then, for $1 \leq k \leq \frac{K}{2}$, the $(k+\frac{K}{2})^{\text{th}}$ term of $P_{\mathcal{A}}(\mathbf{u})$ given by Equation (2.79) is

    $$\mathbf{d}_{k+\frac{K}{2}}(\mathbf{u}\cdot\mathbf{n}_{k+\frac{K}{2}})e^{-j2\pi\mathbf{u}\cdot\mathbf{c}_{k+\frac{K}{2}}}\operatorname{sinc}(\mathbf{u}\cdot(\mathbf{a}_{k+\frac{K}{2}+1}-\mathbf{a}_{k+\frac{K}{2}})) = \mathbf{d}(-\mathbf{u}\cdot\mathbf{n}_k)e^{j2\pi\mathbf{u}\cdot\mathbf{c}_k}\operatorname{sinc}(\mathbf{u}\cdot(\mathbf{a}_{k+1}-\mathbf{a}_k)).$$

    Adding terms for k and $k+\frac{K}{2}$ gives

    $$\mathbf{d}(\mathbf{u}\cdot\mathbf{n}_k)(-2j\sin(2\pi\mathbf{u}\cdot\mathbf{c}_k))\operatorname{sinc}(\mathbf{u}\cdot(\mathbf{a}_{k+1}-\mathbf{a})).$$

    Inserting this in Equation (2.79), combining the terms for k and $k+\frac{K}{2}$ and substituting $\mathbf{c}_k = (\mathbf{a}_{k+1}+\mathbf{a}_k)/2$, we obtain Equation (2.80).

    $$P_{\mathcal{A}}(\mathbf{u}) = \frac{1}{\pi\|\mathbf{u}\|^2}\sum_{k=1}^{K/2} d_k(\mathbf{u}\cdot\mathbf{n}_k)\sin(\pi\mathbf{u}\cdot(\mathbf{a}_{k+1}+\mathbf{a}_k))\operatorname{sinc}(\mathbf{u}\cdot(\mathbf{a}_{k+1}-\mathbf{a}_k)).$$

11. Use the expression in Equation (2.80) to compute the Fourier transform of the rect function.

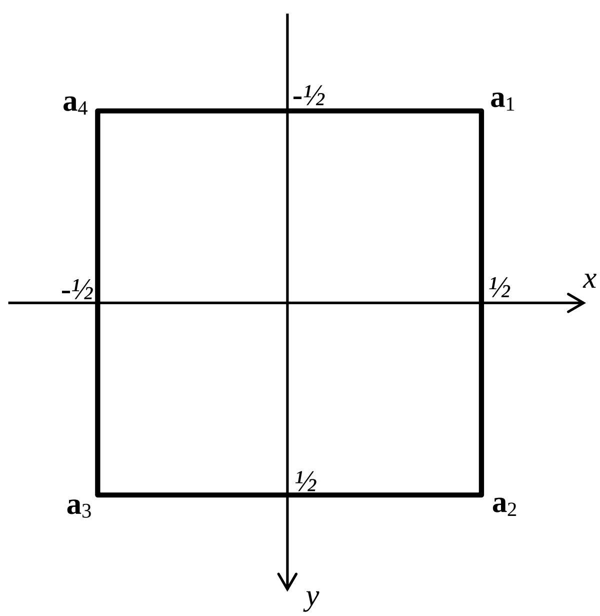
    Solution: The rect function corresponds to a region $\mathcal{A}$ that is a square of unit side centered at the origin, with $K = 4$. It is symmetric about the origin, so we can use Equation (2.80). Referring to the figure below, $d_1 = d_2 = 1$ and

$$\mathbf{n}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{a}_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \qquad \mathbf{a}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad \mathbf{a}_3 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

    Substituting into Equation (2.80), we obtain

$$\begin{aligned} P_{\mathcal{A}}(u,v) &= \frac{1}{\pi(u^2+v^2)}(u\sin(\pi u)\,\mathrm{sinc}(v) + v\sin(\pi v)\,\mathrm{sinc}(-u)) \\ &= \frac{1}{\pi(u^2+v^2)}\left(u\sin\pi u\frac{\sin\pi v}{\pi v} + v\sin\pi v\frac{\sin\pi u}{\pi u}\right) \\ &= \frac{1}{\pi^2(u^2+v^2)}\sin\pi u\sin\pi v\left(\frac{u}{v}+\frac{v}{u}\right) \\ &= \frac{\sin\pi u\sin\pi v}{\pi^2 uv}. \end{aligned}$$

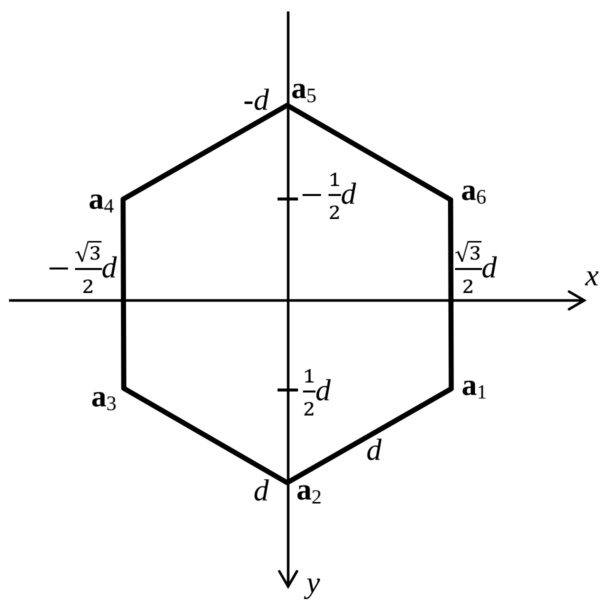
    in agreement with the standard result in Table 2.2.



Region of support of the rect function with vertices $\mathbf{a}_1$ to $\mathbf{a}_4$.

12. Use the expression in Equation (2.80) to compute the Fourier transform of a zero-one function with a region $\mathcal{A}$ that is a regular hexagon of unit area, with vertices on the y axis.

    Solution: Let the length of each size of the regular hexagon be d. Then the area is six times the area of an equilateral triangle of side d, i.e., $6 \times \frac{\sqrt{3}}{4}d^2 = \frac{3\sqrt{3}}{2}d^2$. Thus, for unit area, $d = \sqrt{\frac{2}{3\sqrt{3}}}$. The desired regular hexagon with vertices on the y axis is shown in the figure below. The vertices $\mathbf{a}_1$ to $\mathbf{a}_6$ are indicated, where we arbitrarily select $\mathbf{a}_1$ to be in the first quadrant.



Region of support of the regular hexagon of unit area with vertices $\mathbf{a}_1$ to $\mathbf{a}_6$.

Since this hexagon is symmetric about the origin, we can use Equation (2.80) to compute the Fourier transform. The quantities needed to compute $P_{\mathcal{A}}(u,v)$ are enumerated below.

$$\mathbf{a}_1 = d\begin{bmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{bmatrix} \qquad \mathbf{a}_2 = d\begin{bmatrix}0\\ 1\end{bmatrix} \qquad \mathbf{a}_3 = d\begin{bmatrix}-\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{bmatrix} \qquad \mathbf{a}_4 = d\begin{bmatrix}-\frac{\sqrt{3}}{2}\\ -\frac{1}{2}\end{bmatrix} = -\mathbf{a}_1.$$

For the required normals, $\mathbf{n}_1$ is a unit vector perpendicular to $(\mathbf{a}_2 - \mathbf{a}_1)/d = [-\frac{\sqrt{3}}{2}, \frac{1}{2}]^T$

which is easily seen to be $[\frac{1}{2}, \frac{\sqrt{3}}{2}]^T$. Similarly we can find $\mathbf{n}_2$, and $\mathbf{n}_3$ is obvious. In summary

$$\mathbf{n}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \qquad \mathbf{n}_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \qquad \mathbf{n}_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

The other derived quantities needed for the formula are

$$\mathbf{a}_1 + \mathbf{a}_2 = d \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{3}{2} \end{bmatrix} \qquad \mathbf{a}_2 + \mathbf{a}_3 = d \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{3}{2} \end{bmatrix} \qquad \mathbf{a}_3 + \mathbf{a}_4 = d \begin{bmatrix} -\sqrt{3} \\ 0 \end{bmatrix}.$$

$$\mathbf{a}_2 - \mathbf{a}_1 = d \begin{bmatrix} \frac{-\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \qquad \mathbf{a}_3 - \mathbf{a}_2 = d \begin{bmatrix} \frac{-\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix} \qquad \mathbf{a}_4 - \mathbf{a}_3 = d \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

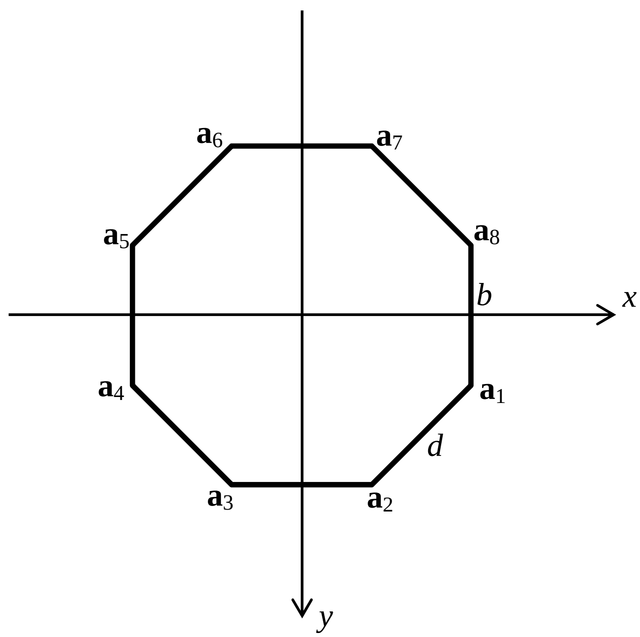
Substituting these quantities into Equation (2.80), we obtain

$$\begin{aligned} P_{\mathcal{A}}(u,v) = \frac{d}{\pi(u^2+v^2)} \Bigg(& \frac{u+\sqrt{3}v}{2} \sin \pi d(\tfrac{\sqrt{3}}{2}u + \tfrac{3}{2}v) \operatorname{sinc} d(-\tfrac{\sqrt{3}}{2}u + \tfrac{1}{2}v) \\ & + \frac{-u+\sqrt{3}v}{2} \sin \pi d(-\tfrac{\sqrt{3}}{2}u + \tfrac{3}{2}v) \operatorname{sinc} d(-\tfrac{\sqrt{3}}{2}u - \tfrac{1}{2}v) \\ & + (-u) \sin \pi u(-\sqrt{3}d) \operatorname{sinc}(-dv) \Bigg). \end{aligned}$$

This formula could be simplified in various ways. Note the similarity with the result on page 2-31. The result of this problem could be obtained from that expression by scaling by d and rotating by $\pi/6$. Verify this.

13. Use the expression in Equation (2.80) to compute the Fourier transform of a zero-one function with a region $\mathcal{A}$ that is a regular octagon of unit area, with two sides parallel to the x axis.

    Solution: Let the length of each side of the regular octagon be d. From the figure below, we see that the coordinate b is given by $\frac{d}{2b} = \tan\pi/8 = 1/(1+\sqrt{2})$, so that $b = d(1+\sqrt{2})/2 \approx 1.207d$. Then the area of the octagon is seen to be $16bd/4 = 2(1+\sqrt{2})d^2$. Thus, for unit area, we need $d^2 = 1/(2+2\sqrt{2})$, or $d = 1/\sqrt{2+2\sqrt{2}} \approx 0.455$. The solution is presented in terms of the side length d which is assumed to have this value.



Region of support of the regular octagon of unit area with vertices $\mathbf{a}_1$ to $\mathbf{a}_8$.

Since this octagon is symmetric about the origin, we can use Equation (2.80) to compute the Fourier transform. The quantities needed to compute $P_{\mathcal{A}}(u,v)$ are enumerated below.

$$\mathbf{a}_1 = d\begin{bmatrix}\frac{1+\sqrt{2}}{2}\\ \frac{1}{2}\end{bmatrix} \qquad \mathbf{a}_2 = d\begin{bmatrix}\frac{1}{2}\\ \frac{1+\sqrt{2}}{2}\end{bmatrix} \qquad \mathbf{a}_3 = d\begin{bmatrix}-\frac{1}{2}\\ \frac{1+\sqrt{2}}{2}\end{bmatrix} \qquad \mathbf{a}_4 = d\begin{bmatrix}-\frac{1+\sqrt{2}}{2}\\ \frac{1}{2}\end{bmatrix} \qquad \mathbf{a}_5 = -\mathbf{a}_1.$$

The unit normals are either horizontal, vertical or at 45 degrees, and are easily seen to be

$$\mathbf{n}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{n}_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \mathbf{n}_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Substituting these into Equation (2.80), we obtain

$$\begin{aligned} P_{\mathcal{A}}(u,v) = \frac{d}{\pi(u^2+v^2)} \Bigg(& \frac{u+v}{\sqrt{2}} \sin(\pi \tfrac{2+\sqrt{2}}{2}(u+v)d) \operatorname{sinc}(\tfrac{\sqrt{2}}{2}(-u+v)d) \\ & + v \sin(\pi(1+\sqrt{2})vd) \operatorname{sinc}(-ud) \\ & + \frac{-u+v}{\sqrt{2}} \sin(\pi \tfrac{2+\sqrt{2}}{2}(-u+v)d) \operatorname{sinc}(-\tfrac{\sqrt{2}}{2}(u+v)d) \\ & +(-u) \sin(\pi(1+\sqrt{2})(-u)d) \operatorname{sinc}(-vd) \Bigg). \end{aligned}$$

This formula can be simplified in various ways, using for example the fact that sine is odd and sinc is even, using trigonometric identities, etc.

14. Consider a continuous-domain Laplacian of Gaussian (LoG) filter with impulse response

$$h(x,y) = c\frac{x^2+y^2-2r_0^2}{2\pi r_0^6}\exp\left(-\frac{x^2+y^2}{2r_0^2}\right).$$

(a) Show that the magnitude frequency response has a peak at radial frequency

$$\sqrt{u^2+v^2} = \frac{1}{\sqrt{2}\pi r_0}.$$

Solution: From Equation (2.90)

$$H(u,v) = -c(2\pi)^2(u^2+v^2)\exp(-2\pi^2 r_0^2(u^2+v^2))$$

which is circularly symmetric. Thus, we can find the peak magnitude frequency response by searching along the u axis:

$$|H(u,0)| = |c|(2\pi)^2u^2\exp(-2\pi^2r_0^2u^2).$$

This function is continuous, has value 0 at the origin, is positive elsewhere and tends to 0 as $u \to \infty$. Thus we can find the maximum by setting the derivative to zero.

$$\frac{d}{du}|H(u,0)| = |c|(2\pi^2)(2u\exp(-2\pi^2r_0^2u^2) - 4\pi^2r_0^2u^3\exp(-2\pi^2r_0^2u^2)) = 0$$

which has a solution at $u = 0$. For $u \neq 0$ we can cancel non-zero terms to obtain $1 - 2\pi^2r_0^2u^2 = 0$, or $u^2 = 1/(2\pi^2r_0^2)$. Thus, the peak magnitude response lies on the circle

$$u^2+v^2 = \frac{1}{2\pi^2r_0^2}.$$

(b) What is the value of c such that the peak magnitude frequency response is 1.0, i.e.,

$$|H(u,v)| = 1 \text{ when } u^2+v^2 = \frac{1}{2\pi^2r_0^2}?$$

Solution: At the radial frequency given in (a)

$$|H(u,v)| = \frac{|c|(2\pi)^2}{2\pi^2r_0^2}\exp(-1) = \frac{2|c|}{r_0^2e}$$

so $c = \pm r_0^2e/2$.

(c) Compute the values found in (a) and (b) when $r_0 = 0.0025$ ph.

Solution: If $r_0 = 0.0025$ ph, the peak radial frequency is $1/(\sqrt{2}\pi(0.0025)) = 90.03$ c/ph. Then, $c = \pm(0.0025)^2 e/2 = \pm 0.849 \times 10^{-6}$. These are the values used in the example in Section 2.7.3. We can take $c = -0.849 \times 10^{-6}$ to have $H(u, v) \geq 0$.

15. Find the D-dimensional Fourier transform of the following function:

(a) A D-dimensional Gaussian $f(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}} \exp(-\|\mathbf{x}\|^2/2)$.

Solution: Written out explicitly, we see that this function is separable

$$\begin{aligned} f(x_1, \ldots, x_D) &= \frac{1}{(2\pi)^{D/2}} \exp(-(x_1^2 + \cdots + x_D^2)/2) \\ &= f_0(x_1) \cdots f_0(x_D), \end{aligned}$$

where

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$$

is a one-dimensional Gaussian. Thus, by property 2.11, $F(u_1, \ldots, u_D) = F_0(u_1) \cdots F_0(u_D)$, where $F_0(u)$ is the one-dimensional Fourier transform of $f_0(x)$. We showed in the solution to Problem 9(b) that the one-dimensional Fourier transform of $\exp(-x^2)$ is $\sqrt{\pi} \exp(-\pi^2 u^2)$. Applying Properties 2.1 and 2.6 (with $A = 1/\sqrt{2}$), we find

$$F_0(u) = \frac{1}{\sqrt{2\pi}} \sqrt{2} \sqrt{\pi} \exp(-2\pi^2 u^2).$$

Thus,

$$F(\mathbf{u}) = \exp(-2\pi^2 |\mathbf{u}|^2).$$

This is the result given in Equation (2.87) with $r_0 = 1$.

15. Find the D-dimensional Fourier transform of the following function:

(b) A D-dimensional circularly symmetric exponential $f(\mathbf{x}) = \exp(-2\pi\|\mathbf{x}\|)$.

Answer:

$$F(\mathbf{u}) = c_D \frac{1}{(1+\|\mathbf{u}\|^2)^{(D+1)/2}}$$

where $c_D = \Gamma((D+1)/2)/\pi^{(D+1)/2}$. $\Gamma(\cdot)$ is the Gamma function, which satifies the following properties: $\Gamma(n) = n!$ for $n = 1, 2, \ldots$, $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(0.5) = \sqrt{\pi}$. Hint: The solution can be found on pages 6 and 7 in Stein and Weiss (1971).

Solution: This solution is adapted from the proof of Theorem 1.14 in Stein and Weiss (1971). The Fourier transform we seek is given by

$$F(\mathbf{u}) = \int_{\mathbb{R}^D} \exp(-2\pi\|\mathbf{x}\|)\exp(-j2\pi\mathbf{u}\cdot\mathbf{x})\,d\mathbf{x}.$$

We start with an alternate expression for $\exp(-2\pi\|\mathbf{x}\|)$. For $\beta > 0$

$$e^{-\beta} = \frac{2}{\pi}\int_0^\infty \frac{\cos\beta t}{1+t^2}\,dt.$$

This is a standard result given in integral tables, obtained using residues:

$$\int_{-\infty}^{\infty} \frac{e^{j\beta t}}{1+t^2}\,dt = 2\pi j \mathrm{res}\left[\frac{e^{j\beta z}}{1+z^2}, j\right] = 2\pi j \frac{e^{-\beta}}{2j} = \pi e^{-\beta}.$$

Thus

$$e^{-\beta} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\beta t + j\sin\beta t}{1+t^2}\,dt = \frac{2}{\pi}\int_0^\infty \frac{\cos\beta t}{1+t^2}\,dt$$

since cos is even and sin is odd. Next, we use the expression $\frac{1}{1+t^2} = \int_0^\infty e^{-(1+t^2)s}\,ds$ to write this as

$$\begin{aligned}
e^{-\beta} &= \frac{2}{\pi}\int_0^\infty \cos\beta t\left(\int_0^\infty e^{-s}e^{-st^2}\,ds\right)dt \\
&= \frac{2}{\pi}\int_0^\infty e^{-s}\left(\int_0^\infty e^{-st^2}\cos\beta t\,dt\right)ds \\
&= \frac{2}{\pi}\int_0^\infty e^{-s}\left(\frac{1}{2}\int_{-\infty}^{\infty} e^{-st^2}e^{j\beta t}\,dt\right)ds \\
&= \frac{2}{\pi}\int_0^\infty e^{-s}\left(\pi\int_{-\infty}^{\infty} e^{-4\pi^2 sw^2}e^{-j2\pi\beta w}\,dw\right)ds \qquad (t = -2\pi w).
\end{aligned}$$

The integral in parentheses is seen to be the one-dimensional Fourier transform of a Gaussian function $g(w) = \exp(-4\pi^2 s w^2)$. We showed in the solution to Problem 9(b) that the one-dimensional Fourier transform of $\exp(-x^2)$ is $\sqrt{\pi}\exp(-\pi^2 u^2)$. Thus, applying Property 2.6 with $A = 2\pi\sqrt{s}$, this Fourier transform is $\frac{1}{2\sqrt{\pi s}}\exp(-\beta^2/4s)$. Inserting it in the expression for $e^{-\beta}$, we obtain

$$e^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} \exp(-\beta^2/4s)\, ds.$$

We use this with $\beta = 2\pi\|\mathbf{x}\|$ in the definition of $F(\mathbf{u})$ to obtain

$$\begin{aligned} F(\mathbf{u}) &= \int_{\mathbb{R}^D} \left(\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} \exp\left(-\frac{4\pi^2\|\mathbf{x}\|^2}{4s} \right) ds \right) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{x} \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} \left(\int_{\mathbb{R}^D} \exp\left(-\frac{\pi^2\|\mathbf{x}\|^2}{s} \right) \exp(-j2\pi\mathbf{u}\cdot\mathbf{x})\, d\mathbf{x} \right) ds. \end{aligned}$$

The inner parenthesis is the Fourier transform of a D-dimensional Gaussian, which is given in Equation (2.87) and evaluated in Problem 2.15(a). Using $r_0 = s/2\pi^2$, this Fourier transform is $(s/pi)^{D/2}\exp(-s\|\mathbf{u}\|^2)$, and inserting it in the expression for $F(\mathbf{u})$,

$$F(\mathbf{u}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} \left(\frac{s}{\pi}\right)^{D/2} e^{-\|\mathbf{u}\|^2 s}\, ds.$$

This has the form of a Gamma function

$$\Gamma(w) = \int_0^\infty t^{w-1} e^{-t}\, dt, \qquad w > 0.$$

Let $t = s(1 + \|\mathbf{u}\|^2)$. Then

$$F(\mathbf{u}) = \frac{1}{\pi^{(D+1)/2}} \int_0^\infty e^{-t} t^{(D-1)/2} \frac{1}{(1+\|\mathbf{u}\|^2)^{(D+1)/2}}\, dt = \frac{\Gamma((D+1)/2)}{\pi^{(D+1)/2}} \frac{1}{(1+\|\mathbf{u}\|^2)^{(D+1)/2}}.$$

Chapter 3

Discrete-domain signals and systems

1. Prove the properties of lattices given in Section 3.2.2.

   Solution: These properties are further addressed in Section 13.3. Here we give simple proofs from first principles.

   Let Λ be any lattice.

   (i) $\mathbf{0} \in \Lambda$: the origin belongs to any lattice.

   Proof: For any sampling matrix $\mathbf{V}$, referring to the definition in Equation (3.3), $\mathbf{V0} \in \Lambda$ since $\mathbf{0} = [0, 0, \ldots, 0]^T \in \mathbb{Z}^D$.

   (ii) If $\mathbf{x} \in \Lambda$ and $\mathbf{y} \in \Lambda$ then $\mathbf{x} + \mathbf{y} \in \Lambda$.

   Proof: Let $\mathbf{V}$ be any sampling matrix for Λ. Then there exist integer vectors $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{Z}^D$ such that $\mathbf{x} = \mathbf{V}\mathbf{n}_1$ and $\mathbf{y} = \mathbf{V}\mathbf{n}_2$. It follows that $\mathbf{x} + \mathbf{y} = \mathbf{V}\mathbf{n}_1 + \mathbf{V}\mathbf{n}_2 = \mathbf{V}(\mathbf{n}_1 + \mathbf{n}_2) \in \Lambda$ since $\mathbf{n}_1 + \mathbf{n}_2 \in \mathbb{Z}^D$.

   (iii) If $\mathbf{d} \in \Lambda$ then $\Lambda + \mathbf{d} = \Lambda$ where $\Lambda + \mathbf{d} = \{\mathbf{x} + \mathbf{d} \mid \mathbf{x} \in \Lambda\}$.

   Proof: Let $\mathbf{s}$ be any element of $\Lambda + \mathbf{d}$. Then by definition $\mathbf{s} = \mathbf{x} + \mathbf{d}$ for some $\mathbf{x} \in \Lambda$. Since $\mathbf{d} \in \Lambda$, from (ii), $\mathbf{s} = \mathbf{x} + \mathbf{d} \in \Lambda$ and thus $\Lambda + \mathbf{d} \subset \Lambda$. Conversely, let $\mathbf{x}$ be any element of Λ and let $\mathbf{s} = \mathbf{x} + (-\mathbf{d})$. Suppose $\mathbf{d} = \mathbf{V}\mathbf{n}$ for some $\mathbf{n} \in \mathbb{Z}^D$. Then $-\mathbf{d} = \mathbf{V}(-\mathbf{n}) \in \Lambda$ since $-\mathbf{n} \in \mathbb{Z}^d$. (This is an additional property given in Section 13.3.) Thus $\mathbf{s} \in \Lambda$ and $\mathbf{x} = \mathbf{s} + \mathbf{d} \in \Lambda + \mathbf{d}$ so $\Lambda \subset \Lambda + \mathbf{d}$. From these two inclusions, we can conclude that $\Lambda + \mathbf{d} = \Lambda$.

2. Prove that convolution on a lattice is commutative, $f * h = h * f$.

   Solution: Let $g = f * h$. Then by definition (Equation (3.25))

$$g[\mathbf{x}] = \sum_{\mathbf{y}\in\Lambda} f[\mathbf{y}]h[\mathbf{x}-\mathbf{y}] \qquad \text{for all } \mathbf{x}\in\Lambda.$$

   For any given $\mathbf{x} \in \Lambda$, let $\mathbf{z} = \mathbf{x} - \mathbf{y}$, and so $\mathbf{y} = \mathbf{x} - \mathbf{z}$. Now $\{-\mathbf{y} \mid \mathbf{y} \in \Lambda\} = \Lambda$ since $\mathbf{y} \in \Lambda \implies -\mathbf{y} \in \Lambda$ (see Problem 3.1), and so $\{\mathbf{x} - \mathbf{y} \mid \mathbf{y} \in \Lambda\} = \Lambda$ (Property (iii) in Section 3.2.2). In other words, as $\mathbf{y}$ ranges over all of Λ, $\mathbf{z} = \mathbf{x} - \mathbf{y}$ also ranges over all of Λ for any $\mathbf{x} \in \Lambda$. Thus

$$\begin{aligned} g[\mathbf{x}] &= \sum_{\mathbf{z}\in\Lambda} f[\mathbf{x}-\mathbf{z}]h[\mathbf{z}] \\ &= \sum_{\mathbf{z}\in\Lambda} h[\mathbf{z}]f[\mathbf{x}-\mathbf{z}] \end{aligned}$$

   i.e., $g = h * f$.

3. A linear shift-invariant filter defined on the hexagonal lattice

$$\Lambda = \mathrm{LAT}\left(\begin{bmatrix} 2X & X \\ 0 & 1.5X \end{bmatrix}\right)$$

has unit-sample response given by

$$h[\mathbf{x}] = \begin{cases} \frac{1}{4} & \mathbf{x} = (0,0) \\ \frac{1}{8} & \mathbf{x} = (X, 1.5X) \text{ or } (-X, -1.5X) \text{ or } (0, 3X) \text{ or } (0, -3X) \\ \frac{1}{16} & \mathbf{x} = (2X, 0) \text{ or } (-2X, 0) \text{ or } (X, -1.5X) \text{ or } (-X, 1.5X) \\ 0 & \text{otherwise} \end{cases}$$

Determine the frequency response $H(u, v)$ of this filter. Express it in real form. What is the DC gain of this filter?

