Properties of Multidimensional Fourier Transforms

Domain	**Continuous-domain, non-periodic**	**Discrete-domain (Λ), non-periodic**	**Continuous-domain, periodic (Γ)**	**Discrete-domain (Λ), periodic ($\Gamma \subset \Lambda$)**
Name of the transform	Continuous-domain Fourier transform (CDFT)	Discrete-domain Fourier transform (DDFT)	Continuous-domain Fourier series (CDFS)	Discrete-domain Fourier series (DDFS)
Signals and domains	$f_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} F_c(\mathbf{u}) \quad \mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^D$ $g_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} G_c(\mathbf{u}) \quad \mathbf{u}, \mathbf{u}_0 \in \mathbb{R}^D$	$f[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} F(\mathbf{u}) \quad \mathbf{x}, \mathbf{x}_0 \in \Lambda$ $g[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} G(\mathbf{u}) \quad \mathbf{u}, \mathbf{u}_0 \in \mathbb{R}^D$	$\tilde{f}_c(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{F}_c[\mathbf{u}] \quad \mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^D$ $\tilde{g}_c(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{G}_c[\mathbf{u}] \quad \mathbf{u}, \mathbf{u}_0 \in \Gamma^*$	$\tilde{f}[\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{F}[\mathbf{u}] \quad \mathbf{x}, \mathbf{x}_0 \in \Lambda$ $\tilde{g}[\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{G}[\mathbf{u}] \quad \mathbf{u}, \mathbf{u}_0 \in \Gamma^*$
Periodicity	none	$F(\mathbf{u}+\mathbf{r}) = F(\mathbf{u}), \qquad \mathbf{r} \in \Lambda^*$	$\tilde{f}_c(\mathbf{x}+\mathbf{s}) = \tilde{f}_c(\mathbf{x}), \qquad \mathbf{s} \in \Gamma$	$\tilde{f}[\mathbf{x}+\mathbf{s}] = \tilde{f}[\mathbf{x}], \qquad \mathbf{s} \in \Gamma$ $\tilde{F}[\mathbf{u}+\mathbf{r}] = \tilde{F}[\mathbf{u}], \qquad \mathbf{r} \in \Lambda^*$
Period	none	$\mathbf{u} : \mathcal{P}_{\Lambda^*}$, $\lvert\mathcal{P}_{\Lambda^*}\rvert = 1/d(\Lambda)$	$\mathbf{x} : \mathcal{P}_{\Gamma}$, $\lvert\mathcal{P}_{\Gamma}\rvert = d(\Gamma)$	$\mathbf{x} : \mathcal{B}$, $\mathbf{u} : \mathcal{D}$, $\lvert\mathcal{B}\rvert = \lvert\mathcal{D}\rvert = K$
Analysis	$F_c(\mathbf{u}) = \int_{\mathbb{R}^D} f_c(\mathbf{x}) \exp(-j2\pi \mathbf{u}\cdot\mathbf{x})\, d\mathbf{x}$	$F(\mathbf{u}) = \sum_{\mathbf{x}\in\Lambda} f[\mathbf{x}] \exp(-j2\pi \mathbf{u}\cdot\mathbf{x})$	$\tilde{F}_c[\mathbf{u}] = \int_{\mathcal{P}_\Gamma} \tilde{f}_c(\mathbf{x}) \exp(-j2\pi \mathbf{u}\cdot\mathbf{x})\, d\mathbf{x}$	$\tilde{F}[\mathbf{u}] = \sum_{\mathbf{x}\in\mathcal{B}} \tilde{f}[\mathbf{x}] \exp(-j2\pi \mathbf{u}\cdot\mathbf{x})$
Synthesis	$f_c(\mathbf{x}) = \int_{\mathbb{R}^D} F_c(\mathbf{u}) \exp(j2\pi \mathbf{u}\cdot\mathbf{x})\, d\mathbf{u}$	$f[\mathbf{x}] = d(\Lambda) \int_{\mathcal{P}_{\Lambda^*}} F(\mathbf{u}) \exp(j2\pi \mathbf{u}\cdot\mathbf{x})\, d\mathbf{u}$	$\tilde{f}_c(\mathbf{x}) = \frac{1}{d(\Gamma)} \sum_{\mathbf{u}\in\Gamma^*} \tilde{F}_c[\mathbf{u}] \exp(j2\pi \mathbf{u}\cdot\mathbf{x})$	$\tilde{f}[\mathbf{x}] = \frac{1}{K} \sum_{\mathbf{u}\in\mathcal{D}} \tilde{F}[\mathbf{u}] \exp(j2\pi \mathbf{u}\cdot\mathbf{x})$
Linearity	$Af_c(\mathbf{x}) + Bg_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} AF_c(\mathbf{u}) + BG_c(\mathbf{u})$	$Af[\mathbf{x}] + Bg[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} AF(\mathbf{u}) + BG(\mathbf{u})$	$A\tilde{f}_c(\mathbf{x}) + B\tilde{g}_c(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} A\tilde{F}_c[\mathbf{u}] + B\tilde{G}_c[\mathbf{u}]$	$A\tilde{f}[\mathbf{x}] + B\tilde{g}[\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} A\tilde{F}[\mathbf{u}] + B\tilde{G}[\mathbf{u}]$
Shift	$f_c(\mathbf{x}-\mathbf{x}_0) \stackrel{\text{CDFT}}{\longleftrightarrow} F_c(\mathbf{u}) \exp(-j2\pi \mathbf{u}\cdot\mathbf{x}_0)$	$f[\mathbf{x}-\mathbf{x}_0] \stackrel{\text{DDFT}}{\longleftrightarrow} F(\mathbf{u}) \exp(-j2\pi \mathbf{u}\cdot\mathbf{x}_0)$	$\tilde{f}_c(\mathbf{x}-\mathbf{x}_0) \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{F}_c[\mathbf{u}] \exp(-j2\pi \mathbf{u}\cdot\mathbf{x}_0)$	$\tilde{f}[\mathbf{x}-\mathbf{x}_0] \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{F}[\mathbf{u}] \exp(-j2\pi \mathbf{u}\cdot\mathbf{x}_0)$
Modulation	$f_c(\mathbf{x}) \exp(j2\pi \mathbf{u}_0\cdot\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} F_c(\mathbf{u}-\mathbf{u}_0)$	$f[\mathbf{