Solution: Applying the definition of the frequency response and combining terms for $\mathbf{x}$ and $-\mathbf{x}$,

$$H(u, v) = \tfrac{1}{4} + \tfrac{1}{4}\cos(2\pi(u + 1.5v)X) + \tfrac{1}{4}\cos(6\pi vX) + \tfrac{1}{8}\cos(4\pi uX) + \tfrac{1}{8}\cos(2\pi(u - 1.5v)X).$$

The DC gain is $H(0, 0) = 1.0$. The reciprocal lattice is

$$\Lambda^* = \mathrm{LAT}\left(\begin{bmatrix} 2X & X \\ 0 & 1.5X \end{bmatrix}^{-T}\right) = \mathrm{LAT}\left(\begin{bmatrix} \frac{1}{2X} & 0 \\ -\frac{1}{3X} & \frac{2}{3X} \end{bmatrix}\right).$$

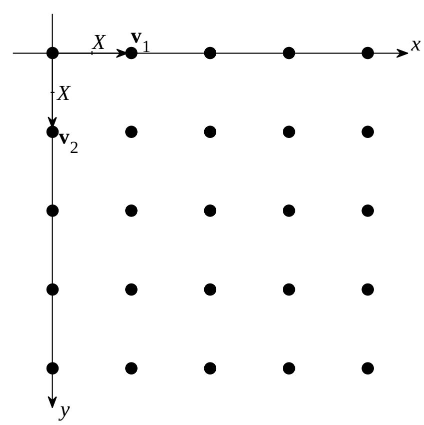
Thus,

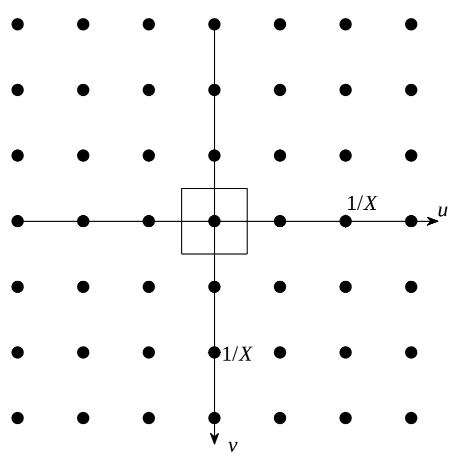
$$H(u, v) = H(u + \tfrac{k_1}{2X}, v - \tfrac{k_1}{3X} + \tfrac{2k_2}{3X}) \quad \text{for all } k_1, k_2 \in \mathbb{Z}.$$

4. For each of the following two-dimensional lattices Λ given by their sampling matrix, sketch the lattice to scale in the space domain, determine and sketch the reciprocal lattice and a Voronoi unit cell of the reciprocal lattice.

   (a) $V_\Lambda = \begin{bmatrix} 2X & 0 \\ 0 & 2X \end{bmatrix}$

   Solution: The reciprocal lattice is given by $V_{\Lambda^*} = \begin{bmatrix} \frac{1}{2X} & 0 \\ 0 & \frac{1}{2X} \end{bmatrix}$. The lattice in the space domain and the reciprocal lattice with Voronoi cell are as follows:

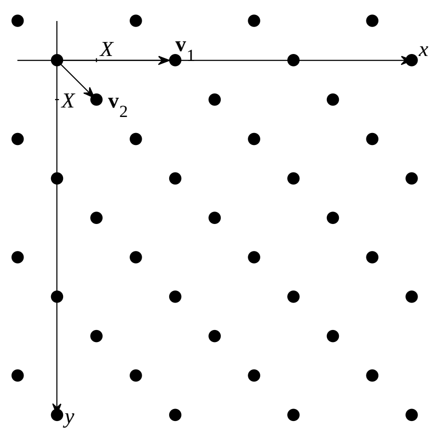


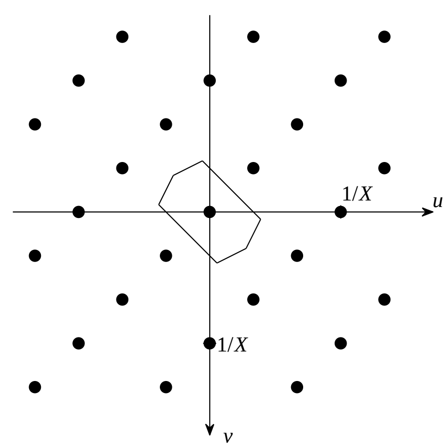


4 For each of the following two-dimensional lattices Λ given by their sampling matrix, sketch the lattice to scale in the space domain, determine and sketch the reciprocal lattice and a Voronoi unit cell of the reciprocal lattice.

(b) $V_\Lambda = \begin{bmatrix} 3X & X \\ 0 & X \end{bmatrix}$

Solution: The reciprocal lattice is given by $V_{\Lambda^*} = V_\Lambda^{-T} = \begin{bmatrix} \frac{1}{3X} & 0 \\ -\frac{1}{3X} & \frac{1}{X} \end{bmatrix}$. The lattice in the space domain and the reciprocal lattice with Voronoi cell are as follows:

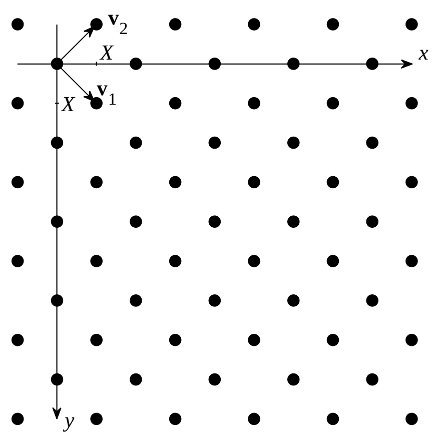


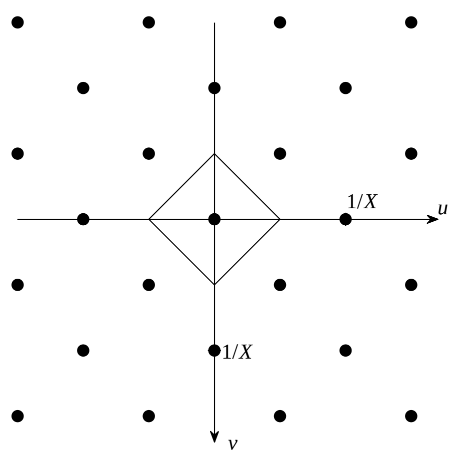


4 For each of the following two-dimensional lattices Λ given by their sampling matrix, sketch the lattice to scale in the space domain, determine and sketch the reciprocal lattice and a Voronoi unit cell of the reciprocal lattice.

(c) $V_\Lambda = \begin{bmatrix} X & X \\ X & -X \end{bmatrix}$

Solution: The reciprocal lattice is given by $V_{\Lambda^*} = V_\Lambda^{-T} = \begin{bmatrix} \frac{1}{2X} & \frac{1}{2X} \\ \frac{1}{2X} & -\frac{1}{2X} \end{bmatrix}$. The lattice in the space domain and the reciprocal lattice with Voronoi cell are as follows:





5. A two-dimensional FIR filter defined on the rectangular lattice $\Lambda = \mathrm{LAT}(\mathrm{diag}(X, X))$ has unit-sample response shown in Figure P3.5.

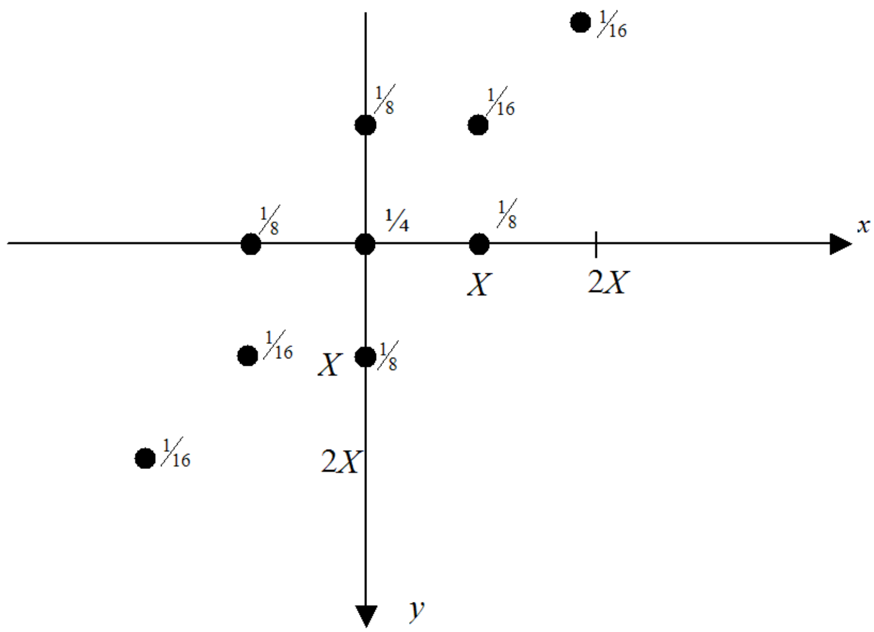


Figure P3.5 Unit-sample response $h[x, y]$. Non-zero values are shown; all others are zero.

(a) Compute the frequency response $H(u, v)$. Express it in real form.

(b) What is the output $g[x, y]$ of this filter if the input is

$$f[x, y] = \delta[x - X, y + X] - \delta[x + X, y - X]?$$

Carefully sketch the output signal $g[x, y]$ to scale in the same manner as in Figure P3.5.

Solution:

(a)

$$\begin{aligned}
H(u,v) &= \sum_{(x,y)\in\Lambda} h[x,y]\exp(-j2\pi(ux+vy)) \\
&= \frac{1}{4}\exp(0) + \frac{1}{8}\exp(-j2\pi(uX)) + \frac{1}{8}\exp(-j2\pi(-uX)) \\
&\quad + \frac{1}{8}\exp(-j2\pi(vX)) + \frac{1}{8}\exp(-j2\pi(-vX)) \\
&\quad + \frac{1}{16}\exp(-j2\pi(uX-vX)) + \frac{1}{16}\exp(-j2\pi(-uX+vX)) \\
&\quad + \frac{1}{16}\exp(-j2\pi(u2X-v2X)) + \frac{1}{16}\exp(-j2\pi(-u2X+v2X)) \\
&= \frac{1}{4} + \frac{1}{4}\cos(2\pi uX) + \frac{1}{4}\cos(2\pi vX) \\
&\quad + \frac{1}{8}\cos(2\pi(u-v)X) + \frac{1}{8}\cos(4\pi(u-v)X)
\end{aligned}$$

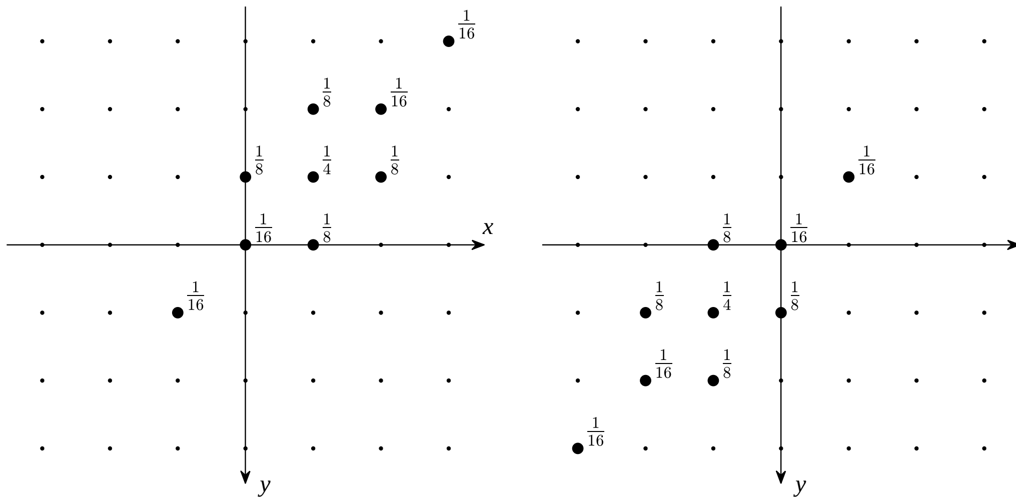
(b) By linearity and shift invariance, $g[x,y] = h[x-X, y+X] - h[x+X, y-X]$. This can be done graphically (see the following page).

The solution can also be obtained by writing out an explicit expression for $h[x,y]$ similar to the one given for $f[x,y]$.

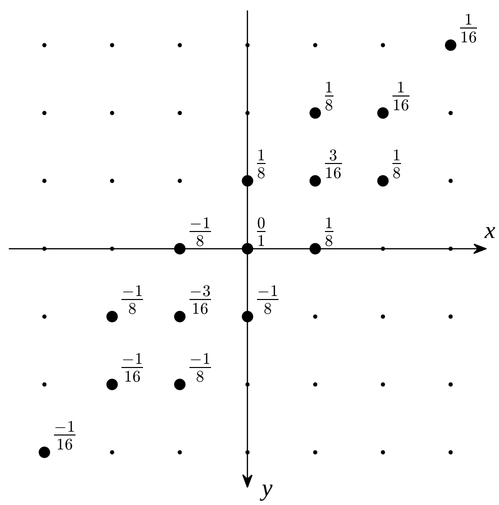
$$\begin{aligned}
h[x,y] &= \tfrac{1}{4}\delta[x,y] + \tfrac{1}{8}\delta[x,y-X] + \tfrac{1}{8}\delta[x,y+X] + \tfrac{1}{8}\delta[x-X,y] + \tfrac{1}{8}\delta[x+X,y] \\
&\quad + \tfrac{1}{16}\delta[x-X,y+X] + \tfrac{1}{16}\delta[x+X,y-X] + \tfrac{1}{16}\delta[x-2X,y+2X] \\
&\quad + \tfrac{1}{16}\delta[x+2X,y-2X]
\end{aligned}$$

Then, simply applying the previous result and simplifying,

$$\begin{aligned}
g[x,y] &= h[x-X,y+X] - h[x+X,y-X] \\
&= \tfrac{1}{8}\delta[x-X,y] - \tfrac{1}{8}\delta[x+X,y] + \tfrac{1}{8}\delta[x,y+X] - \tfrac{1}{8}\delta[x,y-X] \\
&\quad + \tfrac{3}{16}\delta[x-X,y+X] - \tfrac{3}{16}\delta[x+X,y-X] + \tfrac{1}{8}\delta[x-2X,y+X] \\
&\quad - \tfrac{1}{8}\delta[x+2X,y-X] + \tfrac{1}{8}\delta[x-X,y+2X] - \tfrac{1}{8}\delta[x+X,y-2X] \\
&\quad + \tfrac{1}{16}\delta[x-2X,y+2X] - \tfrac{1}{16}\delta[x+2X,y-2X] + \tfrac{1}{16}\delta[x-3X,y+3X] \\
&\quad - \tfrac{1}{16}\delta[x+3X,y-3X]
\end{aligned}$$



$$h[x-X,y+X] \quad - \quad h[x+X,y-X]$$



$$= g[x,y]$$

6. Consider an ideal discrete-space circularly symmetric lowpass filter defined on the rectangular lattice with horizontal and vertical sample spacing X and Y. The passband is $C_W = \{(u,v) \mid u^2+v^2 \le W^2\}$ and the unit cell of the reciprocal lattice is $\mathcal{P}^* = \{(u,v) \mid -1/2X \le u < 1/2X, -1/2Y \le v < 1/2Y\}$. Assume that $W < \min(1/2X, 1/2Y)$.

$$H(u,v) = \begin{cases} 1 & (u,v) \in C_W \\ 0 & (u,v) \in \mathcal{P}^* \backslash C_W \end{cases}$$

where of course $H(u+k/X, v+l/Y) = H(u,v)$ for all integers k,l. Show that the unit sample response of this filter is given by

$$h[mX,nY] = \frac{WXY}{\sqrt{X^2m^2+Y^2n^2}} J_1(2\pi W\sqrt{X^2m^2+Y^2n^2})$$

where $J_1(s)$ is the Bessel function of the first kind and first order. You may use the following identities:

$$J_0(s) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left[js\cos(\theta+\phi)\right]\, d\theta, \qquad \text{for any } \phi$$

$$\int sJ_0(s)\, ds = sJ_1(s)$$

Simplify the expression in the case $X = Y$.

Solution: The solution to this problem is similar to the steps in Example 2.4. This problem could be solved directly and simply using the result of Example 2.4 along with duality and sampling results to be seen later. However, here we present the full direct solution. The desired unit sample response is given by the inverse Fourier transform of the frequency response

$$h[mX,nY] = XY\iint_{C_W} \exp(j2\pi(uXm+vYn))\, du\, dv.$$

Change to polar coordinates with $u = r\cos\theta$, $v = r\sin\theta$, with Jacobian r. Thus

$$h[mX,nY] = XY\int_0^W \int_0^{2\pi} \exp(j2\pi r(Xm\cos\theta + Yn\sin\theta))r\, dr\, d\theta.$$

Now

$$Xm\cos\theta + Yn\sin\theta = \sqrt{X^2m^2+Y^2n^2}\cos(\theta+\phi_{mn}), \qquad \text{where } \phi_{mn} = \tan^{-1}(-\tfrac{Yn}{Xm}).$$

Substituting,

$$h[mX, nY] = XY \int_0^W \int_0^{2\pi} \exp(j2\pi r\sqrt{X^2m^2 + Y^2n^2}\cos(\theta + \phi_{mn}))r\,dr\,d\theta.$$

Using the given formula for $J_0(s)$ with $s = 2\pi r\sqrt{X^2m^2 + Y^2n^2}$, we obtain

$$h[mX, nY] = XY \int_0^W 2\pi J_0(2\pi r\sqrt{X^2m^2 + Y^2n^2})r\,dr.$$

Making the change of variables $s = 2\pi r\sqrt{X^2m^2 + Y^2n^2}$, we find

$$\begin{aligned} h[mX, nY] &= \frac{XY}{2\pi(X^2m^2 + Y^2n^2)} \int_0^{2\pi\sqrt{X^2m^2+Y^2n^2}W} sJ_0(s)\,ds \\ &= \frac{XY}{2\pi(X^2m^2 + Y^2n^2)} sJ_1(s)\Big|_0^{2\pi\sqrt{X^2m^2+Y^2n^2}W} \\ &= \frac{WXY}{\sqrt{X^2m^2 + Y^2n^2}} J_1(2\pi\sqrt{X^2m^2 + Y^2n^2}W). \end{aligned}$$

as required. If $X = Y$, then

$$h[mX, nX] = \frac{WX}{\sqrt{m^2 + n^2}} J_1(2\pi X\sqrt{m^2 + n^2}W).$$

Chapter 4

Discrete-Domain Periodic Signals

1. For each of the following pairs of lattice Λ and sublattice Γ given by their sampling matrices $\mathbf{V}_\Lambda$ and $\mathbf{V}_\Gamma$ respectively: (i) verify that Γ is indeed a sublattice of Λ; (ii) compute the index K of Γ in Λ; (iii) enumerate a set of K coset representatives for Γ in Λ; (iv) find sampling matrices for the reciprocal lattices Λ^* and Γ^*; (v) enumerate a set of K coset representatives for Λ^* in Γ^*.

(a) $\mathbf{V}_\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{V}_\Gamma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$.

Solution: (i) To verify that Γ is a sublattice of Λ we compute

$$\mathbf{V}_\Lambda^{-1}\mathbf{V}_\Gamma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

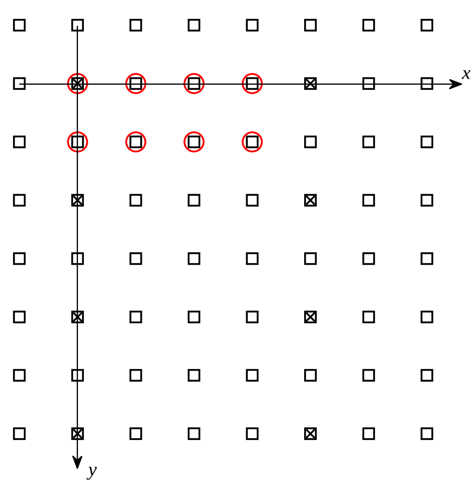
which is an integer matrix as required.

(ii) The index of Γ in Λ is $K = d(\Gamma)/d(\Lambda) = 8/1 = 8$.

(iii) We can choose as coset representatives, the points of Λ in the fundamental parallelepiped unit cell of Γ. These are given by the columns of the matrix

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The following figure shows the points of Λ ($\square$) and of Γ ($\times$) over a portion of $\mathbb{R}^2$. The selected coset representatives are shown as red circles.



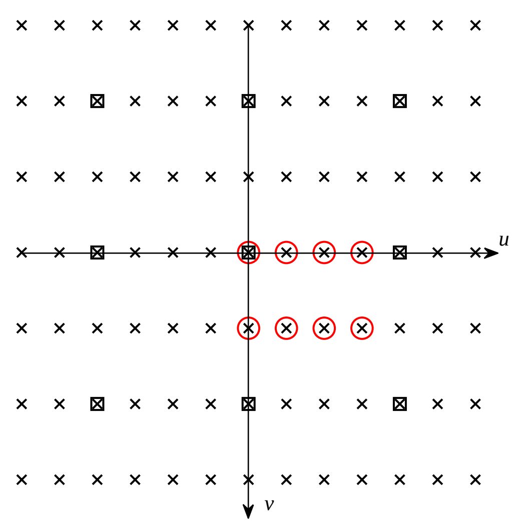
(iv) Sampling matrices for Λ^* and Γ^* are given by

$$\mathbf{V}_{\Lambda^*} = \mathbf{V}_{\Lambda}^{-T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{V}_{\Gamma^*} = \mathbf{V}_{\Gamma}^{-T} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

(v) We can choose as coset representatives the points of Γ^* within the fundamental parallelepiped unit cell of Λ^*. These are given by the columns of the matrix

$$\mathbf{D} = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The following figure shows the points of Λ^* ($\square$) and of Γ^* ($\times$) over a portion of $\mathbb{R}^2$. The selected coset representatives are shown as red circles.



1. For each of the following pairs of lattice Λ and sublattice Γ given by their sampling matrices $\mathbf{V}_\Lambda$ and $\mathbf{V}_\Gamma$ respectively: (i) verify that Γ is indeed a sublattice of Λ; (ii) compute the index K of Γ in Λ; (iii) enumerate a set of K coset representatives for Γ in Λ; (iv) find sampling matrices for the reciprocal lattices Λ^* and Γ^*; (v) enumerate a set of K coset representatives for Λ^* in Γ^*.

(b) $\mathbf{V}_\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{V}_\Gamma = \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}$.

Solution: (i) To verify that Γ is a sublattice of Λ we compute

$$\mathbf{V}_\Lambda^{-1}\mathbf{V}_\Gamma = \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}$$

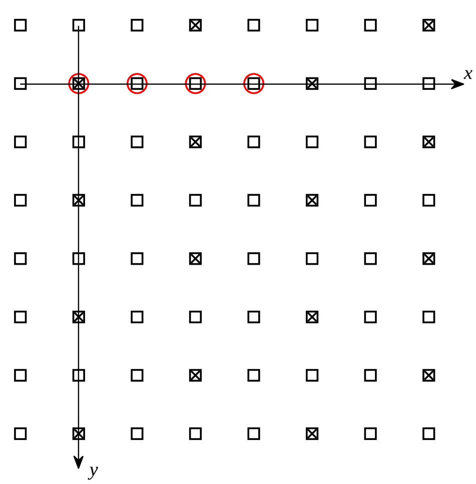
which is an integer matrix as required.

(ii) The index of Γ in Λ is $K = d(\Gamma)/d(\Lambda) = 4/1 = 4$.

(iii) We can choose as coset representatives, the points of Λ in the fundamental parallelepiped unit cell of Γ. These are given by the columns of the matrix

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The following figure shows the points of Λ ($\square$) and of Γ ($\times$) over a portion of $\mathbb{R}^2$. The selected coset representatives are shown as red circles.



(iv) Sampling matrices for Λ^* and Γ^* are given by

$$\mathbf{V}_{\Lambda^*} = \mathbf{V}_{\Lambda}^{-T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{V}_{\Gamma^*} = \mathbf{V}_{\Gamma}^{-T} = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}.$$

We can see by inspection that an alternate and more convenient sampling matrix for Γ^* is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

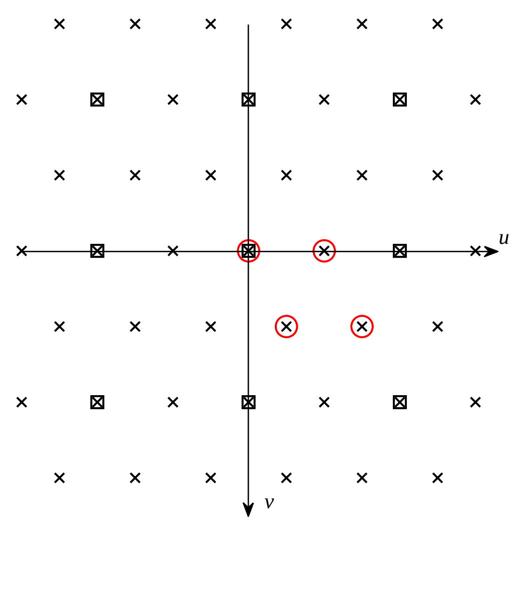
where as required $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ is an integer unimodular matrix with determinant 1.

(v) We can choose as coset representatives the points of Γ^* within the fundamental parallelepiped unit cell of Λ^*. These are given by the columns of the matrix

$$\mathbf{D} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{or} \quad \mathcal{D} = \left\{ k_1 \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} \mid k_1 = 0, 1; k_2 = 0, 1 \right\}.$$

Note that we are using the basis vectors from the alternate sampling matrix for Γ^* to enumerate these coset representatives.

The following figure shows the points of Λ^* ($\square$) and of Γ^* ($\times$) over a portion of $\mathbb{R}^2$. The selected coset representatives are shown as red circles.



1. For each of the following pairs of lattice Λ and sublattice Γ given by their sampling matrices $\mathbf{V}_\Lambda$ and $\mathbf{V}_\Gamma$ respectively: (i) verify that Γ is indeed a sublattice of Λ; (ii) compute the index K of Γ in Λ; (iii) enumerate a set of K coset representatives for Γ in Λ; (iv) find sampling matrices for the reciprocal lattices Λ^* and Γ^*; (v) enumerate a set of K coset representatives for Λ^* in Γ^*.

(c) $\mathbf{V}_\Lambda = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{V}_\Gamma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$.

Solution: (i) To verify that Γ is a sublattice of Λ we compute

$$\mathbf{V}_\Lambda^{-1}\mathbf{V}_\Gamma = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

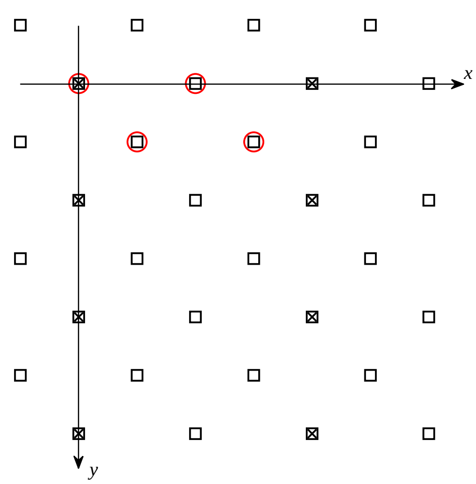
which is an integer matrix as required.

(ii) The index of Γ in Λ is $K = d(\Gamma)/d(\Lambda) = 8/2 = 4$.

(iii) We can choose as coset representatives, the points of Λ in the fundamental parallelepiped unit cell of Γ. These are given by the columns of the matrix

$$\mathbf{B} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{or} \quad \mathcal{B} = \left\{ n_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + n_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mid n_1 = 0,1; n_2 = 0,1 \right\}.$$

The following figure shows the points of Λ ($\square$) and of Γ ($\times$) over a portion of $\mathbb{R}^2$. The selected coset representatives are shown as red circles.



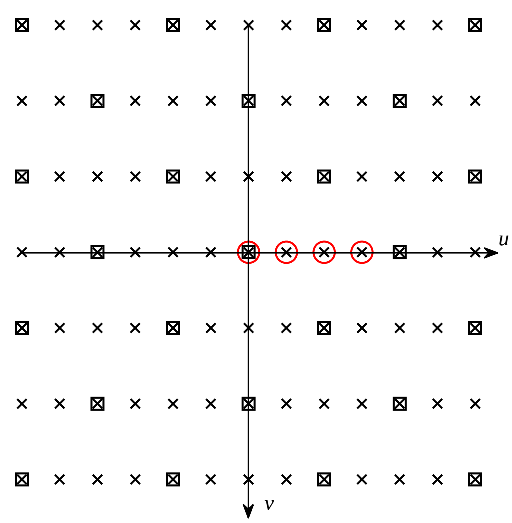
(iv) Sampling matrices for Λ^* and Γ^* are given by

$$\mathbf{V}_{\Lambda^*} = \mathbf{V}_{\Lambda}^{-T} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \qquad \mathbf{V}_{\Gamma^*} = \mathbf{V}_{\Gamma}^{-T} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

(v) We can choose as coset representatives the points of Γ^* within the fundamental parallelepiped unit cell of Λ^*. These are given by the columns of the matrix

$$\mathbf{D} = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The following figure shows the points of Λ^* ($\square$) and of Γ^* ($\times$) over a portion of $\mathbb{R}^2$. The selected coset representatives are shown as red circles.



1. For each of the following pairs of lattice Λ and sublattice Γ given by their sampling matrices $\mathbf{V}_\Lambda$ and $\mathbf{V}_\Gamma$ respectively: (i) verify that Γ is indeed a sublattice of Λ; (ii) compute the index K of Γ in Λ; (iii) enumerate a set of K coset representatives for Γ in Λ; (iv) find sampling matrices for the reciprocal lattices Λ^* and Γ^*; (v) enumerate a set of K coset representatives for Λ^* in Γ^*.

(d) $\mathbf{V}_\Lambda = \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}$, $\mathbf{V}_\Gamma = \begin{bmatrix} 4 & 2 \\ 0 & 3 \end{bmatrix}$.

Solution: (i) To verify that Γ is a sublattice of Λ we compute

$$\mathbf{V}_\Lambda^{-1}\mathbf{V}_\Gamma = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$$

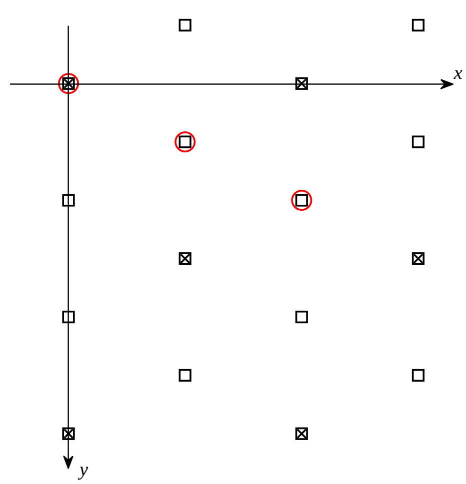
which is an integer matrix as required.

(ii) The index of Γ in Λ is $K = d(\Gamma)/d(\Lambda) = 12/4 = 3$.

(iii) We can choose as coset representatives, the points of Λ in the fundamental parallelepiped unit cell of Γ. These are given by the columns of the matrix

$$\mathbf{B} = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{or} \quad \mathcal{B} = \left\{ n \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mid n = 0, 1, 2 \right\}.$$

The following figure shows the points of Λ ($\square$) and of Γ ($\times$) over a portion of $\mathbb{R}^2$. The selected coset representatives are shown as red circles.



(iv) Sampling matrices for Λ^* and Γ^* are given by

$$\mathbf{V}_{\Lambda^*} = \mathbf{V}_{\Lambda}^{-T} = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \qquad \mathbf{V}_{\Gamma^*} = \mathbf{V}_{\Gamma}^{-T} = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}.$$

We can see by inspection that more convenient but equivalent sampling matrices for Λ^* and Γ^* are given by

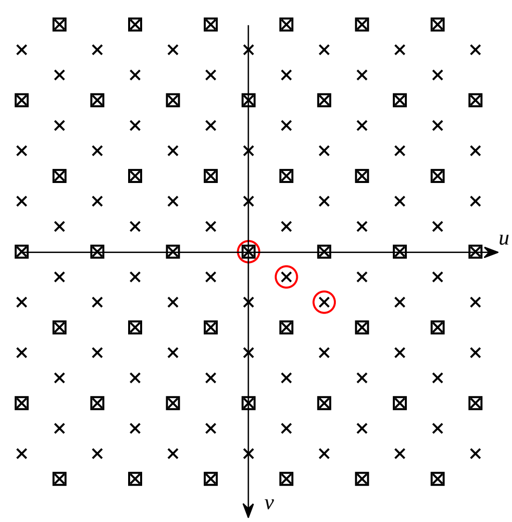
$$\mathbf{V}'_{\Lambda^*} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{bmatrix} \qquad \mathbf{V}'_{\Gamma^*} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{6} \end{bmatrix}.$$

We can verify that $\mathbf{V}'_{\Lambda^*} = \mathbf{V}_{\Lambda^*} \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and that $\mathbf{V}'_{\Gamma^*} = \mathbf{V}_{\Gamma^*} \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ where $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ is an integer unimodular matrix.

(v) We can choose as coset representatives the points of Γ^* within the fundamental parallelepiped unit cell of Λ^*. These are given by the columns of the matrix

$$\mathbf{D} = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{6} & \frac{1}{3} \end{bmatrix} \quad \text{or} \quad \mathcal{D} = \left\{ k \begin{bmatrix} \frac{1}{4} \\ \frac{1}{6} \end{bmatrix} \mid k = 0, 1, 2 \right\}.$$

The following figure shows the points of Λ^* ($\square$) and of Γ^* ($\times$) over a portion of $\mathbb{R}^2$. The selected coset representatives are shown as red circles.



2. Find the analysis and synthesis equations for the discrete-domain Fourier series representation of the following signals, using the signal lattices and periodicity lattices of corresponding problems 1(a)-(d) respectively.

(a) $\tilde{f}[\mathbf{x}] = \begin{cases} 1 & \mathbf{x} \in \Gamma \\ 0 & \mathbf{x} \in \Lambda \backslash \Gamma \end{cases}$. Note that here, $\tilde{f} = \tilde{\delta}_{\Lambda/\Gamma}$.

Solution: Referring to Problem 1(a) and its solution,

$$\Lambda = \mathrm{LAT}\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right), \quad \Gamma = \mathrm{LAT}\left(\left[\begin{smallmatrix} 4 & 0 \\ 0 & 2 \end{smallmatrix}\right]\right), \quad K = 8.$$

Sets of coset representatives in the signal domain and in the frequency domain are

$$\mathcal{B} = \left\{ n_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + n_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid n_1 = 0, \ldots, 3; n_2 = 0, 1 \right\},$$
$$\mathcal{D} = \left\{ k_1 \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \mid k_1 = 0, \ldots, 3; k_2 = 0, 1 \right\}.$$

The analysis equation is

$$\tilde{F}[\mathbf{u}] = \sum_{\mathbf{x} \in \mathcal{B}} \tilde{f}[\mathbf{x}] \exp(-j2\pi \mathbf{u} \cdot \mathbf{x}), \qquad \mathbf{u} \in \mathcal{D}.$$

Substituting explicit expressions for $\tilde{f}$, $\mathcal{B}$ and $\mathcal{D}$, we obtain

$$\begin{aligned} \tilde{F}[\tfrac{k_1}{4}, \tfrac{k_2}{2}] &= \sum_{n_1=0}^{3} \sum_{n_2=0}^{1} \tilde{f}[n_1, n_2] \exp(-j2\pi(\tfrac{k_1 n_1}{4} + \tfrac{k_2 n_2}{2})) \\ &= 1 \qquad k_1 = 0, \ldots, 3; k_2 = 0, 1, \end{aligned}$$

since $\tilde{f}[0, 0] = 1$, $\tilde{f}[n_1, n_2] = 0$ for $(n_1, n_2) \in \mathcal{B} \backslash (0, 0)$, and $\exp(0) = 1$.

The synthesis equation is

$$\tilde{f}[\mathbf{x}] = \frac{1}{K} \sum_{\mathbf{u} \in \mathcal{D}} \tilde{F}[\mathbf{u}] \exp(j2\pi \mathbf{u} \cdot \mathbf{x}), \qquad \mathbf{x} \in \mathcal{B}.$$

Substituting explicit expressions for $\tilde{F}$, $\mathcal{B}$ and $\mathcal{D}$, we obtain

$$\begin{aligned}\tilde{f}[n_1, n_2] &= \frac{1}{8}\sum_{k_1=0}^{3}\sum_{k_2=0}^{1}\exp(j2\pi(\tfrac{k_1 n_1}{4} + \tfrac{k_2 n_2}{2})) \\ &= \frac{1}{8}\sum_{k_1=0}^{3}\sum_{k_2=0}^{1}(j)^{k_1 n_1}(-1)^{k_2 n_2}, \qquad n_1 = 0, \ldots, 3; n_2 = 0, 1.\end{aligned}$$

2. Find the analysis and synthesis equations for the discrete-domain Fourier series representation of the following signals, using the signal lattices and periodicity lattices of corresponding problems 1(a)-(d) respectively.

(b) $\tilde{f}[\mathbf{x}] = \begin{cases} 1 & \mathbf{x} \in \Gamma \\ 0 & \mathbf{x} \in \Lambda \backslash \Gamma \end{cases}$. Note that here again, $\tilde{f} = \tilde{\delta}_{\Lambda/\Gamma}$.

Solution: Referring to Problem 1(b) and its solution,

$$\Lambda = \mathrm{LAT}\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right), \quad \Gamma = \mathrm{LAT}\left(\left[\begin{smallmatrix} 4 & 2 \\ 0 & 1 \end{smallmatrix}\right]\right), \quad K = 4.$$

Sets of coset representatives in the signal domain and in the frequency domain are

$$\mathcal{B} = \left\{ n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mid n = 0, \dots, 3 \right\},$$

$$\mathcal{D} = \left\{ k_1 \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} \mid k_1 = 0, 1; k_2 = 0, 1 \right\}.$$

The analysis equation is

$$\tilde{F}[\mathbf{u}] = \sum_{\mathbf{x} \in \mathcal{B}} \tilde{f}[\mathbf{x}] \exp(-j2\pi \mathbf{u} \cdot \mathbf{x}), \qquad \mathbf{u} \in \mathcal{D}.$$

Substituting explicit expressions for $\tilde{f}$, $\mathcal{B}$ and $\mathcal{D}$, we obtain

$$\begin{aligned} \tilde{F}[\tfrac{k_1}{2} + \tfrac{k_2}{4}, \tfrac{k_2}{2}] &= \sum_{n=0}^{3} \tilde{f}[n, 0] \exp(-j2\pi(\tfrac{k_1 n}{2} + \tfrac{k_2 n}{4})) \\ &= 1 \qquad k_1 = 0, 1; k_2 = 0, 1, \end{aligned}$$

since $\tilde{f}[0, 0] = 1$, $\tilde{f}[n_1, n_2] = 0$ for $(n_1, n_2) \in \mathcal{B} \backslash (0, 0)$, and $\exp(0) = 1$.

The synthesis equation is

$$\tilde{f}[\mathbf{x}] = \frac{1}{K} \sum_{\mathbf{u} \in \mathcal{D}} \tilde{F}[\mathbf{u}] \exp(j2\pi \mathbf{u} \cdot \mathbf{x}), \qquad \mathbf{x} \in \mathcal{B}.$$

Substituting explicit expressions for $\tilde{F}$, $\mathcal{B}$ and $\mathcal{D}$, we obtain

$$\begin{aligned}\tilde{f}[n,0] &= \frac{1}{4}\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\exp(j2\pi(\tfrac{k_1 n}{2}+\tfrac{k_2 n}{4})) \\ &= \frac{1}{4}\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}(-1)^{k_1 n}(j)^{k_2 n}, \qquad n = 0,\dots,3.\end{aligned}$$

2. Find the analysis and synthesis equations for the discrete-domain Fourier series representation of the following signals, using the signal lattices and periodicity lattices of corresponding problems 1(a)-(d) respectively.

(c) $\tilde{f}[\mathbf{x}] = \begin{cases} 1 & \mathbf{x} \in \Gamma \text{ or } \mathbf{x} \in [2,0]^T + \Gamma \\ 0 & \text{otherwise} \end{cases}.$

Solution: Referring to Problem 1(c) and its solution,

$$\Lambda = \mathrm{LAT}\left(\left[\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right]\right), \quad \Gamma = \mathrm{LAT}\left(\left[\begin{smallmatrix} 4 & 0 \\ 0 & 2 \end{smallmatrix}\right]\right), \quad K = 4.$$

Sets of coset representatives in the signal domain and in the frequency domain are

$$\mathcal{B} = \left\{ n_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + n_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mid n_1 = 0, 1; n_2 = 0, 1 \right\},$$
$$\mathcal{D} = \left\{ k \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} \mid k = 0, \dots, 3 \right\}.$$

The analysis equation is

$$\tilde{F}[\mathbf{u}] = \sum_{\mathbf{x} \in \mathcal{B}} \tilde{f}[\mathbf{x}] \exp(-j2\pi \mathbf{u} \cdot \mathbf{x}), \qquad \mathbf{u} \in \mathcal{D}.$$

Substituting explicit expressions for $\tilde{f}$, $\mathcal{B}$ and $\mathcal{D}$, noting that there are two nonzero values of $\tilde{f}$ for $(n_1, n_2) = (0, 0)$ and $(1, 0)$, we obtain

$$\begin{aligned} \tilde{F}[\tfrac{k}{4}, 0] &= \sum_{n_1=0}^{1} \sum_{n_2=0}^{1} \tilde{f}[2n_1 + n_2, n_2] \exp(-j2\pi(\tfrac{kn_1}{2} + \tfrac{kn_2}{4})) \\ &= 1 + \exp(-j2\pi \tfrac{k}{2}) \\ &= 1 + (-1)^k, \qquad k = 0, \dots, 3. \end{aligned}$$

The synthesis equation is

$$\tilde{f}[\mathbf{x}] = \frac{1}{K} \sum_{\mathbf{u} \in \mathcal{D}} \tilde{F}[\mathbf{u}] \exp(j2\pi \mathbf{u} \cdot \mathbf{x}), \qquad \mathbf{x} \in \mathcal{B}.$$

Substituting explicit expressions for $\tilde{F}$, $\mathcal{B}$ and $\mathcal{D}$, we obtain

$$\begin{aligned}
\tilde{f}[2n_1 + n_2, n_2] &= \frac{1}{4}\sum_{k=0}^{3} \tilde{F}[\tfrac{k}{4}, 0] \exp(j2\pi(\tfrac{kn_1}{2} + \tfrac{kn_2}{4})) \\
&= \frac{1}{4}\sum_{k=0}^{3} (1 + (-1)^k)(-1)^{kn_1}(j)^{kn_2} \\
&= \tfrac{1}{2}(1 + (-1)^{n_2}), \qquad n_1 = 0, 1; n_2 = 0, 1.
\end{aligned}$$

In the end, the synthesis equation says that $\tilde{f}[2n_1 + n_2, n_2]$ is 1 when $n_2 = 0$ and 0 when $n_2 = 1$ in agreement with the definition of $\tilde{f}$ in this question.

2. Find the analysis and synthesis equations for the discrete-domain Fourier series representation of the following signals, using the signal lattices and periodicity lattices of corresponding problems 1(a)-(d) respectively.

   (d) $\tilde{f}[\mathbf{x}] = \begin{cases} 1 & \mathbf{x} \in \Gamma \\ 2 & \mathbf{x} \in [2,1]^T + \Gamma \\ 3 & \mathbf{x} \in [0,2]^T + \Gamma \end{cases}$.

   Solution: Referring to Problem 1(d) and its solution,

$$\Lambda = \mathrm{LAT}\left(\left[\begin{smallmatrix} 4 & 2 \\ 0 & 1 \end{smallmatrix}\right]\right), \quad \Gamma = \mathrm{LAT}\left(\left[\begin{smallmatrix} 4 & 2 \\ 0 & 3 \end{smallmatrix}\right]\right), \quad K = 3.$$

   Sets of coset representatives in the signal domain and in the frequency domain are

$$\mathcal{B} = \left\{ n \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mid n = 0, 1, 2 \right\},$$
$$\mathcal{D} = \left\{ k \begin{bmatrix} \frac{1}{4} \\ \frac{1}{6} \end{bmatrix} \mid k = 0, 1, 2 \right\}.$$

   The analysis equation is

$$\tilde{F}[\mathbf{u}] = \sum_{\mathbf{x} \in \mathcal{B}} \tilde{f}[\mathbf{x}] \exp(-j2\pi \mathbf{u} \cdot \mathbf{x}), \qquad \mathbf{u} \in \mathcal{D}.$$

   Substituting explicit expressions for $\tilde{f}$, $\mathcal{B}$ and $\mathcal{D}$, we obtain

$$\begin{aligned} \tilde{F}[\tfrac{k}{4}, \tfrac{k}{6}] &= \sum_{n=0}^{2} \tilde{f}[2n, n] \exp(-j2\pi(\tfrac{kn}{2} + \tfrac{kn}{6})) \\ &= 1 + 2\exp(-j2\pi\tfrac{2k}{3}) + 3\exp(-j2\pi\tfrac{4k}{3}), \qquad k = 0, 1, 2. \end{aligned}$$

   Here we note that $\tilde{f}[4,2] = \tilde{f}[0,2] = 3$ since (4,2) and (0,2) both belong to the same coset.

   The synthesis equation is

$$\tilde{f}[\mathbf{x}] = \frac{1}{K} \sum_{\mathbf{u} \in \mathcal{D}} \tilde{F}[\mathbf{u}] \exp(j2\pi \mathbf{u} \cdot \mathbf{x}), \qquad \mathbf{x} \in \mathcal{B}.$$

Substituting explicit expressions for $\tilde{F}$, $\mathcal{B}$ and $\mathcal{D}$, we obtain

$$\begin{aligned}\tilde{f}[2n, n] &= \frac{1}{3}\sum_{k=0}^{2} \tilde{F}[\tfrac{k}{4}, \tfrac{k}{6}] \exp(j2\pi(\tfrac{kn}{2} + \tfrac{kn}{6})) \\ &= \frac{1}{3}(\tilde{F}[0,0] + \tilde{F}[\tfrac{1}{4}, \tfrac{1}{6}] \exp(j2\pi\tfrac{2n}{3}) + \tilde{F}[\tfrac{1}{2}, \tfrac{1}{3}] \exp(j2\pi\tfrac{4n}{3})), \qquad n = 0, 1, 2.\end{aligned}$$

3. Let Λ be a lattice and Γ a sublattice. Let $\mathcal{B}$ be any set of coset representatives for Γ in Λ.

(a) Let $\mathbf{x}_0$ be any element of Λ. Show that $\mathcal{B} - \mathbf{x}_0 = \{\mathbf{b} - \mathbf{x}_0 \mid \mathbf{b} \in \mathcal{B}\}$ is also a set of coset representatives for Γ in Λ. This is used in the proof of Property 4.2.

Solution: Let $\mathcal{B} = \{\mathbf{b}_0, \dots, \mathbf{b}_{K-1}\}$ where $K = d(\Gamma)/d(\Lambda)$. Then $\mathcal{B} - \mathbf{x}_0 = \{\mathbf{b}_0 - \mathbf{x}_0, \dots, \mathbf{b}_{K-1} - \mathbf{x}_0\}$, which has the correct number of elements. It is sufficient to show that no two elements of this set belong to the same coset. Suppose that $\mathbf{b}_i - \mathbf{x}_0$ and $\mathbf{b}_j - \mathbf{x}_0$ do belong to the same coset for some *different* values of i and j in $[0, K-1]$. Then $(\mathbf{b}_i - \mathbf{x}_0) - (\mathbf{b}_j - \mathbf{x}_0) \in \Gamma$, i.e., $\mathbf{b}_i - \mathbf{b}_j \in \Gamma$. But this is not possible since by assumption $\mathbf{b}_i$ and $\mathbf{b}_j$ belong to different cosets. Thus, no two elements of $\mathcal{B} - \mathbf{x}_0$ can belong to the same coset, and so they must form a set of coset representatives.

(b) Let $\mathbf{A}$ be a transformation of $\mathbb{R}^D$ such that $\mathbf{A}\Lambda = \Lambda$ and $\mathbf{A}\Gamma = \Gamma$. Show that $\mathbf{A}\mathcal{B} = \{\mathbf{Ab} \mid \mathbf{b} \in \mathcal{B}\}$ is also a set of coset representatives for Γ in Λ. This is used in the proof of Property 4.6.

Solution: Let $\mathcal{B} = \{\mathbf{b}_0, \dots, \mathbf{b}_{K-1}\}$ where $K = d(\Gamma)/d(\Lambda)$. Then $\mathbf{A}\mathcal{B} = \{\mathbf{Ab}_0, \dots, \mathbf{Ab}_{K-1}\}$, which has K elements that belong to Λ since $\mathbf{A}\Lambda = \Lambda$. It is sufficient to show that no two elements of this set belong to the same coset. Suppose that$\mathbf{Ab}_i$ and $\mathbf{Ab}_j$ do belong to the same coset for some *different* values of i and j in $[0, K-1]$. Then $\mathbf{Ab}_i - \mathbf{Ab}_j \in \Gamma$, i.e., $\mathbf{A}(\mathbf{b}_i - \mathbf{b}_j) \in \Gamma$. Now, since $\mathbf{A}\Gamma = \Gamma$, for any $\mathbf{x} \in \Gamma$ there exists some $\mathbf{y} \in \Gamma$ such that $\mathbf{Ay} = \mathbf{x}$, and thus $\mathbf{A}^{-1}\mathbf{x} = \mathbf{y} \in \Gamma$. Thus from above, $\mathbf{A}(\mathbf{b}_i - \mathbf{b}_j) \in \Gamma$ implies that $\mathbf{b}_i - \mathbf{b}_j \in \Gamma$. But this is not possible since by assumption $\mathbf{b}_i$ and $\mathbf{b}_j$ belong to different cosets. Thus, no two elements of $\mathbf{A}\mathcal{B}$ can belong to the same coset, and so they must form a set of coset representatives.

Chapter 5

Continuous-Domain Periodic Signals

1. Find the continuous-domain Fourier series representation of the following periodic signals.

(a) A circ function with rectangular periodicity

$$\tilde{f}(x,y) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \operatorname{circ}\left(\frac{x-k_1 Z}{A}, \frac{y-k_2 Z}{A}\right)$$

where $A, Z > 0$ and $A < Z$.

Solution: Assume in fact that $A < Z/2$. Then, the shifted versions of the circ function do not overlap. The signal $\tilde{f}$ has periodicity lattice $\Gamma = \mathrm{LAT}(\left[\begin{smallmatrix} Z & 0 \\ 0 & Z \end{smallmatrix}\right])$ since

$$\begin{aligned}
\tilde{f}(x+\ell_1 Z, y+\ell_2 Z) &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \operatorname{circ}\left(\frac{x+\ell_1 Z-k_1 Z}{A}, \frac{y+\ell_2 Z-k_2 Z}{A}\right) \\
&= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \operatorname{circ}\left(\frac{x-m_1 Z}{A}, \frac{y-m_2 Z}{A}\right) \qquad (m_1 = k_1-\ell_1, m_2 = k_2-\ell_2) \\
&= \tilde{f}(x,y).
\end{aligned}$$

A unit cell of the periodicity lattice is $\mathcal{P}_\Gamma = \{(x,y) \mid -Z/2 < x \leq Z/2, -Z/2 < y \leq Z/2\}$.

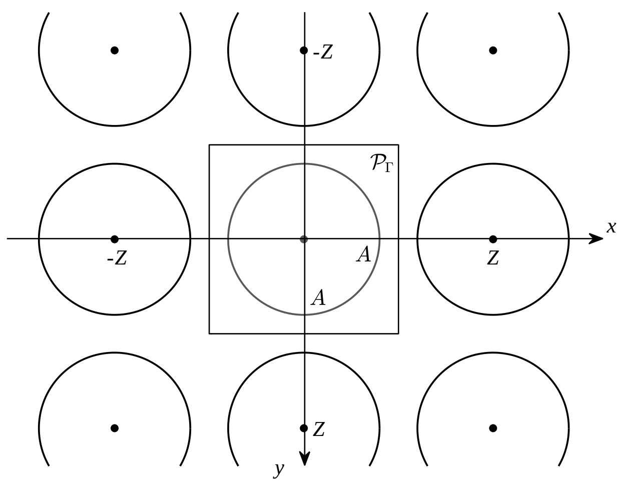


Figure P5.1a Periodic circ signal with periodicity lattice diag(Z,Z) and $A < Z/2$. $\mathcal{P}_\Gamma$ is a unit cell of Γ. The periodic signal value is 1.0 inside the circles and 0.0 elsewhere.

The reciprocal lattice is $\Gamma^* = \mathrm{LAT}(\left[\begin{smallmatrix} 1/Z & 0 \\ 0 & 1/Z \end{smallmatrix}\right])$.

The Fourier transform is given by

$$\begin{aligned}\tilde{F}[\tfrac{k_1}{Z},\tfrac{k_2}{Z}] &= \iint_{\mathcal{P}_\Gamma} \tilde{f}(x,y)\exp(-j2\pi(\tfrac{k_1 x}{Z}+\tfrac{k_2 y}{Z}))\,dxdy \\ &= \iint_{x^2+y^2\le A^2} \exp(-j2\pi(\tfrac{k_1 x}{Z}+\tfrac{k_2 y}{Z}))\,dxdy \\ &= \frac{A}{\sqrt{(\frac{k_1}{Z})^2+(\frac{k_2}{Z})^2}} J_1\left(2\pi A\sqrt{(\tfrac{k_1}{Z})^2+(\tfrac{k_2}{Z})^2}\right), \qquad (k_1,k_2)\in\mathbb{Z}^2,\end{aligned}$$

where we refer to Example 2.4 for the evaluation of this integral.

The continuous-domain Fourier series representation of $\tilde{f}$ is then

$$\begin{aligned}\tilde{f}(x,y) &= \frac{1}{Z^2}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} \frac{A}{\sqrt{(\frac{k_1}{Z})^2+(\frac{k_2}{Z})^2}} J_1\left(2\pi A\sqrt{(\tfrac{k_1}{Z})^2+(\tfrac{k_2}{Z})^2}\right)\exp(j2\pi(\tfrac{k_1 x}{Z}+\tfrac{k_2 y}{Z})) \\ &= \frac{1}{Z}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} \frac{A}{k_1^2+k_2^2} J_1(2\pi\tfrac{A}{Z}\sqrt{k_1^2+k_2^2})\exp(j\tfrac{2\pi}{Z}(k_1 x+k_2 y)).\end{aligned}$$

If $Z/2 < A < Z$, this solution method does not apply. However, as will be seen in Chapter 6, the solution will be the same as above and can be obtained with little effort. Thus, we will not present a direct solution here.

(b) A hexagonal function with rectangular periodicity

$$\tilde{f}(x,y) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \mathrm{hex}\left(\frac{x-k_1Z}{A}, \frac{y-k_2Z}{A}\right)$$

where $A, Z > 0$, $A < Z/2$ and $\mathrm{hex}(x,y)$ is the the zero-one function with hexagonal region of support of unit side as defined in Example 2.7.

Solution: Since $A < Z/2$, the shifted versions of the hex function do not overlap. The signal $\tilde{f}$ has periodicity lattice $\Gamma = \mathrm{LAT}(\left[\begin{smallmatrix} Z & 0 \\ 0 & Z \end{smallmatrix}\right])$ since

$$\begin{aligned}
\tilde{f}(x+\ell_1 Z, y+\ell_2 Z) &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \mathrm{hex}\left(\frac{x+\ell_1 Z-k_1Z}{A}, \frac{y+\ell_2 Z-k_2Z}{A}\right) \\
&= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \mathrm{hex}\left(\frac{x-m_1Z}{A}, \frac{y-m_2Z}{A}\right) \qquad (m_1 = k_1-\ell_1, m_2 = k_2-\ell_2) \\
&= \tilde{f}(x,y).
\end{aligned}$$

A unit cell of the periodicity lattice is $\mathcal{P}_\Gamma = \{(x,y) \mid -Z/2 < x \le Z/2, -Z/2 < y \le Z/2\}$.

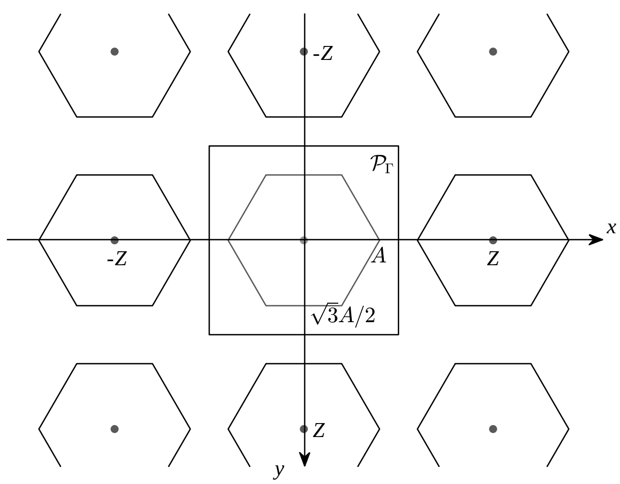


Figure P5.1b Periodic hex signal with periodicity lattice $\mathrm{diag}(Z,Z)$ and $A < Z/2$. $\mathcal{P}_\Gamma$ is a unit cell of Γ. The periodic signal value is 1.0 inside the hexagons and 0.0 elsewhere.

The reciprocal lattice is $\Gamma^* = \mathrm{LAT}(\begin{bmatrix} 1/Z & 0 \\ 0 & 1/Z \end{bmatrix})$.

The Fourier transform is given by

$$\tilde{F}[\tfrac{k_1}{Z},\tfrac{k_2}{Z}] = \iint_{\mathcal{P}_\Gamma} \tilde{f}(x,y)\exp(-j2\pi(\tfrac{k_1x}{Z}+\tfrac{k_2y}{Z}))\,dxdy$$
$$= \iint_{(x,y)\in\mathcal{A}_H} \exp(-j2\pi(\tfrac{k_1x}{Z}+\tfrac{k_2y}{Z}))\,dxdy$$

where $\mathcal{A}_H$ is the region of support of the basic hex function $\mathrm{hex}(\frac{x}{A},\frac{y}{A})$, shown within $\mathcal{P}_\Gamma$ in Figure P5.1b. This is the same as the Fourier transform of the unit hex function evaluated in Example 2.7, but scaled by A and only evaluated at the points of Γ^*. Using Property 2.6 with $\mathbf{A} = \mathrm{diag}(1/A, 1/A)$, and evaluating on $\Gamma*$,

$$\tilde{F}[\tfrac{k_1}{Z},\tfrac{k_2}{Z}] = \frac{A^2Z^2}{\pi(k_1^2+k_2^2)}\left(\frac{A(\sqrt{3}k_1+k_2)}{2Z}\sin\left(\frac{\pi A(3u+\sqrt{3}v)}{2Z}\right)\operatorname{sinc}\left(\frac{A(-k_1+\sqrt{3}k_2)}{2Z}\right)\right.$$
$$+ A\frac{k_2}{Z}\sin(\pi A\sqrt{3}\frac{k_2}{Z})\operatorname{sinc}(A\frac{k_1}{Z}) + \frac{A(-\sqrt{3}k_1+k_2)}{2Z}\sin\left(\frac{\pi A(-3k_1+\sqrt{3}k_2)}{2Z}\right)$$
$$\left.\times\operatorname{sinc}\left(\frac{A(k_1+\sqrt{3}k_)}{2Z}\right)\right).$$

The continuous-domain Fourier series is then given by

$$\tilde{f}(x,y) = \frac{1}{Z^2}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\tilde{F}[\tfrac{k_1}{Z},\tfrac{k_2}{Z}]\exp(j2\pi(\tfrac{k_1x}{Z}+\tfrac{k_2y}{Z})),$$

where $\tilde{F}[\frac{k_1}{Z},\frac{k_2}{Z}]$ is as given above.

Chapter 6

Sampling, Reconstruction and Sampling Theorems for Multidimensional Signals

1. A two-dimensional continuous-domain signal $f_c(x, y)$ has Fourier transform

$$F_c(u,v) = \begin{cases} ce^{-\alpha(|u|+|v|)} & u^2+v^2 < W^2 \\ 0 & u^2+v^2 \geq W^2 \end{cases}$$

for some real number W. The signal is sampled on a hexagonal lattice Λ with sampling matrix

$$\mathbf{V} = \begin{bmatrix} X & X/2 \\ 0 & \sqrt{3}X/2 \end{bmatrix}$$

to give the sampled signal $f[x, y]$, $(x, y) \in \Lambda$, with Fourier transform $F(u, v)$.

(a) What is the expression for $F(u, v)$ in terms of $F_c(u, v)$?

Solution:

$$\begin{aligned} \Lambda^* &= \mathrm{LAT}(\mathbf{V}^{-T}) \\ &= \mathrm{LAT}\left(\begin{bmatrix} \frac{1}{X} & 0 \\ -\frac{1}{\sqrt{3}X} & \frac{2}{\sqrt{3}X} \end{bmatrix}\right) \end{aligned}$$

Also, $d(\Lambda) = \frac{\sqrt{3}X^2}{2}$.

$$\begin{aligned} F(u,v) &= \frac{2}{\sqrt{3}X^2} \sum_{(r_1,r_2)\in\Lambda^*} F_c(u+r_1, v+r_2) \\ &= \frac{2}{\sqrt{3}X^2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} F_c\left(u + \frac{k_1}{X}, v - \frac{k_1}{\sqrt{3}X} + \frac{2k_2}{\sqrt{3}X}\right) \end{aligned}$$

(b) Find the largest possible value of X such that there is no aliasing? Sketch the region of support of the Fourier transform of the sampled signal in this case (including all replicas), and also indicate a unit cell of the reciprocal lattice Λ^*.

Solution: There will be no aliasing if all the nearest neighbors to 0 in Λ^* are at least distance $2W$ from the origin. Looking at the Figure P6.1.1 below, we see that the distance z to the nearest neighbors at $(\pm\frac{1}{X}, \pm\frac{1}{\sqrt{3}X})$ is given by

$$z^2 = \left(\frac{1}{X}\right)^2 + \left(\frac{1}{\sqrt{3}X}\right)^2 = \frac{4}{3X^2}, \qquad \Rightarrow z = \frac{2}{\sqrt{3}X}.$$

The nearest neighbors on the v-axis are also at distance $\frac{2}{\sqrt{3}X}$ from the origin. Thus, for no aliasing, we require

$$W < \frac{z}{2} = \frac{1}{\sqrt{3}X}$$
$$\text{i.e., } X < \frac{1}{\sqrt{3}W}$$

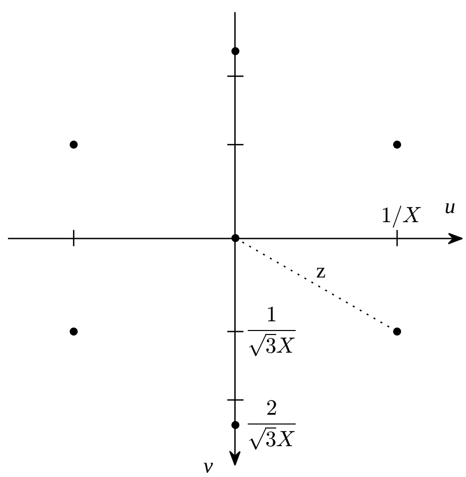


Figure P6.1.1

Figure P6.1.2 illustrates a few replicas of the region of support of the Fourier transform of the sampled signal, for X slightly less than $\frac{1}{\sqrt{3}W}$. The Voronoi unit cell of the reciprocal lattice is also shown.

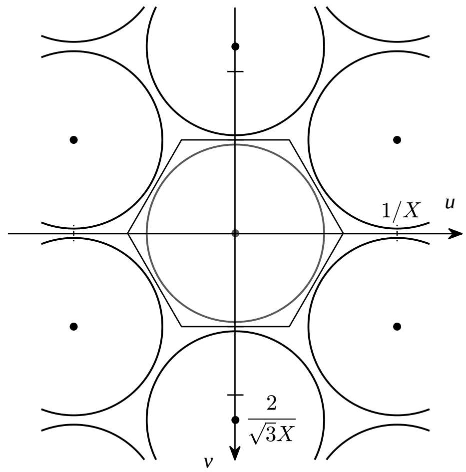


Figure P6.1.2

2. The face-centered cubic lattice is the most efficient lattice for the packing of spheres in three dimensions. A sampling matrix for this lattice is given by

$$\mathbf{V} = K \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where K is some real constant. Suppose that a bandlimited three-dimensional signal $f(\mathbf{x})$ satisfying $F(\mathbf{u}) = 0$ for $|\mathbf{u}| > W$ is sampled on a lattice whose reciprocal lattice is face-centered cubic. Find the least dense lattice such that there is no aliasing. Compare the resulting sampling density with the best orthogonal sampling for which there is no aliasing.

Solution: Figure P6.2.1 shows a perspective view of several replicas of the spherical support of $F(u_1, u_2, u_3)$ when the reciprocal lattice is face-center cubic and the sampling density is well above the critical value. The cube edges are drawn just to help visualization. From the figure, we see that the replicas at cube corners and center of cube face will be the first to touch as the sampling density decreases.

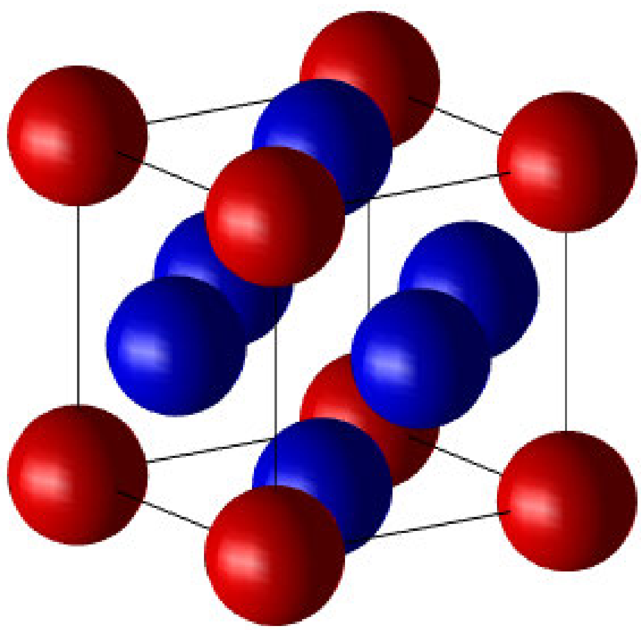


Figure P6.2.1 Illustration of spheres on points of a face-center cubic lattice. Cube edges and different colors for spheres at corners and face edges are to help visualize.

At critical sampling, the replicated spherical support of the baseband Fourier transform will touch on the $u_1 - u_2$ plane, as well as on the $u_1 - u_3$ plane and the $u_2 - u_3$ plane. The situation on the $u_1 - u_2$ plane is shown in Figure P6.6.2.

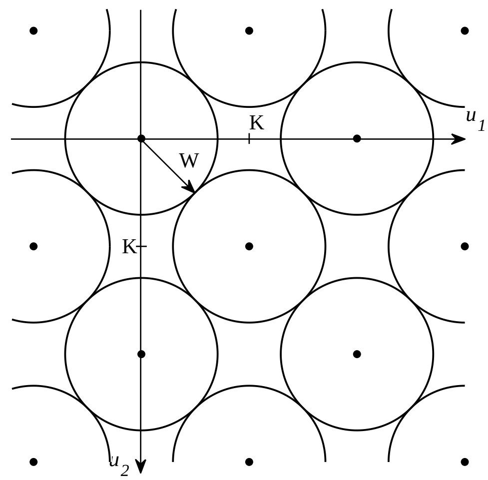


Figure P6.6.2 Slice of frequency domain on $u_1 - u_2$ plane, showing situation for critical sampling.

From the geometry, we see that $K^2 + K^2 = (2W)^2$ so that $K = \sqrt{2}W$. Thus the reciprocal lattice at critical sampling is

$$\Lambda^* = \mathrm{LAT}\left(\sqrt{2}W \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right).$$

To obtain the sampling density, $d(\Lambda^*) = |\det \mathbf{V}| = (\sqrt{2}W)^3 \cdot 2 = 4\sqrt{2}W^3$. It follows that the sampling density is

$$\frac{1}{d(\Lambda)} = d(\Lambda^*) = 4\sqrt{2}W^3.$$

The corresponding sampling lattice is the body-centered cubic lattice with sampling matrix

$\mathbf{V}^{-T}$, i.e.,

$$\Lambda = \mathrm{LAT}\left(\frac{1}{\sqrt{2}W}\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}\right) = \mathrm{LAT}\left(\frac{1}{\sqrt{2}W}\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\right),$$

where the latter matrix is an equivalent but more convenient sampling matrix.

For orthogonal sampling, the reciprocal lattice is

$$\mathbf{V}_O = K_O \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case, it is clear $K_O = 2W$, so that the sampling density for critical sampling is $d(\Lambda_O^*) = 8W^3$. The ratio of sampling densities is $\frac{8W^3}{4\sqrt{2}W^3} = \sqrt{2} \approx 1.414$. Thus orthogonal sampling requires about 41.4% more samples per unit volume for alias-free sampling.

Figure P6.2.3 shows a similar view of the frequency domain to Figure P6.2.1 at critical sampling.

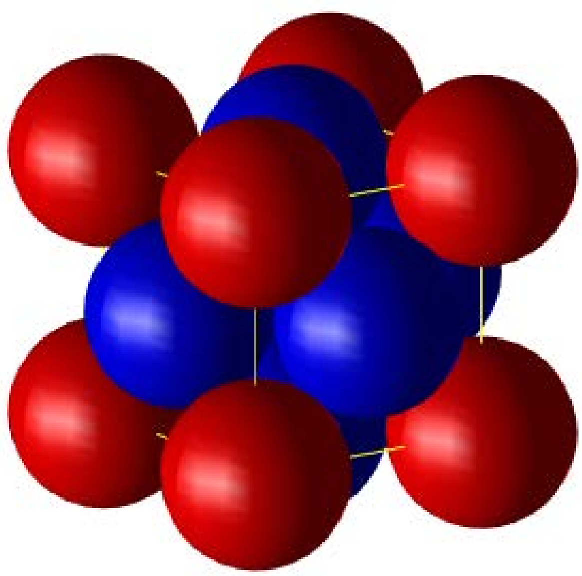


Figure P6.2.3 Illustration of spheres on points of a face-center cubic lattice at critical sampling.

3. Fig. 6.5 illustrates the sensor in a hypothetical digital camera using a sensor element which is hexagonal in shape. There are $M = 740$ sensor elements in each horizontal row and there are $N = 480$ rows of sensor elements, for a total of 480×740 sensor elements. The centers of the sensor elements lie on a hexagonal lattice Λ, and each sensor element is a unit cell of this lattice. The output of each sensor element is the integral of light irradiance over the sensor element for some arbitrary exposure time, and it is associated with the lattice point at the center of the sensor element. Assume that the picture width is MX and the picture height (ph) is NY. We use the picture height as the unit of length. The sensor element is a regular hexagon with $Y = \sqrt{3}X/2$. (Note that Fig. 6.5 is just a sketch and is not drawn to scale.)

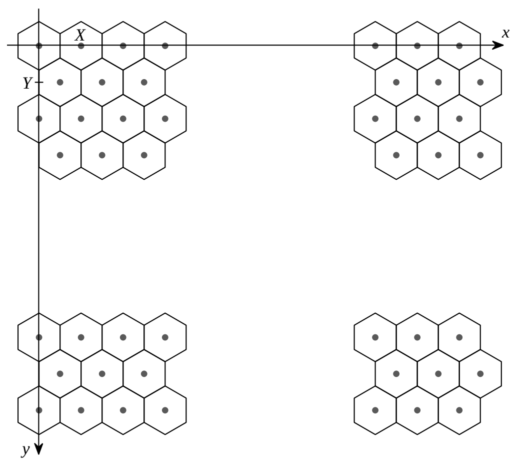


Figure 6.5 An image sensor with hexagonal sensor elements.

(a) Give a sampling matrix for the lattice shown in Fig. 6.5 in units of ph.

Solution: By inspection of Figure 6.5, a suitable sampling matrix for Λ is

$$\mathbf{V}_\Lambda = \begin{bmatrix} X & \frac{X}{2} \\ 0 & Y \end{bmatrix} = \begin{bmatrix} X & \frac{X}{2} \\ 0 & \frac{\sqrt{3}X}{2} \end{bmatrix}.$$

Since there are 480 rows, each of height Y, $480Y = 1$ ph, so that $Y = \frac{1}{480}$ ph ≈ 0.0021 ph.

Thus $X = \frac{2Y}{\sqrt{3}} = \frac{1}{240\sqrt{3}}$ ph ≈ 0.0012 ph. Thus, using numerical values

$$\mathbf{V}_\Lambda = \begin{bmatrix} \frac{1}{240\sqrt{3}} & \frac{1}{480\sqrt{3}} \\ 0 & \frac{1}{480} \end{bmatrix} \text{ ph} \approx \begin{bmatrix} 0.0024 & 0.0012 \\ 0 & 0.0021 \end{bmatrix} \text{ ph.}$$

(b) What is the area of a sensor element, with correct units? What is the sampling density, with correct units?

Solution: The area of a sensor element is

$$d(\Lambda) = |\det \mathbf{V}_\Lambda| = \frac{\sqrt{3}X^2}{2} = \frac{\sqrt{3}}{2}\left(\frac{1}{240\sqrt{3}}\right)^2 \text{ ph}^2 \approx 5.01 \times 10^{-6} \text{ ph}^2.$$

The sampling density is

$$\frac{1}{d(\Lambda)} = \frac{2}{\sqrt{3}X^2} \approx 2.00 \times 10^5 \text{ samples per ph}^2.$$

(c) What is the aspect ratio of the sensor? Is it approximately 4/3 or approximately 16/9?

Solution: The picture width is $740X$ and the picture height is $480Y = 240\sqrt{3}X$. Thus, the aspect ratio is

$$ar = \frac{\text{picture width}}{\text{picture height}} = \frac{740X}{240\sqrt{3}X} = \frac{37}{12\sqrt{3}} \approx 1.78.$$

Since $4/3 = 1.\overline{3}$ and $16/9 = 1.\overline{7}$, the aspect ratio is approximately 16/9.

(d) The sampling process carried out by this sensor can be modeled by a linear shift-invariant (LSI) continuous-space filter followed by ideal sampling on Λ. Give an expression for the impulse response $h_a(x, y)$ of this LSI filter with the correct gain. Assume that if the image irradiance is a constant value over a sensor element (in arbitrary normalized units), the sampled value is that same value, i.e., the DC gain of $h_a(x, y)$ is 1.0.

Solution: The sampling aperture is $a(x, y) = c\,\mathrm{hexa}(x, y)$ where $\mathrm{hexa}(x, y)$ is a zero-one function with regular hexagonal region of support $\mathcal{A}_H$ shown centered at the origin in Figure 6.5 and c is selected to get a DC gain of 1. Since $a(x, y)$ is symmetric about the origin, $h_a(x, y) = a(x, y)$. To have a DC gain of 1, we require $\iint_{\mathbb{R}^2} h_a(x, y)\,dxdy = 1$, or $\iint_{\mathcal{A}_H} c\,dxdy = 1$. Thus,

$$c = \frac{1}{\text{area}(\mathcal{A}_H)} = \frac{1}{d(\Lambda)} = \frac{2}{\sqrt{3}X^2} \approx 2.00 \times 10^5$$

(see (b) above).

We can relate this function to the hexagonal zero-one function with unit side given in Example 2.7 (which has been denoted $\mathrm{hex}(x, y)$) by scaling and rotating by $\pi/6$. The length S of one side of the regular hexagon $\mathcal{A}_H$ is given by $S^2 = (X/2)^2 + (S/2)^2$, or $S = X/\sqrt{3} = 1/720$ ph. Let

$$\mathbf{A}_{\pi/6} = \begin{bmatrix} \cos\pi/6 & \sin\pi/6 \\ -\sin\pi/6 & \cos\pi/6 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \qquad \mathbf{A}_S = \begin{bmatrix} \frac{1}{S} & 0 \\ 0 & \frac{1}{S} \end{bmatrix} = \begin{bmatrix} 720 & 0 \\ 0 & 720 \end{bmatrix}.$$

Then $h_a(\mathbf{x}) = c\,\mathrm{hex}(\mathbf{A}_{\pi/6}\mathbf{A}_S\mathbf{x})$ or explicitly

$$h_a(x, y) = 2 \times 10^{-5}\mathrm{hex}(360(\sqrt{3}x + y), 360(-x + \sqrt{3})).$$

(e) Give an expression for the frequency response $H_a(u, v)$ corresponding to the camera aperture impulse response $h_a(x, y)$.

Solution: The requested frequency response $H_a(u, v)$ is the continuous-domain Fourier transform of $h_a(x, y)$ given in part (d) of this problem. Since $h_a(\mathbf{x}) = c\,\mathrm{hex}(\mathbf{A}\mathbf{x})$ where $\mathbf{A} = \mathbf{A}_{\pi/6}\mathbf{A}_S$, we can obtain $H_a(u, v)$ using Fourier transform properties and the Fourier transform of $\mathrm{hex}(\mathbf{x})$ found in Example 2.7, which we denote $\mathrm{HEX}(\mathbf{u})$. Then, using Properties (2.1) and (2.6),

$$H_a(\mathbf{u}) = \frac{c}{|\det \mathbf{A}|}\mathrm{HEX}(\mathbf{A}^{-T}\mathbf{u}),$$

where

$$\mathbf{A} = \begin{bmatrix} 360\sqrt{3} & 360 \\ -360 & 360\sqrt{3} \end{bmatrix} \qquad \mathbf{A}^{-T} = \begin{bmatrix} \frac{\sqrt{3}}{1440} & \frac{1}{1440} \\ -\frac{1}{1440} & \frac{\sqrt{3}}{1440} \end{bmatrix}$$

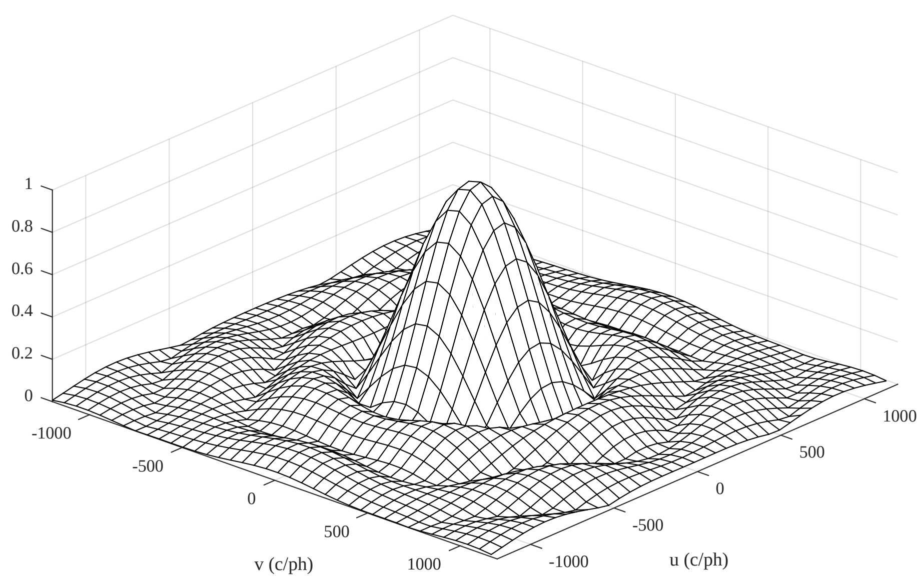
and $\det \mathbf{A} = 4(360)^2$, so that $\frac{c}{|\det \mathbf{A}|} = \frac{2\sqrt{3}}{9}$. Explicitly,

$$H_a(u, v) = \frac{2\sqrt{3}}{9}\mathrm{HEX}\left(\frac{\sqrt{3}u + v}{1440}, \frac{-u + \sqrt{3}v}{1440}\right).$$

Substituting into the expression for $\mathrm{HEX}(u, v)$ given in Example 2.7 and simplifying, we obtain

$$H_a(u,v) = \frac{80\sqrt{3}}{\pi(u^2+v^2)}\left((u+\sqrt{3}v)\sin\left(\pi\frac{\sqrt{3}u+3v}{1440}\right)\operatorname{sinc}\left(\frac{-\sqrt{3}u+v}{1440}\right)\right.$$
$$+(-u+\sqrt{3}v)\sin\left(\pi\frac{-\sqrt{3}u+3v}{1440}\right)\operatorname{sinc}\left(\frac{\sqrt{3}u+v}{1440}\right)$$
$$\left.+2u\sin\left(\pi\frac{2\sqrt{3}u}{1440}\right)\operatorname{sinc}\left(\frac{2v}{1440}\right)\right).$$

We can verify that $H_a(0,0) = 1$ as required. A perspective view of the frequency response $H_a(u,v)$ is shown below in the range $-1200 \leq u, v \leq 1200$ c/ph. The hexagonal symmetry is apparent in this plot.



(f) Assume that the continuous-space input light irradiance $f_c(x, y)$ has a Fourier transform $F_c(u, v)$. Give an expression for the Fourier transform of the sampled image $f[x, y], (x, y) \in \Lambda$ in terms of in terms of $F_c(u, v)$ and $H_a(u, v)$; you should explicitly evaluate the reciprocal lattice Λ^*.

Solution: From Equation (6.16),

$$F(\mathbf{u}) = \frac{1}{d(\Lambda)} \sum_{\mathbf{r} \in \Lambda^*} H_a(\mathbf{u} - \mathbf{r}) F_c(\mathbf{u} - \mathbf{r}),$$

where $H_a(\mathbf{u})$ was found in part (d) and $d(\Lambda) = \frac{\sqrt{3}X^2}{2} = \frac{1}{2\sqrt{3}(240)^2}$ was found in part (b). A sampling matrix for Λ^* is

$$\mathbf{V}_{\Lambda^*} = \mathbf{V}_\Lambda^{-T} = \begin{bmatrix} \frac{1}{X} & 0 \\ -\frac{1}{\sqrt{3}X} & \frac{2}{\sqrt{3}X} \end{bmatrix} = \begin{bmatrix} 240\sqrt{3} & 0 \\ -240 & 480 \end{bmatrix}.$$

An equivalent sampling matrix in upper triangular form is

$$\mathbf{V}_{\Lambda^*} = \begin{bmatrix} \frac{2}{X} & \frac{1}{X} \\ 0 & \frac{1}{\sqrt{3}X} \end{bmatrix} = \begin{bmatrix} 480\sqrt{3} & 240\sqrt{3} \\ 0 & 240 \end{bmatrix}.$$

Thus, explicitly

$$\begin{aligned} F(u, v) = 2\sqrt{3}(240)^2 \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} & H_a(u - 480\sqrt{3}k_1 - 240\sqrt{3}k_2, v - 240k_2) \\ & \times F_c(u - 480\sqrt{3}k_1 - 240\sqrt{3}k_2, v - 240k_2). \end{aligned}$$

Chapter 7

Light and Color Representation in Imaging Systems

1. The web color *goldenrod* that we will denote $[\mathbf{Q}]$ is specified by the RGB values 218, 165, 32, on a scale from 0 to 255. Thus they can be assumed to be $Q'_R = 0.8549$, $Q'_G = 0.6471$, $Q'_B = 0.1255$ on a scale from 0 to 1. We assume that these are gamma-corrected values, according to the sRGB gamma law (Section 7.5.5), and that the primaries are the Rec. 709/sRGB primaries, normalized with respect to reference white D_{65} (Section 7.5.3). The goal of this problem is to determine representations of this color in other color coordinate representations. Determine the following (show your work):

(a) The tristimulus values Q_R, Q_G, Q_B in the Rec. 709/sRGB color representation (Section 7.5.3);

Solution: We use the sRGB gamma law given in Equation (7.70). Since all gamma-corrected values on the scale from 0 to 1 are greater than 0.04045, the tristimulus values are obtained by the formula

$$Q_i = \left(\frac{Q'_i + 0.055}{1.055}\right)^{2.4}, \qquad i = R, G, B,$$

giving the result $Q_R = 0.7011$, $Q_G = 0.3763$, $Q_B = 0.0144$.

(b) The luminance Q_L and the chromaticities q_R, q_G, q_B in the Rec. 709/sRGB representation;

Solution: From Equation (7.55), the luminance $Q_L = Q_Y = Q_R R_Y + Q_G G_Y + Q_B B_Y$, where R_Y, G_Y and B_Y are the relative luminance of the sRGB primaries, given in Equation (7.63) as $[R_Y, G_Y, B_Y] = [0.2126, 0.7152, 0.0722]$. Substituting these values gives $Q_L = 0.4192$. The chromaticities are given by Equation (7.56) $q_i = Q_i/(Q_R + Q_G + Q_B)$, $i = R, G, B$, giving $q_R = 0.6421$, $q_G = 0.3446$, $q_B = 0.0132$.

(c) The XYZ tristimulus values Q_X, Q_Y, Q_Z and the corresponding chromaticities q_X, q_Y, q_Z (Section 7.4.7);

Solution: The XYZ tristimulus values are obtained from the sRGB tristimulus values using Equation (7.65), giving the result $Q_X = 0.4263$, $Q_Y = 0.4192$, $Q_Z = 0.0721$. The corresponding chromaticities are obtained by $q_i = Q_i/(Q_X + Q_Y + Q_Z)$, $i = X, Y, Z$, giving $q_X = 0.4646$, $q_Y = 0.4568$, $q_Z = 0.0786$.

(d) The 1976 $U'V'W'$ tristimulus values $Q_{U'}$, $Q_{V'}$, $Q_{W'}$ and the corresponding chromaticities $q_{U'}$, $q_{V'}$, $q_{W'}$ (Section 7.4.8);

Solution: The 1976 $U'V'W'$ tristimulus values are obtained from the XYZ tristimulus values using Equation (7.49), giving the result $Q_{U'} = 0.1895$, $Q_{V'} = 0.4192$, $Q_{W'} = 0.1614$. The chromaticities are obtained by $q_i = Q_i/(Q_{U'} + Q_{V'} + Q_{W'})$, $i = U', V', W'$, giving the result $q_{U'} = 0.2460$, $q_{V'} = 0.5444$, $q_{W'} = 0.2096$.

(e) The CIELAB coordinates Q_{L^*}, Q_{a^*}, Q_{b^*} (Section 7.5.4);

Solution: The CIELAB coordinates are obtained using Equation (7.68), with the function f given by Equation (7.67) with $[\mathbf{W}] = [\mathbf{D}_{65}]$, i.e. $\mathbf{W}_{\mathcal{XYZ}} = [0.9505, 1.0, 1.0891]'$. In this case, the values Q_i/W_i, $i = X, Y, Z$, are all greater than $(\frac{6}{29})^3$ so f takes the cube root form in all expressions. Thus

$$\begin{aligned}
Q_{L^*} &= 116 Q_Y^{1/3} - 16 = 70.8157 \\
Q_{a^*} &= 500((Q_X/0.9505)^{1/3} - Q_Y^{1/3}) = 8.5209 \\
Q_{b^*} &= 200(Q_Y^{1/3} - (Q_Z/1.0891)^{1/3}) = 68.7714.
\end{aligned}$$

(f) The Luma and color differences $Q_{Y'}$, Q_{P_B}, Q_{P_R} (Section 7.5.6).

Solution:The luma and color differences are obtained from the gamma-corrected RGB values using the equation in Section 7.5.6, giving the result $Q_{Y'} = 0.6497$, $Q_{P_B} = -0.2959$, $Q_{P_R} = 0.1462$.

* You can visualize this color in any Windows program that lets you specify the RGB values of a color. For example, in Microsoft Word, draw a shape like a rectangle and set the fill color using "More Colors – Custom" and enter the gamma-corrected red, green and blue values in the boxes.

The following pages give MATLAB code and output for this problem.

Problem 7.1

Solution using MATLAB. All sets of coordinates are displayed as a row vector to save space.

Contents

- Input
- Part (a)
- Part (b)
- Part (c)
- Part (d)
- Part (e)
- Part (f)
- Display the color

Input

Given, gamma-corrected Rec. 709 RGB values for goldenrod. This can be changed to any other input gamma-corrected color

```
clear all; close all;
QRGBprime = [218 165 32]/255
```

```
QRGBprime =

    0.8549    0.6471    0.1255
```

Part (a)

RGB tristimulus values using sRGB gamma law (section 7.5.5).

```
QRGB = ((QRGBprime+0.055)/1.055).^(2.4);
QRGB(QRGBprime < 0.04045)=QRGBprime(QRGBprime < 0.04045)/12.92;
QRGB
```

```
QRGB =

    0.7011    0.3763    0.0144
```

Part (b)

Luminance and Rec. 709 chromaticities.

```
RGBL = [.2126 .7152 .0722];  %(section 7.5.3)
QL = RGBL * QRGB'    % luminance of Q
qRGB = QRGB/([1 1 1]*QRGB')  % RGB chromaticities of Q
```

```
QL =

    0.4192


qRGB =

    0.6421    0.3446    0.0132
```

Part (c)

XYZ tristimulus values and chromaticities

```
A_rgb2xyz = [.4124 .3576 .1805; .2126 .7152 .0722; .0193 .1192 .9505]; %(section 7.5.3)
QXYZ = (A_rgb2xyz*QRGB')'  %XYZ tristimulus values
qXYZ = QXYZ/([1 1 1]*QXYZ')  %XYZ chromaticities
```

```
QXYZ =

    0.4263    0.4192    0.0721


qXYZ =

    0.4646    0.4568    0.0786
```

Part (d)

1976 U'V'W' tristimulus values and chromaticities

```
A_xyz2uvw = [4/9 0 0; 0 1 0; -1/3 2/3 1/3];  %(section 7.4.8)
QUVW = (A_xyz2uvw*QXYZ')'   %U'V'W' tristimulus values
qUVW = QUVW/([1 1 1]*QUVW')  %U'V'W' chromaticities
```

```
QUVW =

    0.1895    0.4192    0.1614


qUVW =

    0.2460    0.5444    0.2096
```

Part (e)

CIELAB coordinates (section 7.5.4). Uses functions XYZ2LAB and f_xyz2lab available on the book web site in the utilities section

```
QLAB = XYZ2LAB(QXYZ)
```

```
QLAB =

   70.8157    8.5209   68.7714
```

Part (f)

Luma and color differences

```
A3 = [.299 .587 .114; -.169 -.331 .5; .5 -.419 -.081]; %(section 7.5.6)
QYPBPR = (A3 * QRGBprime')'
```

```
QYPBPR =

    0.6497   -0.2959    0.1462
```

Display the color

```
box(:,:,1) = QRGBprime(1)*ones(128,256);
box(:,:,2) = QRGBprime(2)*ones(128,256);
box(:,:,3) = QRGBprime(3)*ones(128,256);
imshow(box)
```

Published with MATLAB® R2017b

2. As stated at the end of Section 7.4.8, the spectral absorption curves of the three types of cone photoreceptors in the human retina should be a linear combination of any set of three color-matching functions. The outputs of these receptors can be considered to be tristimulus values with respect to some set of primaries that we will call [**L**], [**M**], [**S**] (which stands for long, medium and short). It has been found that the tristimulus values with respect to these primaries for a color [**C**], denoted C_l, C_m, C_s, can be obtained from the XYZ tristimulus values by

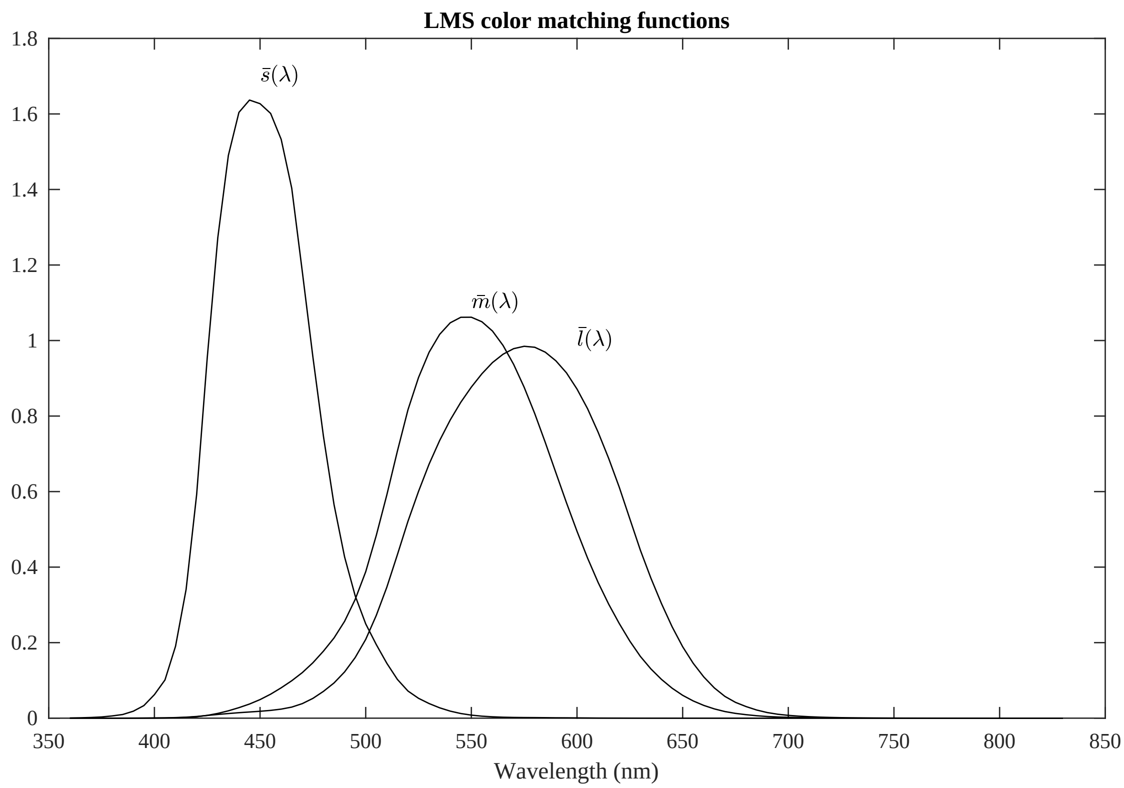
$$\begin{bmatrix} C_l \\ C_m \\ C_s \end{bmatrix} = \begin{bmatrix} 0.4002 & 0.7076 & -0.0808 \\ -0.2263 & 1.1653 & 0.0457 \\ 0.0 & 0.0 & 0.9182 \end{bmatrix} \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix}$$

(a) Determine and plot the color matching functions for the LMS primaries, denoted $\bar{l}(\lambda)$, $\bar{m}(\lambda)$, $\bar{s}(\lambda)$. The data for the xyz color-matching functions are given on the CVRL website (www.cvrl.org).

Solution: Using the notation from Chapter 7, the above equation can also be written as $\mathbf{C}_{\mathcal{LMS}} = \mathbf{A}_{\mathcal{XYZ}\to\mathcal{LMS}}\mathbf{C}_{\mathcal{XYZ}}$. Referring to Table 7.1, we see that the LMS color matching functions can be obtained using a similar equation

$$\begin{bmatrix} \bar{l}(\lambda) \\ \bar{m}(\lambda) \\ \bar{s}(\lambda) \end{bmatrix} = \begin{bmatrix} 0.4002 & 0.7076 & -0.0808 \\ -0.2263 & 1.1653 & 0.0457 \\ 0.0 & 0.0 & 0.9182 \end{bmatrix} \begin{bmatrix} \bar{x}(\lambda) \\ \bar{y}(\lambda) \\ \bar{z}(\lambda) \end{bmatrix}.$$

Using the xyz color-matching function data available from the CVRL web site and also the book web site in the data section, we obtain LMS color matching functions shown below.



(b) Express the primaries [$\mathbf{L}$], [$\mathbf{M}$], [$\mathbf{S}$] in terms of the primaries [$\mathbf{X}$], [$\mathbf{Y}$], [$\mathbf{Z}$]. What color is $[\mathbf{L}] + [\mathbf{M}] + [\mathbf{S}]$?

Solution: Referring to Table 7.1, the primaries [$\mathbf{L}$], [$\mathbf{M}$], [$\mathbf{S}$] can be obtaind from the primaries [$\mathbf{X}$], [$\mathbf{Y}$], [$\mathbf{Z}$] by multiplying by the matrix $\mathbf{A}^{-T}_{\mathcal{XYZ}\rightarrow\mathcal{LMS}}$. Thus, evaluating the inverse transpose, we obtain

$$\begin{bmatrix} [\mathbf{L}] \\ [\mathbf{M}] \\ [\mathbf{S}] \end{bmatrix} = \begin{bmatrix} 1.8601 & 0.3612 & 0 \\ -1.1295 & 0.6388 & 0 \\ 0.2199 & 0 & 1.0891 \end{bmatrix} \begin{bmatrix} [\mathbf{X}] \\ [\mathbf{Y}] \\ [\mathbf{Z}] \end{bmatrix}.$$

Using this equation, $[\mathbf{L}] + [\mathbf{M}] + [\mathbf{S}] = 0.9505[\mathbf{X}] + [\mathbf{Y}] + 1.0891[\mathbf{Z}] = [\mathbf{D}_{65}]$.

(c) Why are these primaries called [$\mathbf{L}$], [$\mathbf{M}$] and [$\mathbf{S}$]?

Solution: The primaries are called [$\mathbf{L}$], [$\mathbf{M}$] and [$\mathbf{S}$]b because the color matching functions associated with them are sensitive to the longest wavelengths, the middle range of wavelengths and the shortest wavelengths respectively, as can be seen from the graph of the color matching functions. These color matching functions are essentially the spectral sensitivities of the corresponding types of cones in the retina.

(d) Determine the LMS tristimulus values of the color *goldenrod* of Problem 1. Can you give a physical interpretation (in terms of your eye) of these tristimulus values?

Solution: The XYZ tristimulus values of $[\mathbf{GR}]$ (goldenrod) were found in question 1, giving $[\mathbf{GR}] = 0.4263[\mathbf{X}] + 0.4192[\mathbf{Y}] + 0.0721[\mathbf{Z}]$. From (b), we can find the LMS tristimulus values. Converting with $\mathbf{A}_{\mathcal{XYZ}\to\mathcal{LMS}}$ given above, we obtain

$$[\mathbf{GR}] = 0.4614[\mathbf{L}] + 0.3953[\mathbf{M}] + 0.0662[\mathbf{S}].$$

These would roughly represent the relative strength of the response of the L, M and S cones in the eye when observing the goldenrod color.

3. The Bayer color sampling strategy induces a new set of color signals from the original RGB values (assume Rec. 709) as follows:

$$\begin{bmatrix} f_L \\ f_{C1} \\ f_{C2} \end{bmatrix} = \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ -0.25 & 0.5 & -0.25 \\ -0.25 & 0.0 & 0.25 \end{bmatrix} \begin{bmatrix} f_R \\ f_G \\ f_B \end{bmatrix}$$

These can be considered to be tristimulus values with respect to a new set of primaries denoted [**L**], [**C1**], [**C2**].

(a) Determine and plot the color matching functions for the LC1C2 primaries, denoted $\bar{l}(\lambda)$, $\bar{c1}(\lambda)$, $\bar{c2}(\lambda)$. The data for the XYZ color-matching functions are given on the CVRL website (www.cvrl.org). Note that this data is also available on the book web site in the data section.

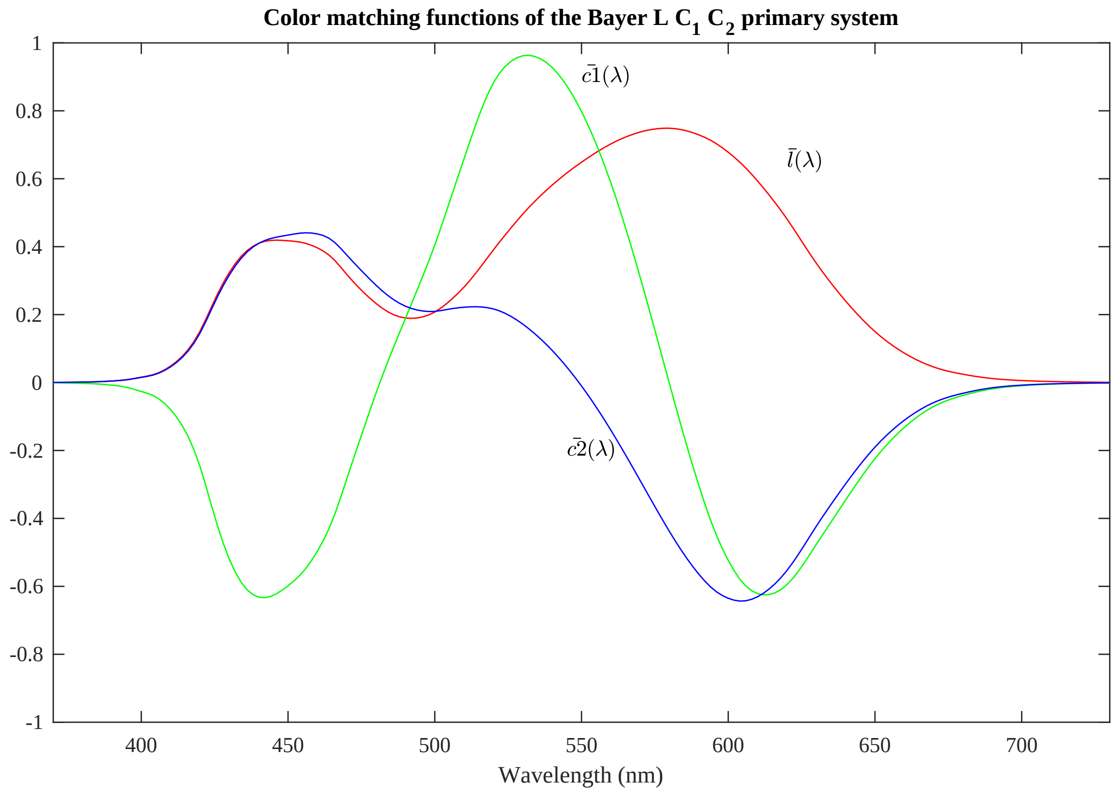
Solution: The matrix given above is $\mathbf{A}_{\mathcal{RGB}\to\mathcal{LC}1\mathcal{C}2}$. To convert known XYZ color-matching functions, we first convert them to RGB using $\mathbf{A}_{\mathcal{XYZ}\to\mathcal{RGB}}$ given in Equation (7.66), and then to LC1C2 using $\mathbf{A}_{\mathcal{RGB}\to\mathcal{LC}1\mathcal{C}2}$. This can be done in one step using the matrix

$$\mathbf{A}_{\mathcal{XYZ}\to\mathcal{LC}1\mathcal{C}2} = \mathbf{A}_{\mathcal{RGB}\to\mathcal{LC}1\mathcal{C}2}\mathbf{A}_{\mathcal{XYZ}\to\mathcal{RGB}} = \begin{bmatrix} 0.3396 & 0.5026 & 0.1604 \\ -1.3085 & 1.3732 & -0.1189 \\ -0.7962 & 0.3333 & 0.3889 \end{bmatrix}.$$

Thus we have explicitly that

$$\begin{bmatrix} \bar{l}(\lambda) \\ \bar{c1}(\lambda) \\ \bar{c2}(\lambda) \end{bmatrix} = \begin{bmatrix} 0.3396 & 0.5026 & 0.1604 \\ -1.3085 & 1.3732 & -0.1189 \\ -0.7962 & 0.3333 & 0.3889 \end{bmatrix} \begin{bmatrix} \bar{x}(\lambda) \\ \bar{y}(\lambda) \\ \bar{z}(\lambda) \end{bmatrix}.$$

These color matching functions are graphed in the following figure.



(b) Express the primaries $[\mathbf{L}]$, $[\mathbf{C1}]$, $[\mathbf{C2}]$ in terms of the primaries $[\mathbf{R}]$, $[\mathbf{G}]$, $[\mathbf{B}]$ and in terms of the primaries $[\mathbf{X}]$, $[\mathbf{Y}]$, $[\mathbf{Z}]$.

Solution: Referring to Table 7.1,

$$\begin{bmatrix} [\mathbf{L}] \\ [\mathbf{C1}] \\ [\mathbf{C2}] \end{bmatrix} = \mathbf{A}^{-T}_{\mathcal{RGB}\to\mathcal{LC}1\mathcal{C}2} \begin{bmatrix} [\mathbf{R}] \\ [\mathbf{G}] \\ [\mathbf{B}] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} [\mathbf{R}] \\ [\mathbf{G}] \\ [\mathbf{B}] \end{bmatrix}.$$

$$\begin{bmatrix} [\mathbf{L}] \\ [\mathbf{C1}] \\ [\mathbf{C2}] \end{bmatrix} = \mathbf{A}^{-T}_{\mathcal{XYZ}\to\mathcal{LC}1\mathcal{C}2} \begin{bmatrix} [\mathbf{X}] \\ [\mathbf{Y}] \\ [\mathbf{Z}] \end{bmatrix} = \begin{bmatrix} 0.9505 & 1.0000 & 1.0890 \\ -0.2353 & 0.4304 & -0.8506 \\ -0.4638 & -0.2808 & 1.8624 \end{bmatrix} \begin{bmatrix} [\mathbf{X}] \\ [\mathbf{Y}] \\ [\mathbf{Z}] \end{bmatrix}.$$

(c) Determine the LC1C2 tristimulus values of the color *goldenrod* of problem 1 and of the reference white D_{65}.

Solution: The RGB tristimulus values of Goldenrod were found in Problem 1 to be $Q_R = 0.7011$, $Q_G = 0.3763$, $Q_B = 0.0144$. Applying the transformation matrix $\mathbf{A}_{\mathcal{RGB}\to\mathcal{LC}1\mathcal{C}2}$, we find $Q_L = 0.3670$, $Q_{C1} = 0.0092$, $Q_{C2} = -0.1717$.

$[\mathbf{D}_{65}] = [\mathbf{R}] + [\mathbf{G}] + [\mathbf{B}] = [\mathbf{L}]$, so that $D65_L = 1$, $D65_{C1} = 0$, $D65_{C2} = 0$.

4. The recommendation 709 RGB primaries can be expressed in terms of the CIE XYZ primaries by

$$\begin{bmatrix} [\mathbf{R}] \\ [\mathbf{G}] \\ [\mathbf{B}] \end{bmatrix} = \begin{bmatrix} 0.4125 & 0.2127 & 0.0193 \\ 0.3576 & 0.7152 & 0.1192 \\ 0.1804 & 0.0722 & 0.9502 \end{bmatrix} \begin{bmatrix} [\mathbf{X}] \\ [\mathbf{Y}] \\ [\mathbf{Z}] \end{bmatrix}.$$

Consider the cyan, magenta and yellow (CMY) primaries used in printing. These are given by $[\mathbf{C}] = [\mathbf{B}] + [\mathbf{G}]$, $[\mathbf{M}] = [\mathbf{R}] + [\mathbf{B}]$ and $[\mathbf{YE}] = [\mathbf{R}] + [\mathbf{G}]$.

(a) Determine the *tristimulus values* of $[\mathbf{C}]$, $[\mathbf{M}]$ and $[\mathbf{YE}]$ with respect to the XYZ primaries. Compute the XYZ *chromaticities* of $[\mathbf{C}]$, $[\mathbf{M}]$ and $[\mathbf{YE}]$ and plot them on an xy chromaticity diagram. Comment on the suitability of cyan, magenta and yellow as primaries for an additive color display device like a cathode ray tube (CRT).

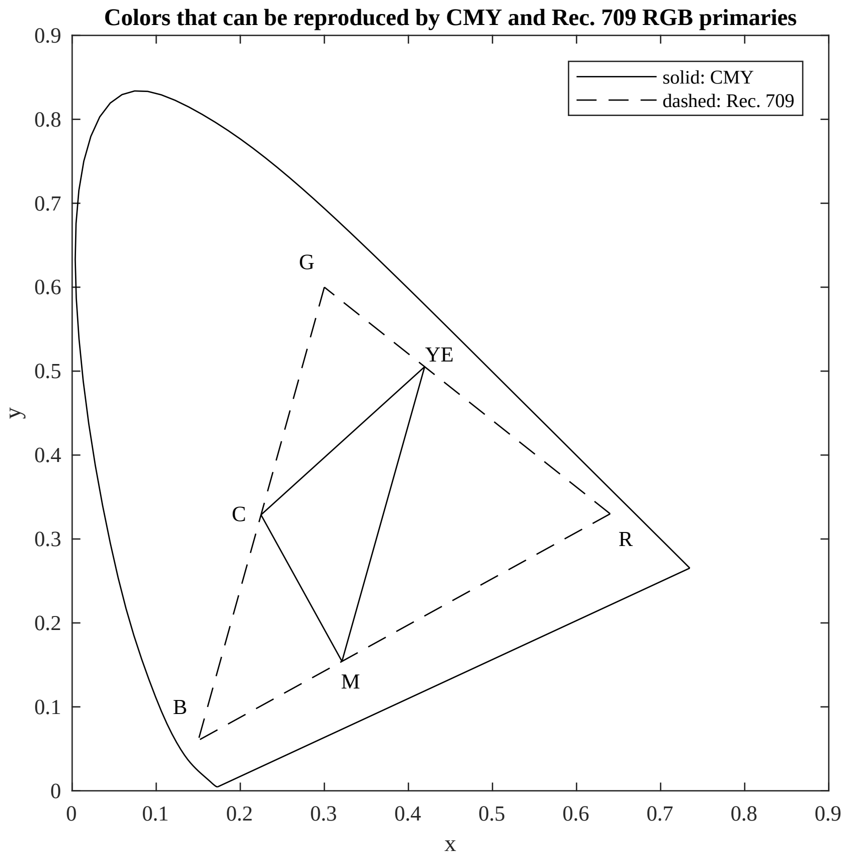
Solution: From the given equation, we find that

$$\begin{aligned} [\mathbf{C}] &= [\mathbf{B}] + [\mathbf{G}] = 0.5380[\mathbf{X}] + 0.7874[\mathbf{Y}] + 1.0694[\mathbf{Z}] \\ [\mathbf{M}] &= [\mathbf{R}] + [\mathbf{B}] = 0.5929[\mathbf{X}] + 0.2849[\mathbf{Y}] + 0.9695[\mathbf{Z}] \\ [\mathbf{YE}] &= [\mathbf{R}] + [\mathbf{G}] = 0.7701[\mathbf{X}] + 0.9279[\mathbf{Y}] + 0.1385[\mathbf{Z}] \end{aligned}$$

from which the tristimulus values of $[\mathbf{C}]$, $[\mathbf{M}]$ and $[\mathbf{YE}]$ with respect to the XYZ primaries are evident, e.g. $C_X = 0.5380$, etc. The chromaticities are found using Equation (7.56), e.g., $c_X = 0.5380/(0.5380+0.7874+1.0694) = 0.2247$, etc. Doing this for the three colors, we find

$$\begin{array}{lll} c_X = 0.2247 & c_Y = 0.3288 & c_Z = 0.4466 \\ m_X = 0.3210 & m_Y = 0.1542 & m_Z = 0.5248 \\ ye_X = 0.4193 & ye_Y = 0.5052 & ye_Z = 0.0752 \end{array}$$

These are plotted on the following xy chromaticity diagram, along with the rec709/sRGB primaries. The chromaticities that can be reproduced by a linear combination of $[\mathbf{C}]$, $[\mathbf{M}]$ and $[\mathbf{YE}]$ with positive coefficients lie in the interior of the small triangle with vertices identified C, M, YE. This is clearly a much smaller gamut than what can be reproduced with R, G and B, so cyan, magenta and yellow are not good display primaries for an additive display.



(b) Suppose that [**C**], [**M**] and [**YE**] as in part (a) are taken as primaries in a color system. Determine the tristimulus values of a monochromatic light $\delta(\lambda - 510\text{nm})$ with respect to these primaries. You will need to use the XYZ color matching functions. Carefully explain all steps. Can the given light be *physically* synthesized as a sum of a positive quantity of the [**C**], [**M**] and [**YE**] primaries?

Solution: Denote $[\mathbf{Q}] = [\delta(\lambda - 510\text{nm})]$. From the XYZ color-matching functions, we find

$$[\mathbf{Q}] = [\delta(\lambda - 510\text{nm})] = 0.0093[\mathbf{X}] + 0.5030[\mathbf{Y}] + 0.1582[\mathbf{X}],$$

i.e., $Q_X = 0.0093$, $Q_Y = 0.5030$, $Q_Z = 0.1582$. We are asked to find Q_C, Q_M and Q_{YE}.

To do this, we need the matrix $\mathbf{A}_{\mathcal{XYZ}\to\mathcal{CMY}}$. From part (a), we indentify

$$\begin{bmatrix}[\mathbf{C}]\\ [\mathbf{M}]\\ [\mathbf{YE}]\end{bmatrix} = \begin{bmatrix}0.5380 & 0.7874 & 1.0694\\ 0.5929 & 0.2849 & 0.9695\\ 0.7701 & 0.9279 & 0.1385\end{bmatrix}\begin{bmatrix}[\mathbf{X}]\\ [\mathbf{Y}]\\ [\mathbf{Z}]\end{bmatrix}$$

$$= \mathbf{A}^T_{\mathcal{CMY}\to\mathcal{XYZ}}\begin{bmatrix}[\mathbf{X}]\\ [\mathbf{Y}]\\ [\mathbf{Z}]\end{bmatrix}.$$

Thus, taking the inverse transpose,

$$\mathbf{A}_{\mathcal{XYZ}\to\mathcal{CMY}} = \begin{bmatrix}-2.0766 & 1.6043 & 0.7986\\ 2.1324 & -1.8084 & 0.2588\\ 1.1076 & 0.2715 & -0.7571\end{bmatrix}.$$

Thus finally

$$\mathbf{Q}_{\mathcal{CMY}} = \begin{bmatrix}-2.0766 & 1.6043 & 0.7986\\ 2.1324 & -1.8084 & 0.2588\\ 1.1076 & 0.2715 & -0.7571\end{bmatrix}\begin{bmatrix}0.0093\\ 0.5030\\ 0.1582\end{bmatrix} = \begin{bmatrix}0.9140\\ -0.8489\\ 0.0271\end{bmatrix}.$$

One tristimulus value is negative, so this light *cannot* be physically synthesized as a positive linear combination of the CMY primaries.

(c) Determine the tristimulus values of the color *goldenrod* of question 1 with respect to the [**C**], [**M**] and [**YE**] primaries. Can this color be *physically* synthesized as a sum of a positive quantity of the [**C**], [**M**] and [**YE**] primaries?

Solution: For the color goldenrod [**GR**], it was found in Problem 1 that $\mathbf{GR}_{\mathcal{XYZ}} = [0.4623, 0.4192, 0.0721]'$. Then, $\mathbf{GR}_{\mathcal{CMY}} = \mathbf{A}_{\mathcal{XYZ}\to\mathcal{CMY}}\mathbf{GR}_{\mathcal{XYZ}} = [-0.2299, 0.2464, 0.5713]'$. This color has a negative CMY tristimulus value and so *cannot* be physically synthesized as a positive linear combination of the CMY primaries.

5. The EBU (European Broadcasting Union) primaries, have the following specification

	Red	Green	Blue	White D_{65}
x	0.640	0.290	0.150	0.3127
y	0.330	0.600	0.060	0.3290
z	0.030	0.110	0.790	0.3582

Assume that the reference white has unit luminance $D_L = 1.0$ and that $[\mathbf{R}] + [\mathbf{G}] + [\mathbf{B}] = [\mathbf{D}_{65}]$.

(a) Find the XYZ tristimulus values of the reference white $[\mathbf{D}_{65}]$, i.e., D_X, D_Y and D_Z.

Solution: This simply reproduces what was done in Section 7.5.3, using Equation (7.58)-(7.60). Repeating that with the values in the above table, referring to reference white as simply $[\mathbf{D}]$ for this problem: $d_X = 0.3127$, $d_Y = 0.3290$, $d_Z = 0.3582$. Thus

$$D_X = \frac{d_X}{d_Y} = 0.9505 \qquad D_Y = 1.0 \qquad D_Z = \frac{d_Z}{d_Y} = 1.0888.$$

(b) Using $[\mathbf{R}] + [\mathbf{G}] + [\mathbf{B}] = [\mathbf{D}_{65}]$, determine the luminances of the three primaries, R_L, G_L and B_L.

Solution: This follows the same procedure used in Section 7.5.3 for the Rec709/sRGB primaries. Use Equations (7.58)-(7.60) to express the tristimulus values of $[\mathbf{R}]$, $[\mathbf{G}]$ and $[\mathbf{B}]$ in terms of R_L, G_L and B_L, e.g. $R_X = \frac{R_L r_X}{r_Y}$, etc, substitute these into $[\mathbf{R}]+[\mathbf{G}]+[\mathbf{B}] = [\mathbf{D}_{65}]$ and write in matrix form to obtain

$$\begin{bmatrix} 1.\overline{93} & 0.48\overline{3} & 2.5 \\ 1.0 & 1.0 & 1.0 \\ 0.\overline{09} & 0.18\overline{3} & 13.1\overline{6} \end{bmatrix} \begin{bmatrix} R_L \\ G_L \\ B_L \end{bmatrix} = \begin{bmatrix} 0.9505 \\ 1.0 \\ 1.0888 \end{bmatrix}.$$

Solving the matrix equation, we obtain $R_L = 0.2220$, $G_L = 0.7066$, $B_L = 0.0713$.

(c) Now find the XYZ tristimulus values of the three primaries, i.e. if $[\mathbf{R}] = R_X[\mathbf{X}] + R_Y[\mathbf{Y}] + R_Z[\mathbf{Z}]$, find R_X, R_Y, R_Z, and similarly for $[\mathbf{G}]$ and $[\mathbf{B}]$.

Solution: Insert the values of R_L, G_L and B_L found in (b) into the expressions $R_X = \frac{R_L r_X}{r_Y}$, etc, to find the requested tristimulus values. Expressed in matrix form

$$\begin{bmatrix}[\mathbf{R}]\\ [\mathbf{G}]\\ [\mathbf{B}]\end{bmatrix} = \begin{bmatrix} R_X & R_Y & R_Z\\ G_X & G_Y & G_Z\\ B_X & B_Y & B_Z\end{bmatrix}\begin{bmatrix}[\mathbf{X}]\\ [\mathbf{Y}]\\ [\mathbf{Z}]\end{bmatrix} = \begin{bmatrix}0.4306 & 0.2220 & 0.0202\\ 0.3415 & 0.7066 & 0.1296\\ 0.1783 & 0.0713 & 0.9390\end{bmatrix}\begin{bmatrix}[\mathbf{X}]\\ [\mathbf{Y}]\\ [\mathbf{Z}]\end{bmatrix}.$$

(d) If an arbitrary color $[\mathbf{Q}]$ is written

$$[\mathbf{Q}] = Q_X[\mathbf{X}] + Q_Y[\mathbf{Y}] + Q_Z[\mathbf{Z}] = Q_R[\mathbf{R}] + Q_G[\mathbf{G}] + Q_B[\mathbf{B}]$$

determine the matrix relations to find Q_X, Q_Y, Q_Z from Q_R, Q_G, Q_B and vice-versa.

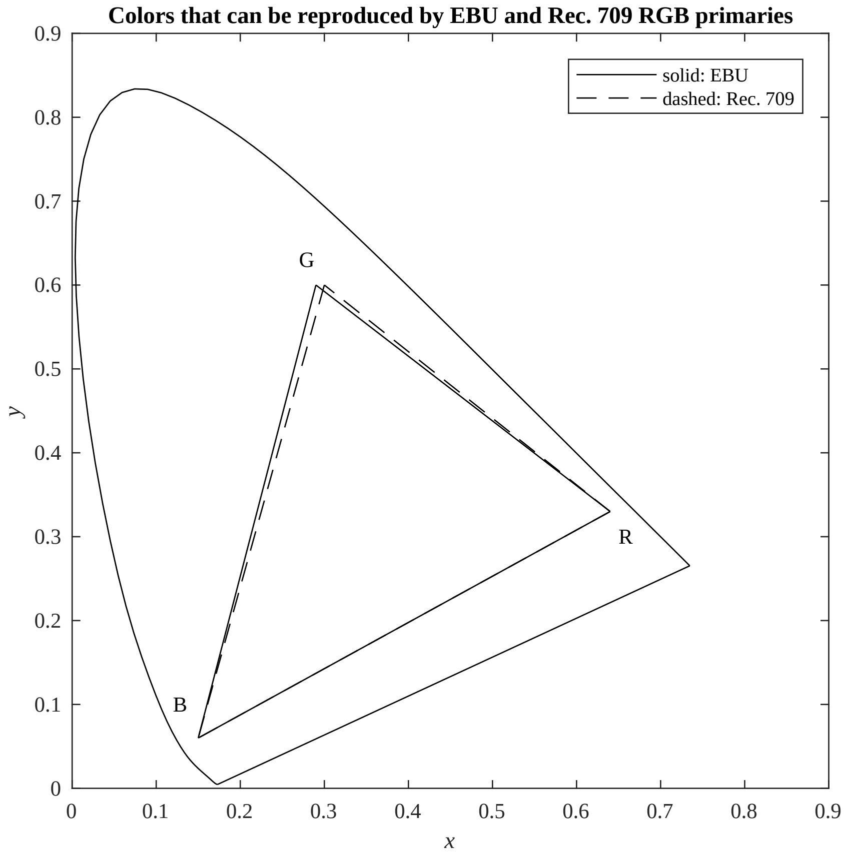
Solution: Referring to Table 7.1, entry 6, the matrix found in (c) can be named $\mathbf{A}^T_{\mathcal{RGB}\to\mathcal{XYZ}}$. This problem asks us to find $\mathbf{A}_{\mathcal{RGB}\to\mathcal{XYZ}}$ and $\mathbf{A}_{\mathcal{XYZ}\to\mathcal{RGB}}$ respectively, where the latter is given by $\mathbf{A}_{\mathcal{XYZ}\to\mathcal{RGB}} = \mathbf{A}^{-1}_{\mathcal{RGB}\to\mathcal{XYZ}}$. Thus, performing the transpose and the inverse, we obtain in matrix form

$$\begin{bmatrix}Q_X\\ Q_Y\\ Q_Z\end{bmatrix} = \begin{bmatrix}0.4306 & 0.3415 & 0.1783\\ 0.2220 & 0.7066 & 0.0713\\ 0.0202 & 0.1296 & 0.9390\end{bmatrix}\begin{bmatrix}Q_R\\ Q_G\\ Q_B\end{bmatrix}$$

$$\begin{bmatrix}Q_R\\ Q_G\\ Q_B\end{bmatrix} = \begin{bmatrix}3.0629 & -1.3932 & -0.4758\\ -0.9693 & 1.8760 & 0.0416\\ 0.0679 & -0.2289 & 1.0694\end{bmatrix}\begin{bmatrix}Q_X\\ Q_Y\\ Q_Z\end{bmatrix}$$

(e) Plot an xy chromaticity diagram showing the triangles of chromaticities reproducible with the EBU RGB primaries.

Solution: The xy chromaticities of $[\mathbf{R}]$, $[\mathbf{G}]$ and $[\mathbf{B}]$ are given in the problem statement. The set of all xy chromaticities reproducible with an additive linear combination (with positive coefficients) of the EBU RGB primaries is the triangle whose vertices are the xy chromaticities of $[\mathbf{R}]$, $[\mathbf{G}]$ and $[\mathbf{B}]$. This triangle is shown on the chromaticity diagram that follows. For comparison, the colors reproducible by the Rec709/sRGB primaries as given in Section 7.5.3 are shown as well.

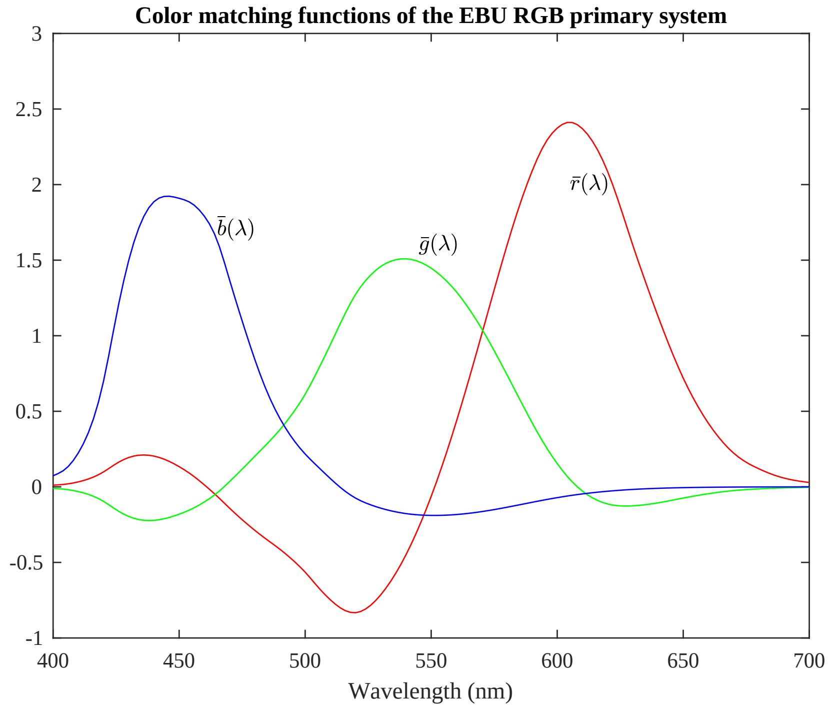


(f) Compute and plot the color matching functions for the EBU RGB primaries by transforming the XYZ color matching functions using the results of (d).

Solution: The EBU color matching functions are obtained by transforming the XYZ color matching functions using $\mathbf{A}_{\mathcal{XYZ}\to\mathcal{RGB}}$ as found in part (d), i.e.,

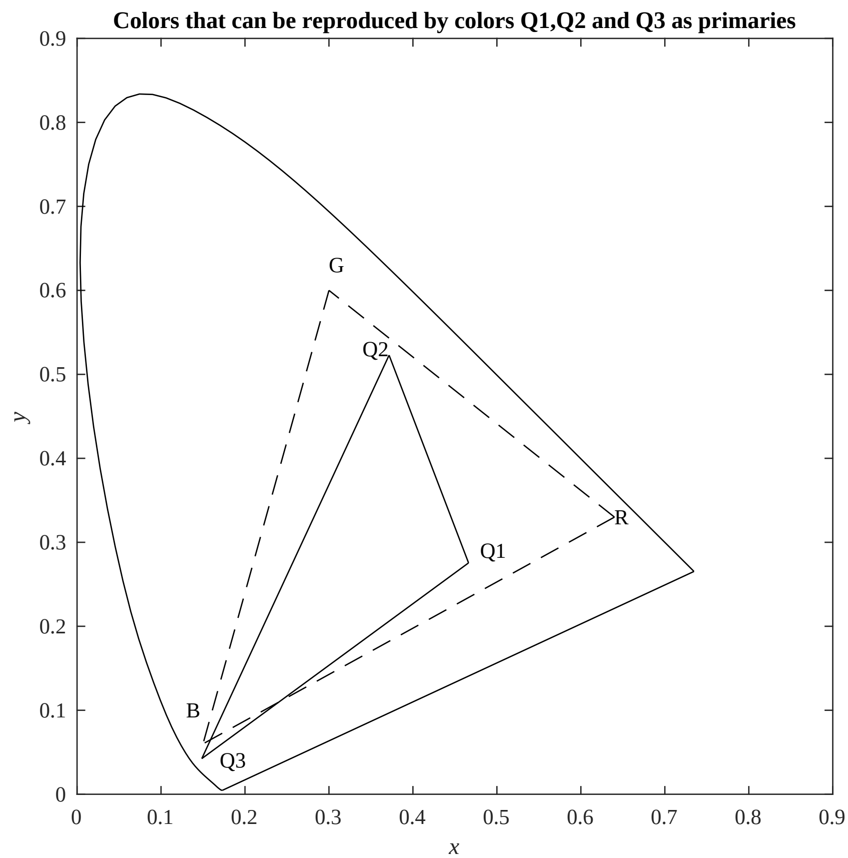
$$\begin{bmatrix} \bar{r}(\lambda) \\ \bar{g}(\lambda) \\ \bar{b}(\lambda) \end{bmatrix} = \begin{bmatrix} 3.0629 & -1.3932 & -0.4758 \\ -0.9693 & 1.8760 & 0.0416 \\ 0.0679 & -0.2289 & 1.0694 \end{bmatrix} \begin{bmatrix} \bar{x}(\lambda) \\ \bar{y}(\lambda) \\ \bar{z}(\lambda) \end{bmatrix}.$$

The result is shown in the following plot.



(g) For the three spectral densities $Q_1(\lambda)$, $Q_2(\lambda)$ and $Q_3(\lambda)$ in the following table, compute their chromaticities and plot them on an xy chromaticity diagram. Would they make good primaries for a physical color image synthesis system such as a CRT? Explain.

Solution: To estimate the chromaticities of the colors $[\mathbf{Q}_1]$, $[\mathbf{Q}_2]$, $[\mathbf{Q}_3]$, we have interpolated the given spectral densities $Q_1(\lambda)$, $Q_2(\lambda)$ and $Q_3(\lambda)$ to a wavelength spacing of 2 nm and also the XYZ color matching functions. We compute unnormalized tristimulus values by expressions such as $KQ_{i,X} = \sum_j Q_i(\lambda_j)\bar{x}(\lambda_j)$ to approximate the integral up to a constant factor K. The constant factor K will cancel when calculating chromaticities. The resulting chromaticities of $[\mathbf{Q}_1]$, $[\mathbf{Q}_2]$, $[\mathbf{Q}_3]$ are shown on the following diagram. Chromaticities of the RGB primaries are shown for comparison. Thus, $[\mathbf{Q}_1]$, $[\mathbf{Q}_2]$, $[\mathbf{Q}_3]$ can reproduce a significantly smaller gamut of chromaticities than $[\mathbf{R}]$, $[\mathbf{G}]$ and $[\mathbf{B}]$.


Colors that can be reproduced by colors Q1,Q2 and Q3 as primaries
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
y
G
Q2
R
Q1
B
Q3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
x

Q_1	Q_2	Q_3	λ
.19	.00	.60	400 nm
.20	.00	.63	
.20	.00	.64	
.20	.00	.63	
.20	.00	.62	
.20	.02	.59	450 nm
.20	.06	.53	
.19	.19	.43	
.18	.31	.31	
.16	.43	.20	
.13	.52	.10	500 nm
.08	.61	.05	
.06	.67	.02	
.04	.69	.01	
.03	.69	.00	
.04	.67	.00	550 nm
.08	.64	.00	
.14	.60	.00	
.22	.55	.00	
.32	.49	.00	
.41	.43	.00	600 nm
.50	.38	.00	
.56	.33	.00	
.63	.28	.00	
.67	.25	.00	
.71	.23	.00	650 nm
.75	.21	.00	
.77	.20	.00	
.79	.19	.00	
.80	.19	.00	
.81	.18	.00	700 nm

Chapter 9

Random Field Models

1. Continuous-domain separable exponential autocorrelation. Show that if

$$R_f(x, y) = \sigma_f^2 \exp(-\gamma_1|x|)\exp(-\gamma_2|y|),$$

then power-density spectrum is given by

$$S_f(u, v) = \sigma_f^2 \frac{2\gamma_1}{\gamma_1^2 + 4\pi^2u^2} \frac{2\gamma_2}{\gamma_2^2 + 4\pi^2v^2}.$$

Solution: Note that we must have $\gamma_1 > 0$, $\gamma_2 > 0$. Then, by the definition

$$\begin{aligned} S_f(u, v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_f^2 \exp(-\gamma_1|x|)\exp(-\gamma_2|y|)\exp(-j2\pi(ux + vy))\, dxdy \\ &= \sigma_f^2 \int_{-\infty}^{\infty} \exp(-\gamma_1|x| - j2\pi ux)\, dx \int_{-\infty}^{\infty} \exp(-\gamma_2|y| - j2\pi vy)\, dy. \end{aligned}$$

Both integrals have the same form. Denote the integral over x as I. Then

$$\begin{aligned} I &= \int_{-\infty}^{0} \exp(\gamma_1 x - j2\pi ux)\, dx + \int_{0}^{\infty} \exp(-\gamma_1 x - j2\pi ux)\, dx \\ &= \left.\frac{\exp(\gamma_1 x - j2\pi ux)}{\gamma_1 - j2\pi u}\right|_{-\infty}^{0} + \left.\frac{\exp(-\gamma_1 x - j2\pi ux)}{-\gamma_1 - j2\pi u}\right|_{0}^{\infty} \\ &= \frac{1}{\gamma_1 - j2\pi u} + \frac{-1}{-\gamma_1 - j2\pi u} \\ &= \frac{\gamma_1 + j2\pi u + \gamma_1 - j2\pi u}{(\gamma_1 - j2\pi u)(\gamma_1 + j2\pi u)} \\ &= \frac{2\gamma_1}{\gamma_1^2 + 4\pi^2u^2}. \end{aligned}$$

With a simiar method for the integral over y, we obtain

$$S_f(u, v) = \sigma_f^2 \frac{2\gamma_1}{\gamma_1^2 + 4\pi^2u^2} \frac{2\gamma_2}{\gamma_2^2 + 4\pi^2v^2}.$$

2. Discrete-domain separable exponential autocorrelation. Show that if

$$R_f[mX, nX] = \sigma_f^2 \rho_1^{|m|} \rho_2^{|n|},$$

then the power density spectrum is separable and is given by

$$S_f(u,v) = \sigma_f^2 \frac{1-\rho_1^2}{(1+\rho_1^2) - 2\rho_1 \cos(2\pi uX)} \frac{1-\rho_2^2}{(1+\rho_2^2) - 2\rho_2 \cos(2\pi vX)}.$$

Each term is the sum of two geometric series for positive and negative values of the independent variable.

Solution: Note that we need $|\rho_i| < 1$ for $i = 1, 2$. Then

$$\begin{aligned} S_f(u,v) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sigma_f^2 \rho_1^{|m|} \rho_2^{|n|} \exp(-j2\pi(umX + vnX)) \\ &= \sigma_f^2 \sum_{m=-\infty}^{\infty} \rho_1^{|m|} \exp(-j2\pi umX) \sum_{n=-\infty}^{\infty} \rho_2^{|n|} \exp(-j2\pi vnX) \end{aligned}$$

Both sums have the same form. Denote the sum over m as Σ. Then

$$\begin{aligned} \Sigma &= \sum_{m=0}^{\infty} \rho_1^m \exp(-j2\pi umX) + \sum_{m=-\infty}^{0} \rho_1^{-m} \exp(-j2\pi umX) - 1 \\ &= \sum_{m=0}^{\infty} (\rho_1 \exp(-j2\pi uX))^m + \sum_{k=0}^{\infty} (\rho_1 \exp(j2\pi uX))^k - 1 \\ &= \frac{1}{1-\rho_1 \exp(-j2\pi uX)} + \frac{1}{1-\rho_1 \exp(j2\pi uX)} - 1 \\ &= \frac{1-\rho_1 \exp(j2\pi uX) + 1 - \rho_1 \exp(-j2\pi uX) - (1+\rho_1^2 - 2\rho_1 \cos(2\pi uX))}{(1-\rho_1 \exp(-j2\pi uX))(1-\rho_1 \exp(j2\pi uX))} \\ &= \frac{2 - 2\rho_1 \cos(2\pi uX) - 1 - \rho_1^2 + 2\rho_1 \cos(2\pi uX)}{1+\rho^2 - 2\rho_1 \cos(2\pi uX)} \\ &= \frac{1-\rho_1^2}{1+\rho_1^2 - 2\rho_1 \cos(2\pi uX)} \end{aligned}$$

Thus, applying the same result to the second sum and combining,

$$S_f(u,v) = \sigma_f^2 \frac{1-\rho_1^2}{(1+\rho_1^2) - 2\rho_1 \cos(2\pi uX)} \frac{1-\rho_2^2}{(1+\rho_2^2) - 2\rho_2 \cos(2\pi vX)}.$$

3. Prove that the spectral density matrix estimate $\widehat{\mathbf{Q}}(\mathbf{u})$ of Eq. (9.46) is equal to the discrete-domain Fourier transform of the windowed sample autocorrelation $\mathbf{Q}[\mathbf{x}]$ given in Eq. (9.47).

   Solution: From Equation (9.46), $\widehat{\mathbf{Q}}(\mathbf{u}) = \frac{1}{K}\widehat{\mathbf{V}}(\mathbf{u})\widehat{\mathbf{V}}^H(\mathbf{u})$, where

$$\widehat{\mathbf{V}}(\mathbf{u}) = \sum_{\mathbf{x}\in\mathcal{B}} w[\mathbf{x}]\mathbf{C}[\mathbf{x}]\exp(-j2\pi\mathbf{u}\cdot\mathbf{x}),$$

   and $\mathcal{B}$ is the region of support of w. We recall that $\mathbf{C}$ is real. From Equation (9.47),

$$\mathbf{Q}[\mathbf{x}] = \frac{1}{K}\sum_{\mathbf{z}\in\mathcal{B}} w[\mathbf{z}+\mathbf{x}]\mathbf{C}[\mathbf{z}+\mathbf{x}]\mathbf{C}^T[\mathbf{z}]w[\mathbf{z}].$$

   We want to show that $Q_{ij}[\mathbf{x}] \overset{\mathsf{DDFT}}{\longleftrightarrow} \widehat{Q}_{ij}(\mathbf{u})$.

   Written out explicitly

$$Q_{ij}[\mathbf{x}] = \frac{1}{K}\sum_{\mathbf{z}\in\Lambda} w[\mathbf{z}+\mathbf{x}]C_i[\mathbf{z}+\mathbf{x}]C_j[\mathbf{z}]w[\mathbf{z}]$$

   and

$$\widehat{Q}_{ij}(\mathbf{u}) = \frac{1}{K}\widehat{V}_i(\mathbf{u})\widehat{V}_j^*(\mathbf{u}) \quad \text{where} \quad \widehat{V}_i(\mathbf{u}) = \sum_{\mathbf{x}\in\Lambda} w[\mathbf{x}]C_i[\mathbf{x}]\exp(-j2\pi\mathbf{u}\cdot\mathbf{x})$$

   since $w[\mathbf{x}] = 0$ for $\mathbf{x} \in \Lambda\backslash\mathcal{B}$. To simplify notation, let $f_i[\mathbf{x}] = w[\mathbf{x}]C_i[\mathbf{x}]$ and let

$$g[\mathbf{x}] = \sum_{\mathbf{z}\in\Lambda} f_i[\mathbf{z}+\mathbf{x}]f_j[\mathbf{z}] = KQ_{ij}[\mathbf{x}].$$

   Taking the discrete-domain Fourier transform, $G(\mathbf{u}) = F_i(\mathbf{u})F_j^*(\mathbf{u}) = \widehat{V}_i(\mathbf{u})\widehat{V}_j^*(\mathbf{u})$. (See Section 9.5.1 where we show that the Fourier transform of $\sum_{\mathbf{z}\in\Lambda} h[\mathbf{z}]q[\mathbf{x}+\mathbf{z}]$ is $H^*(\mathbf{u})Q(\mathbf{u})$ when h is real.) Thus, reverting to the original notation

$$KQ_{ij}[\mathbf{x}] \overset{\mathsf{DDFT}}{\longleftrightarrow} K\widehat{Q}_{ij}(\mathbf{u}).$$

Chapter 10

Analysis and Design of Multidimensional FIR Filters

1. Derive in detail the expression for the frequency response of a moving average filter on a rectangular lattice, with a rectangular region of support, as given in Eq. (10.9).

   Solution:

$$h_{MA}[\mathbf{x}] = \begin{cases} \frac{1}{|\mathcal{B}|} & \text{if } \mathbf{x} \in \mathcal{B}, \\ 0 & \text{otherwise.} \end{cases}$$

   where $\Lambda = \mathrm{LAT}\left(\left[\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right]\right)$, $\mathcal{B} = \{(n_1X, n_2Y) \mid -L \leq n_1 \leq L, -L \leq n_2 \leq L\}$, and $|\mathcal{B}| = (2L+1)^2$.

$$\begin{aligned} H(u,v) &= \frac{1}{(2L+1)^2} \sum_{n_1=-L}^{L} \sum_{n_2=-L}^{L} \exp(-j2\pi(un_1X + vn_2Y)) \\ &= \frac{1}{2L+1} \sum_{n_1=-L}^{L} \exp(-j2\pi un_1X) \frac{1}{2L+1} \sum_{n_2=-L}^{L} \exp(-j2\pi vn_2Y), \end{aligned}$$

   which is separable as expected. Both terms have the same form. Let

$$S = \sum_{n=-L}^{L} \exp(-j2\pi wnZ),$$

   which is seen to be a geometric series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} = \frac{a - rl}{1-r}.$$

   Here, $a = \exp(j2\pi wLZ)$, $r = \exp(-j2\pi wZ)$ and $l = ar^{n-1} = \exp(-j2\pi wLZ)$. Thus

$$\begin{aligned} S &= \frac{\exp(j2\pi wLZ) - \exp(-j2\pi wZ)\exp(-j2\pi wLZ)}{1 - \exp(-j2\pi wZ)} \\ &= \frac{\exp(-j\pi wZ)(\exp(j\pi w(2L+1)Z) - \exp(-j\pi w(2L+1)Z)}{\exp(-j\pi wZ)(\exp(j\pi wZ) - \exp(-j\pi wZ))} \\ &= \frac{2j\sin(\pi w(2L+1)Z)}{2j\sin(\pi wZ)}. \end{aligned}$$

   Substituting in the two terms

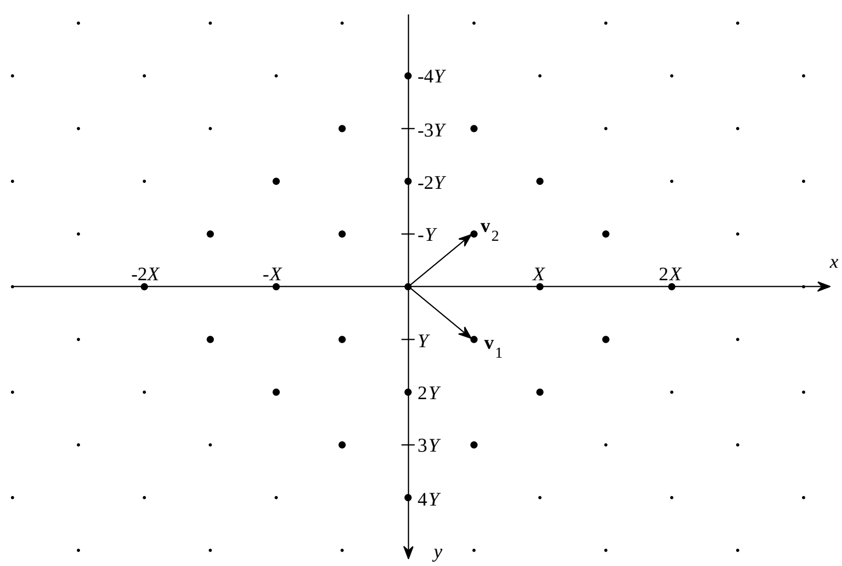
$$H(u,v) = \frac{1}{(2L+1)^2} \frac{\sin(\pi u(2L+1)X)\sin(\pi v(2L+1)Y)}{\sin(\pi uX)\sin(\pi vY)}.$$

2. Determine the frequency response of a moving average filter for a hexagonal lattice with a diamond-shaped region of support.

   Solution: A sampling matrix for a hexagonal lattice as given in Equation (3.5) is

$$\mathbf{V}_H = \begin{bmatrix} X & X/2 \\ 0 & Y \end{bmatrix}.$$

   The points of such a lattice are shown in the figure below, along with an example of a diamond-shaped region of support shown as larger dots.



   To specify the diamond-shaped region of support, it is more convenient to use the equivalent sampling matrix

$$\mathbf{V} = \begin{bmatrix} X/2 & X/2 \\ Y & -Y \end{bmatrix}.$$

   (To check these are equivalent, note that $\mathbf{V}^{-1}\mathbf{V}_H = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ which is an integer matrix with absolute value of determinant equal to 1.) The corresponding basis vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are shown in the figure. With these basis vectors, we can specify the diamond-shaped region of support as

$$\begin{aligned}\mathcal{A} &= \{n_1\mathbf{v}_1 + n_2\mathbf{v}_2 \mid -L \le n_1 \le L, -L \le n_2 \le L\} \\ &= \left\{(n_1+n_2)\tfrac{X}{2}, (n_1-n_2)Y \mid -L \le n_1 \le L, -L \le n_2 \le L\right\}\end{aligned}$$

where L is a positive integer. It is evident that $|\mathcal{A}| = (2L+1)^2$.

The frequency response of the moving average filter is then given by

$$\begin{aligned}H(u,v) &= \frac{1}{(2L+1)^2}\sum_{n_1=-L}^{L}\sum_{n_2=-L}^{L}\exp(-j2\pi(u(n_1+n_2)\tfrac{X}{2} + v(n_1-n_2)Y)) \\ &= \frac{1}{(2L+1)^2}\sum_{n_1=-L}^{L}\exp(-j2\pi(u\tfrac{X}{2}+vY)n_1)\sum_{n_2=-L}^{L}\exp(-j2\pi(u\tfrac{X}{2}-vY)n_2).\end{aligned}$$

Referring to the solution to Problem 1, we have the result

$$S = \sum_{n=-L}^{L}\exp(-j2\pi wZn) = \frac{\sin(\pi(2L+1)wZ)}{\sin(\pi wZ)}.$$

Replacing wZ by $u\frac{X}{2}+vY$ and $u\frac{X}{2}-vY$ respectively in the two sums defining $H(u,v)$, we obtain

$$H(u,v) = \frac{1}{(2L+1)^2}\frac{\sin(\pi(2L+1)(u\frac{X}{2}+vY))\sin(\pi(2L+1)(u\frac{X}{2}-vY))}{\sin(\pi(u\frac{X}{2}+vY))\sin(\pi(u\frac{X}{2}-vY))}.$$

We note that $h[\mathbf{x}]$ is separable along the directions $\mathbf{v}_1$ and $\mathbf{v}_2$,

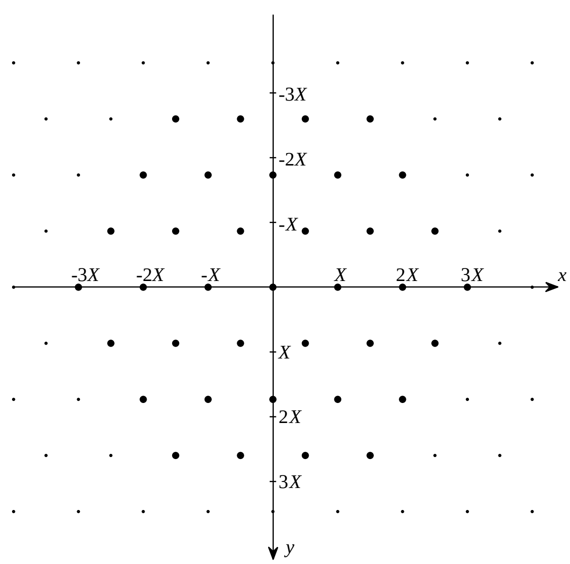
$$\begin{aligned}h[n_1\mathbf{v}_1+n_2\mathbf{v}_2] &= h_1[n_1\mathbf{v}_1]h_2[n_2\mathbf{v}_2] \\ &= \operatorname{rect}(\tfrac{n_1}{2L})\operatorname{rect}(\tfrac{n_2}{2L}).\end{aligned}$$

We also note that $H(\mathbf{u})$ is separable along the directions $\mathbf{w}_1$ and $\mathbf{w}_2$, where $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2] = \mathbf{V}^{-T}$ is this specific reciprocal sampling matrix for Λ^*. This follows since using these directions as basis for $\mathbb{R}^2$

$$\begin{aligned}\begin{bmatrix}u\\v\end{bmatrix} &= \underbrace{\begin{bmatrix}\frac{1}{X} & \frac{1}{X}\\ \frac{1}{2Y} & -\frac{1}{2Y}\end{bmatrix}}_{\mathbf{W}}\begin{bmatrix}u\frac{X}{2}+vY\\ u\frac{X}{2}-vY\end{bmatrix} \\ &= (u\tfrac{X}{2}+vY)\mathbf{w}_1 + (u\tfrac{X}{2}-vY)\mathbf{w}_2.\end{aligned}$$

3. A square image (pw = ph) is sampled on the hexagonal lattice Λ generated by the sampling matrix
$$\mathbf{V} = \begin{bmatrix} X & X/2 \\ 0 & \sqrt{3}X/2 \end{bmatrix}$$
where $X = 1/512$ ph. Design a Gaussian FIR filter with unit sample response $h[\mathbf{x}] = c\exp(-\|\mathbf{x}\|^2/2r^2)$ for $\mathbf{x} \in \mathcal{A}$ having a 3dB bandwidth of $0.2/X$ c/ph. The region of support of the FIR filter is $\mathcal{A} = \{\mathbf{x} \in \Lambda \mid \|\mathbf{x}\| \leq 3X\}$. Having determined the correct values of c and r, give the coefficients of the filter. Give an analytical approximation for the frequency response of the filter. Make a contour plot and a perspective plot of the frequency response of the filter over the frequency range $-2/X \leq u \leq 2/X$, $-4/(\sqrt{3}X) \leq v \leq 4/(\sqrt{3}X)$. Sketch (by hand if you wish) on the contour plot the points of the reciprocal lattice Λ^* and a Voronoi unit cell of Λ^*, and comment on the periodicity of the frequency response. Recall that the Voronoi unit cell consists of all points in Λ^* closer to the origin than to any other point of Λ^*.

   Solution: Note that this filter is used as an example in Section 12.1.2. There are 37 points in Λ such $x^2 + y^2 \leq 9$. These are illustrated as larger dots in the following figure. Thus we need to determine 37 filter coefficients.



Given a 3 dB bandwidth of $u_c = \frac{0.2}{X}$ c/ph, according to Equation (10.13), we should choose $r = \frac{0.1325}{0.2/X} = 0.6625X$. Thus,

$$h[x,y] = c\exp(-(x^2+y^2)/0.8778X^2), \quad (x,y) \in \mathcal{A}.$$

Using Equation (10.11) to set the DC gain to 1, we find $c = 0.31402$. Note that using the approximation for c, we obtain $c \approx d(\Lambda)/2\pi r^2 = 0.31404$, so the approximation is quite accurate. We may now calculate the 37 non-zero coefficients of the Gaussian filter. Due to the circular symmetry, there are only six distinct values corresponding to six distinct values of x^2+y^2 within $\mathcal{A}$. These filter coefficients are enumerated as follows:

$$\begin{aligned}
h[0,0] &= 0.3140 \\
h[X,0] &= h[-X,0] = h[\tfrac{X}{2}, \tfrac{\sqrt{3}X}{2}] = h[-\tfrac{X}{2}, \tfrac{\sqrt{3}X}{2}] = h[\tfrac{X}{2}, -\tfrac{\sqrt{3}X}{2}] = h[-\tfrac{X}{2}, -\tfrac{\sqrt{3}X}{2}] \\
&= 0.3140\exp(-1/0.8778) = 0.1005 \\
h[\tfrac{3X}{2}, \tfrac{\sqrt{3}X}{2}] &= h[-\tfrac{3X}{2}, \tfrac{\sqrt{3}X}{2}] = h[\tfrac{3X}{2}, -\tfrac{\sqrt{3}X}{2}] = h[-\tfrac{3X}{2}, -\tfrac{\sqrt{3}X}{2}] = h[0, \sqrt{3}X] \\
&= h[0, -\sqrt{3}X] = 0.3140\exp(-3/0.8778) = 0.0103 \\
h[2X,0] &= h[-2X,0] = h[X, \sqrt{3}X] = h[-X, \sqrt{3}X] = h[X, -\sqrt{3}X] = h[-X, -\sqrt{3}X] \\
&= 0.3140\exp(-4/0.8778) = 0.00330 \\
h[\tfrac{5X}{2}, \tfrac{\sqrt{3}X}{2}] &= h[-\tfrac{5X}{2}, \tfrac{\sqrt{3}X}{2}] = h[\tfrac{5X}{2}, -\tfrac{\sqrt{3}X}{2}] = h[-\tfrac{5X}{2}, -\tfrac{\sqrt{3}X}{2}] \\
&= h[2X, \sqrt{3}X] = h[-2X, \sqrt{3}X] = h[2X, -\sqrt{3}X] = h[-2X, -\sqrt{3}X] \\
&= h[\tfrac{X}{2}, \tfrac{3\sqrt{3}X}{2}] = h[-\tfrac{X}{2}, \tfrac{3\sqrt{3}X}{2}] = h[\tfrac{X}{2}, -\tfrac{3\sqrt{3}X}{2}] = h[-\tfrac{X}{2}, -\tfrac{3\sqrt{3}X}{2}] \\
&= 0.3140\exp(-7/0.8778) = 1.0808 \times 10^{-4} \\
h[3X,0] &= h[-3X,0] = h[\tfrac{3X}{2}, \tfrac{3\sqrt{3}}{2}] = h[-\tfrac{3X}{2}, \tfrac{3\sqrt{3}}{2}] = h[\tfrac{3X}{2}, -\tfrac{3\sqrt{3}}{2}] \\
&= h[-\tfrac{3X}{2}, -\tfrac{3\sqrt{3}}{2}] = 0.3140\exp(-9/0.8778) = 1.1072 \times 10^{-5}
\end{aligned}$$

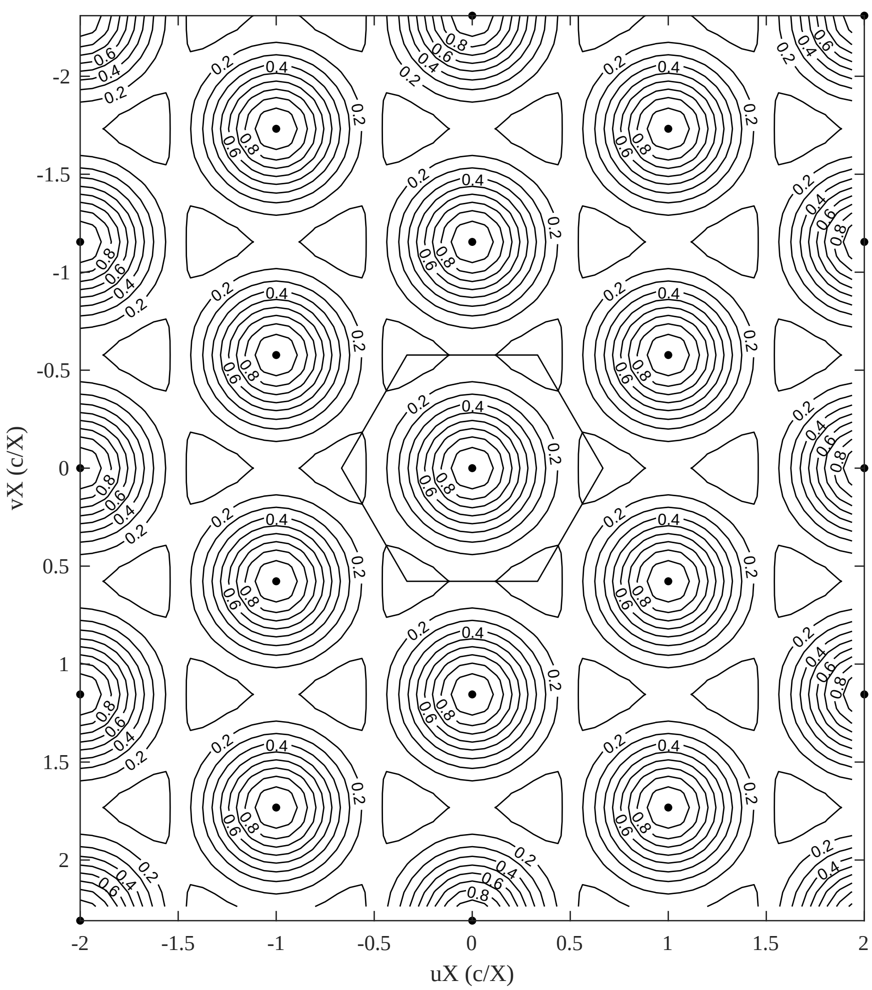
The analytical approximation for the frequency response within a unit cell $\mathcal{P}^*$ of Λ^* is given by Equation (10.12)

$$H(u,v) = \exp(-2\pi^2(u^2+v^2)(0.4389)X^2), \quad (u,v) \in \mathcal{P}^*.$$

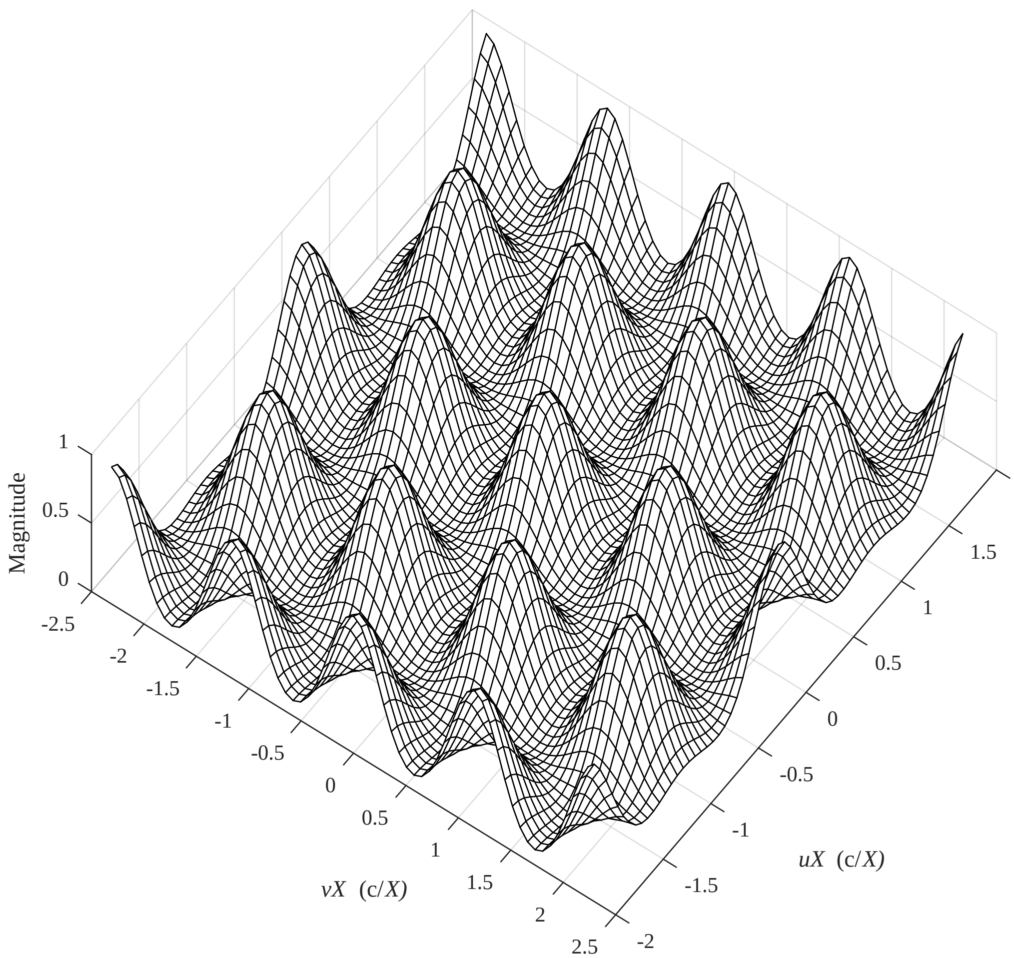
This frequency response is periodic with periodicity lattice Λ^*. A sampling matrix for the reciprocal lattice is

$$\mathbf{V}_{\Lambda^*} = \mathbf{V}^{-T} = \begin{bmatrix} \frac{1}{X} & 0 \\ -\frac{1}{\sqrt{3}X} & \frac{2}{\sqrt{3}X} \end{bmatrix}.$$

A contour plot of the frequency response of the filter over the frequency range $-2/X \leq u \leq 2/X$, $-4/(\sqrt{3}X) \leq v \leq 4/(\sqrt{3}X)$ is shown in the following figure. Points of Λ^* are indicated as filled circles and the Voronoi unit cell of Λ^* is shown.



A perspective view of this frequency response over the same region is as follows. The view angle is chosen to highlight the Λ^* periodicity of the frequency response.



Chapter 11

Changing the Sampling Structure of an Image

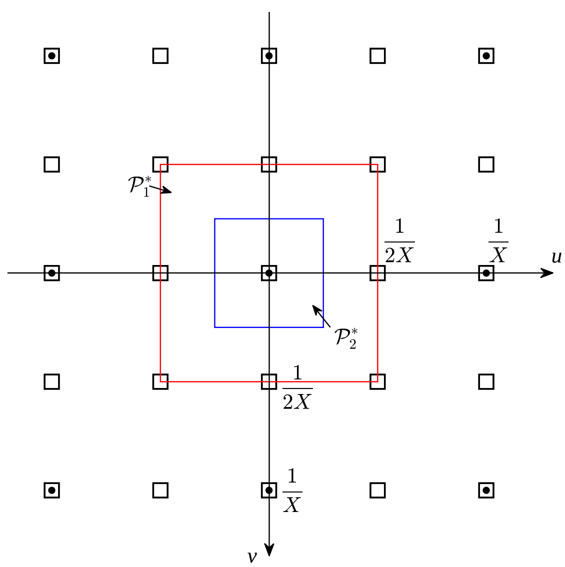
1. For each of the following pairs of lattices Λ_1 and Λ_2, state whether $\Lambda_1 \subset \Lambda_2$, $\Lambda_2 \subset \Lambda_1$ or neither. If neither, find (by inspection) the least dense lattice Λ_3 such that $\Lambda_1 \subset \Lambda_3$ and $\Lambda_2 \subset \Lambda_3$. For each lattice Λ_1, Λ_2 and Λ_3 (if required), determine and sketch the reciprocal lattice and a unit cell of the reciprocal lattice. Specify a sampling structure conversion system to transform a signal $f[\mathbf{x}]$ sampled on Λ_1 to a signal $g[\mathbf{x}]$ sampled on Λ_2. Assume that ideal low-pass filters are used where filters are required, sketch their passband in the frequency domain and indicate the gain.

(a)

$$\mathbf{V}_{\Lambda_1} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \qquad \mathbf{V}_{\Lambda_2} = \begin{bmatrix} 2X & 0 \\ 0 & 2X \end{bmatrix}$$

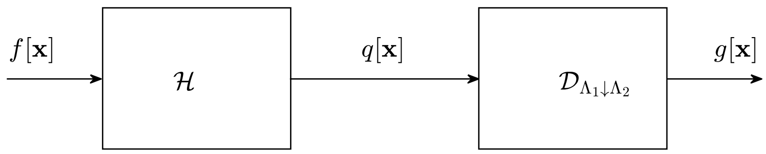
Solution: Λ_2 is a sublattice of Λ_1, with $\mathbf{V}_{\Lambda_2} = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right] \mathbf{V}_{\Lambda_1}$.

$$\Lambda_1^* = \mathrm{LAT}\left(\begin{bmatrix} \frac{1}{X} & 0 \\ 0 & \frac{1}{X} \end{bmatrix}\right) \qquad \Lambda_2^* = \mathrm{LAT}\left(\begin{bmatrix} \frac{1}{2X} & 0 \\ 0 & \frac{1}{2X} \end{bmatrix}\right)$$



Points of reciprocal lattices Λ_1^* ($\bullet$) and Λ_2^* ($\square$), with unit cells $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$.

The sampling structure conversion system is a downsampling system.

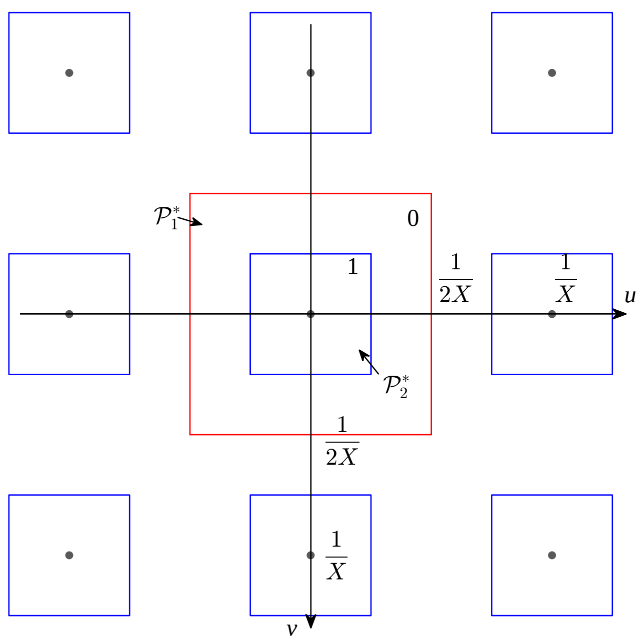


Block diagram of the sample structure conversion system

The filter $\mathcal{H}$ is an antialiasing filter with frequency response in one unit cell of Λ_1^* given by

$$H(u,v) = \begin{cases} 1 & (u,v) \in \mathcal{P}_2^* \\ 0 & (u,v) \in \mathcal{P}_1^* \backslash \mathcal{P}_2^* \end{cases}$$

Its frequency response is Λ_1^* periodic.



Frequency response of the antialiasing prefilter.

1. For each of the following pairs of lattices Λ_1 and Λ_2, state whether $\Lambda_1 \subset \Lambda_2$, $\Lambda_2 \subset \Lambda_1$ or neither. If neither, find (by inspection) the least dense lattice Λ_3 such that $\Lambda_1 \subset \Lambda_3$ and $\Lambda_2 \subset \Lambda_3$. For each lattice Λ_1, Λ_2 and Λ_3 (if required), determine and sketch the reciprocal lattice and a unit cell of the reciprocal lattice. Specify a sampling structure conversion system to transform a signal $f[\mathbf{x}]$ sampled on Λ_1 to a signal $g[\mathbf{x}]$ sampled on Λ_2. Assume that ideal low-pass filters are used where filters are required, sketch their passband in the frequency domain and indicate the gain.

(b)

$$V_{\Lambda_1} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \qquad V_{\Lambda_2} = \begin{bmatrix} 3X & X \\ 0 & X \end{bmatrix}$$

Solution: $d(\Lambda_1) = X^2$ and $d(\Lambda_2) = 3X^2 = 3d(\Lambda_1)$, so it is possible that Λ_2 is a sublattice of Λ_1. Checking the condition,

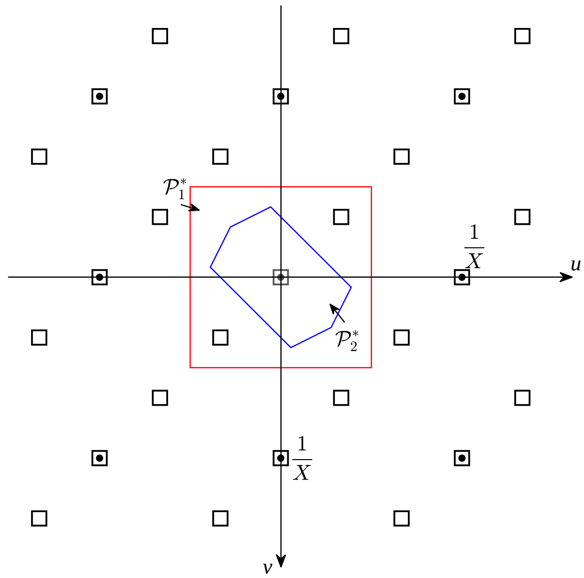
$$\mathbf{V}_{\Lambda_1}^{-1}\mathbf{V}_{\Lambda_2} = \begin{bmatrix} \frac{1}{X} & 0 \\ 0 & \frac{1}{X} \end{bmatrix} \begin{bmatrix} 3X & X \\ 0 & X \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

which is an integer matrix, so indeed $\Lambda_2 \subset \Lambda_1$. The reciprocal lattices are

$$\Lambda_1^* = \mathrm{LAT}(\mathbf{V}_{\Lambda_1}^{-T}) = \mathrm{LAT}\left(\begin{bmatrix} \frac{1}{X} & 0 \\ 0 & \frac{1}{X} \end{bmatrix}\right),$$

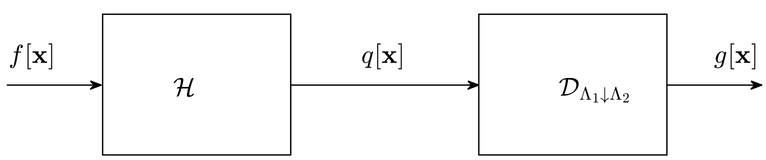
$$\Lambda_2^* = \mathrm{LAT}(\mathbf{V}_{\Lambda_2}^{-T}) = \mathrm{LAT}\left(\begin{bmatrix} \frac{1}{3X} & 0 \\ -\frac{1}{3X} & \frac{1}{X} \end{bmatrix}\right).$$

These are plotted on the following figure, along with the Voronoi unit cells. The Voronoi unit cell of Λ_1^* is a square, and the Voronoi unit cell of Λ_2^* is a hexagon. The exact specification of the Voronoi unit cell of Λ_2^* can be found by finding the equations of the perpendicular bisectors of the lines from the origin to the six nearest lattice points, and finding the points of intersection of these lines. For example, the perpendicular bisector of the line from the origin to $(\frac{2}{3X}, \frac{1}{3X})$ is $y = -2x + \frac{5}{6}$. The perpendicular bisector of the line from the origin to $(\frac{1}{3X}, -\frac{1}{3X})$ is $y = x - \frac{1}{3}$. The point of intersection is $(\frac{7}{18}, \frac{1}{18})$.



Points of reciprocal lattices Λ_1^* (•) and Λ_2^* (□), with unit cells $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$.

The sampling structure conversion system is a downsampling system.

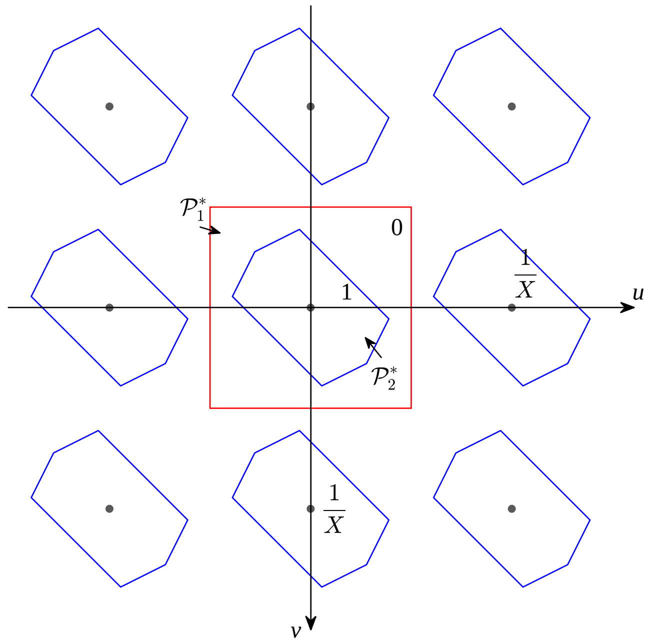


Block diagram of the sample structure conversion system

The filter $\mathcal{H}$ is an antialiasing filter with frequency response in one unit cell of Λ_1^* given by

$$H(u,v) = \begin{cases} 1 & (u,v) \in \mathcal{P}_2^* \\ 0 & (u,v) \in \mathcal{P}_1^* \backslash \mathcal{P}_2^* \end{cases}$$

Its frequency response is Λ_1^* periodic.



Frequency response of the antialiasing prefilter.

1. For each of the following pairs of lattices Λ_1 and Λ_2, state whether $\Lambda_1 \subset \Lambda_2$, $\Lambda_2 \subset \Lambda_1$ or neither. If neither, find (by inspection) the least dense lattice Λ_3 such that $\Lambda_1 \subset \Lambda_3$ and $\Lambda_2 \subset \Lambda_3$. For each lattice Λ_1, Λ_2 and Λ_3 (if required), determine and sketch the reciprocal lattice and a unit cell of the reciprocal lattice. Specify a sampling structure conversion system to transform a signal $f[\mathbf{x}]$ sampled on Λ_1 to a signal $g[\mathbf{x}]$ sampled on Λ_2. Assume that ideal low-pass filters are used where filters are required, sketch their passband in the frequency domain and indicate the gain.

(c)

$$V_{\Lambda_1} = \begin{bmatrix} 2X & 0 \\ 0 & 2X \end{bmatrix} \qquad V_{\Lambda_2} = \begin{bmatrix} X & X \\ X & -X \end{bmatrix}$$

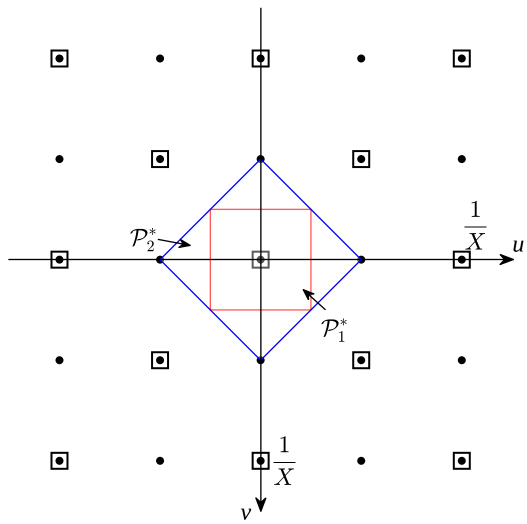
Solution: $d(\Lambda_1) = |\det \mathbf{V}_{\Lambda_1}| = 4X^2$ and $d(\Lambda_2) = |\det \mathbf{V}_{\Lambda_2}| = 2X^2$. Thus $d(\Lambda_1) = 2d(\Lambda_2)$. From Corollary 11.1, we see that $\Lambda_2 \subset \Lambda_1$ is *impossible*, but $\Lambda_1 \subset \Lambda_2$ is possible. Performing the test of Theorem 11.1,

$$\mathbf{V}_{\Lambda_2}^{-1}\mathbf{V}_{\Lambda_1} = \begin{bmatrix} \frac{1}{2X} & \frac{1}{2X} \\ \frac{1}{2X} & -\frac{1}{2X} \end{bmatrix} \begin{bmatrix} 2X & 0 \\ 0 & 2X \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

which is an integer matrix, so indeed $\Lambda_1 \subset \Lambda_2$. The reciprocal lattices are

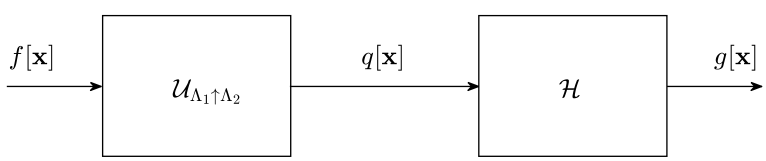
$$\Lambda_1^* = \mathrm{LAT}(\mathbf{V}_{\Lambda_1}^{-T}) = \mathrm{LAT}\left(\begin{bmatrix} \frac{1}{2X} & 0 \\ 0 & \frac{1}{2X} \end{bmatrix}\right),$$
$$\Lambda_2^* = \mathrm{LAT}(\mathbf{V}_{\Lambda_2}^{-T}) = \mathrm{LAT}\left(\begin{bmatrix} \frac{1}{2X} & \frac{1}{2X} \\ \frac{1}{2X} & -\frac{1}{2X} \end{bmatrix}\right).$$

These are plotted on the following figure, along with the Voronoi unit cells. The Voronoi unit cell of Λ_1 is a square of side $\frac{1}{2X}$ while the Voronoi unit cell of Λ_2 is a diamond, which is in fact a square of side $\frac{1}{\sqrt{2}X}$ rotated by 45°.



Points of reciprocal lattices Λ_1^* ($\bullet$) and Λ_2^* ($\square$), with unit cells $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$.

Since Λ_2 is denser than Λ_1, this is an upsampling problem with an upsampling factor of 2. The system block diagram is

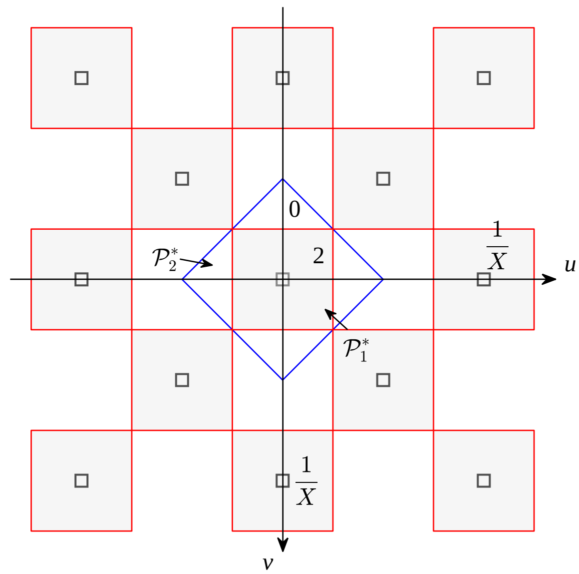


Block diagram of the sample structure conversion system.

The filter $\mathcal{H}$ is an interpolation filter with frequency response in one unit cell of Λ_2^* given by

$$H(u,v) = \begin{cases} 2 & (u,v) \in \mathcal{P}_1^* \\ 0 & (u,v) \in \mathcal{P}_2^* \backslash \mathcal{P}_1^* \end{cases}$$

Its frequency response is Λ_2^* periodic.



Frequency response of the interpolation filter. Periodicity lattice is Λ_2^* ($\square$). Filter response is 2 in the shaded area and 0 in the unshaded area.

1. For each of the following pairs of lattices Λ_1 and Λ_2, state whether $\Lambda_1 \subset \Lambda_2$, $\Lambda_2 \subset \Lambda_1$ or neither. If neither, find (by inspection) the least dense lattice Λ_3 such that $\Lambda_1 \subset \Lambda_3$ and $\Lambda_2 \subset \Lambda_3$. For each lattice Λ_1, Λ_2 and Λ_3 (if required), determine and sketch the reciprocal lattice and a unit cell of the reciprocal lattice. Specify a sampling structure conversion system to transform a signal $f[\mathbf{x}]$ sampled on Λ_1 to a signal $g[\mathbf{x}]$ sampled on Λ_2. Assume that ideal low-pass filters are used where filters are required, sketch their passband in the frequency domain and indicate the gain.

(d)

$$V_{\Lambda_1} = \begin{bmatrix} 1.5X & 0 \\ 0 & 1.5X \end{bmatrix} \qquad V_{\Lambda_2} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$$

Solution: $d(\Lambda_1) = |\det \mathbf{V}_{\Lambda_1}| = \frac{9}{4}X^2$, $d(\Lambda_2) = |\det \mathbf{V}_{\Lambda_2}| = X^2$. Since neither is an integer multiple of the other, neither $\Lambda_1 \subset \Lambda_2$ nor $\Lambda_2 \subset \Lambda_1$. The least common superlattice is seen by inspection to be

$$\Lambda_3 = \mathrm{LAT}\left(\begin{bmatrix} \frac{X}{2} & 0 \\ 0 & \frac{X}{2} \end{bmatrix}\right), \qquad d(\Lambda_3) = \frac{X^2}{4}.$$

This can be verified as follows. As shown in Section 13.9, the least common superlattice is $\Lambda_3 = \Lambda_1 + \Lambda_2$. Let $\Gamma = \mathrm{LAT}\left(\left[\begin{smallmatrix} X/2 & 0 \\ 0 & X/2 \end{smallmatrix}\right]\right)$. We can see that

$$\begin{bmatrix} \frac{3X}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} X \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{X}{2} \\ 0 \end{bmatrix} \qquad \text{and}$$

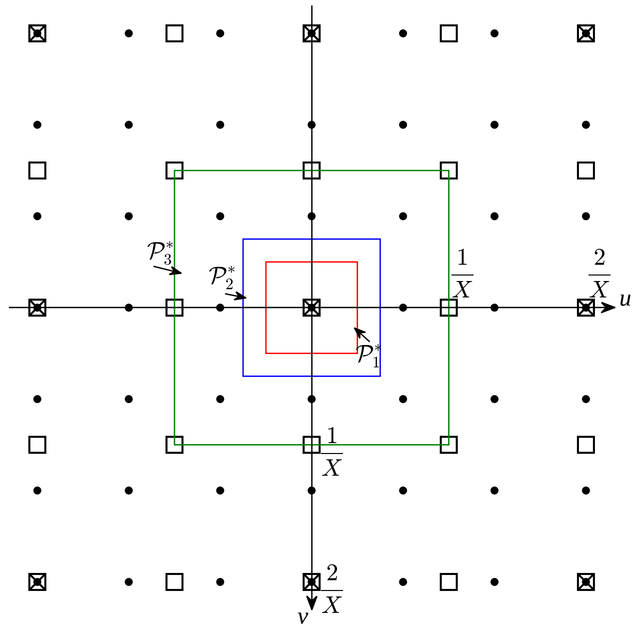
$$\begin{bmatrix} 0 \\ \frac{3X}{2} \end{bmatrix} - \begin{bmatrix} 0 \\ X \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{X}{2} \end{bmatrix}$$

must belong to Λ_3. Thus, all integer linear combinations of these vectors must belong to Λ_3, so $\Gamma \subset \Lambda_3$. But $\Lambda_1 \subset \Gamma$ and $\Lambda_2 \subset \Gamma$ and thus $\Lambda_3 = \Lambda_1 + \Lambda_2 \subset \Gamma$. Thus $\Lambda_3 = \Gamma$ as claimed.

All lattices are square lattices, more simply expressed as $\Lambda_1 = \frac{3}{2}X\mathbb{Z}^2$, $\Lambda_2 = X\mathbb{Z}^2$, $\Lambda_3 = \frac{1}{2}X\mathbb{Z}^2$. The reciprocal lattices are thus

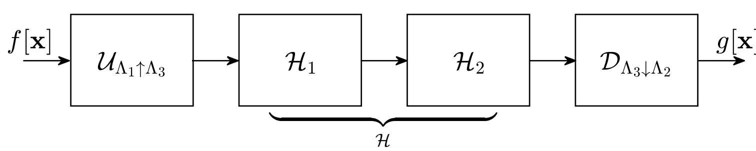
$$\Lambda_1^* = \tfrac{2}{3X}\mathbb{Z}^2 \qquad \Lambda_2^* = \tfrac{1}{X}\mathbb{Z}^2 \qquad \Lambda_3 = \tfrac{2}{X}\mathbb{Z}^2.$$

These are plotted in the following figure, along with the Voronoi cells, which are all squares.



Points of reciprocal lattices Λ_1^* ($\bullet$), Λ_2^* ($\square$) and Λ_3^* ($\times$) with unit cells $\mathcal{P}_1^*$, $\mathcal{P}_2^*$ and $\mathcal{P}_3^*$.

The sampling structure conversion can be implemented by upsampling to Λ_3 followed by downsampling to Λ_2 as follows.



Block diagram of the sample structure conversion system.

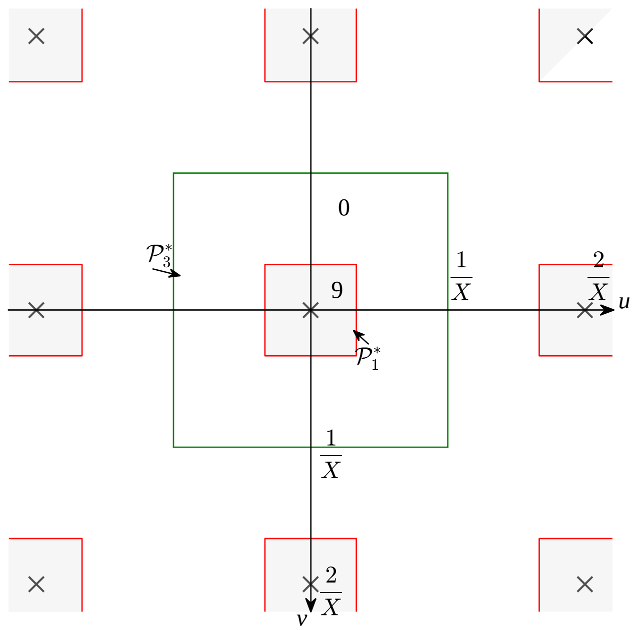
The combined filter defined on Λ_3 has frequency response in one unit cell of Λ_3^* given by

$$H(u,v) = \begin{cases} \frac{d(\Lambda_1)}{d(\Lambda_3)} = 9 & (u,v) \in \mathcal{P}_1^* \cap \mathcal{P}_2^*, \\ 0 & (u,v) \in \mathcal{P}_3^* \backslash (\mathcal{P}_1^* \cap \mathcal{P}_2^*). \end{cases}$$

Since in this example $\mathcal{P}_1^* \subset \mathcal{P}_2^*$, this can be simplified to

$$H(u,v) = \begin{cases} 9 & (u,v) \in \mathcal{P}_1^*, \\ 0 & (u,v) \in \mathcal{P}_3^* \backslash \mathcal{P}_1^*. \end{cases}$$

The frequency response is Λ_3^* periodic.



Frequency response of the conversion filter. Periodicity lattice is Λ_3^* ($\times$). Filter response is 9 in the shaded area and 0 in the unshaded area.

1. For each of the following pairs of lattices Λ_1 and Λ_2, state whether $\Lambda_1 \subset \Lambda_2$, $\Lambda_2 \subset \Lambda_1$ or neither. If neither, find (by inspection) the least dense lattice Λ_3 such that $\Lambda_1 \subset \Lambda_3$ and $\Lambda_2 \subset \Lambda_3$. For each lattice Λ_1, Λ_2 and Λ_3 (if required), determine and sketch the reciprocal lattice and a unit cell of the reciprocal lattice. Specify a sampling structure conversion system to transform a signal $f[\mathbf{x}]$ sampled on Λ_1 to a signal $g[\mathbf{x}]$ sampled on Λ_2. Assume that ideal low-pass filters are used where filters are required, sketch their passband in the frequency domain and indicate the gain.

(e)

$$V_{\Lambda_1} = \begin{bmatrix} X & X \\ X & -X \end{bmatrix} \qquad V_{\Lambda_2} = \begin{bmatrix} 1.5X & 1.5X \\ 1.5X & -1.5X \end{bmatrix}$$

Solution: $d(\Lambda_1) = |\det \mathbf{V}_{\Lambda_1}| = 2X^2$, $d(\Lambda_2) = |\det \mathbf{V}_{\Lambda_2}| = 4.5X^2$. Since neither is an integer multiple of the other, neither $\Lambda_1 \subset \Lambda_2$ nor $\Lambda_2 \subset \Lambda_1$. It can be seen by inspection of lattices Λ_1 and Λ_2 that the least dense superlattice is

$$\Lambda_3 = \mathrm{LAT}\left(\begin{bmatrix} \frac{X}{2} & \frac{X}{2} \\ \frac{X}{2} & -\frac{X}{2} \end{bmatrix}\right) \qquad \text{with} \quad d(\Lambda_3) = \frac{X^2}{2}.$$

This can be verified as follows. As shown in Section 13.9, the least common superlattice is $\Lambda_3 = \Lambda_1 + \Lambda_2$. Let $\Gamma = \mathrm{LAT}\left(\left[\begin{smallmatrix} X/2 & X/2 \\ X/2 & -X/2 \end{smallmatrix}\right]\right)$. We can see that the following elements of Γ,

$$\begin{bmatrix} \frac{3X}{2} \\ \frac{3X}{2} \end{bmatrix} - \begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} \frac{X}{2} \\ \frac{X}{2} \end{bmatrix} \qquad \text{and}$$

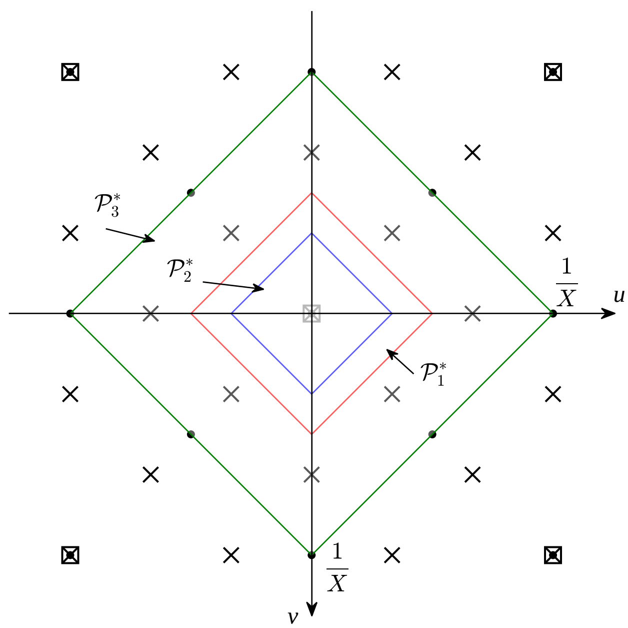
$$\begin{bmatrix} \frac{3X}{2} \\ -\frac{3X}{2} \end{bmatrix} - \begin{bmatrix} X \\ -X \end{bmatrix} = \begin{bmatrix} \frac{X}{2} \\ -\frac{X}{2} \end{bmatrix}$$

must belong to $\Lambda_3 = \Lambda_1 + \Lambda_2$. Thus, all integer linear combinations of these vectors must belong to Λ_3, so $\Gamma \subset \Lambda_3$. But $\Lambda_1 \subset \Gamma$ ($\mathbf{V}_\Gamma^{-1}\mathbf{V}_{\Lambda_1} = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$) and $\Lambda_2 \subset \Gamma$ ($\mathbf{V}_\Gamma^{-1}\mathbf{V}_{\Lambda_2} = \left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$) and so $\Lambda_3 = \Lambda_1 + \Lambda_2 \subset \Gamma$. Thus $\Lambda_3 = \Gamma$ as claimed.

The reciprocal lattices are

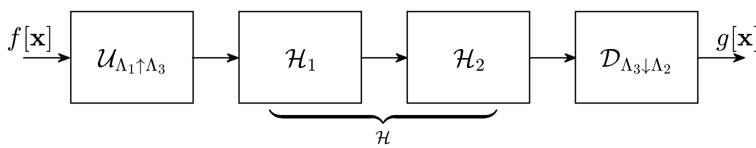
$$\Lambda_1^* = \mathrm{LAT}(\mathbf{V}_{\Lambda_1}^{-T}) = \mathrm{LAT}\left(\begin{bmatrix} \frac{1}{2X} & \frac{1}{2X} \\ \frac{1}{2X} & -\frac{1}{2X} \end{bmatrix}\right)$$
$$\Lambda_2^* = \mathrm{LAT}(\mathbf{V}_{\Lambda_2}^{-T}) = \mathrm{LAT}\left(\begin{bmatrix} \frac{1}{3X} & \frac{1}{3X} \\ \frac{1}{3X} & -\frac{1}{3X} \end{bmatrix}\right)$$
$$\Lambda_3^* = \mathrm{LAT}(\mathbf{V}_{\Lambda_3}^{-T}) = \mathrm{LAT}\left(\begin{bmatrix} \frac{1}{X} & \frac{1}{X} \\ \frac{1}{X} & -\frac{1}{X} \end{bmatrix}\right)$$

These are plotted in the following figure, along with the Voronoi cells which are all diamonds that are squares rotated by 45°.



Points of reciprocal lattices Λ_1^* (•), Λ_2^* (×) and Λ_3^* (□) with unit cells $\mathcal{P}_1^*$, $\mathcal{P}_2^*$ and $\mathcal{P}_3^*$.

The sampling structure can be implemented by upsampling to Λ_3 followed by downsampling to Λ_2 as follows.



Block diagram of the sample structure conversion system.

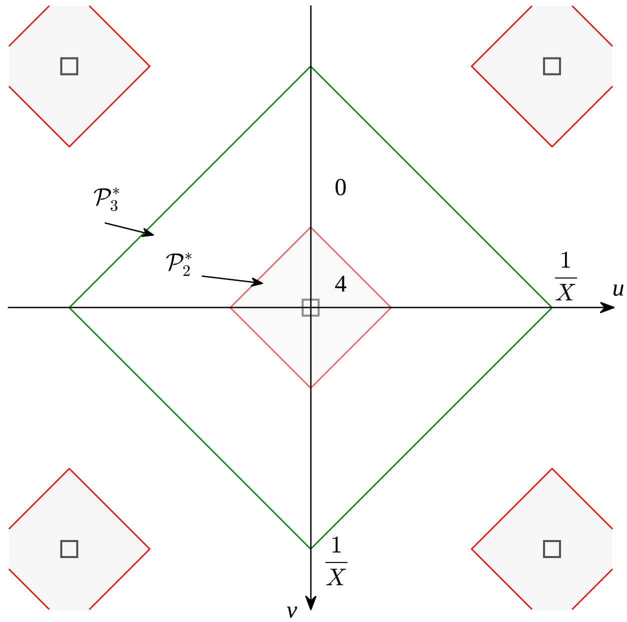
The combined filter defined on Λ_3 has frequency response in one unit cell of Λ_3^* given by

$$H(u,v) = \begin{cases} \frac{d(\Lambda_1)}{d(\Lambda_3)} = 4 & (u,v) \in \mathcal{P}_1^* \cap \mathcal{P}_2^*, \\ 0 & (u,v) \in \mathcal{P}_3^* \backslash (\mathcal{P}_1^* \cap \mathcal{P}_2^*). \end{cases}$$

Since in this example $\mathcal{P}_2^* \subset \mathcal{P}_1^*$, this can be simplified to

$$H(u,v) = \begin{cases} 4 & (u,v) \in \mathcal{P}_2^*, \\ 0 & (u,v) \in \mathcal{P}_3^* \backslash \mathcal{P}_2^*. \end{cases}$$

The frequency response is Λ_3^* periodic.



Frequency response of the conversion filter. Periodicity lattice is Λ_3^* ($\square$). Filter response is 4 in the shaded area and 0 in the unshaded area.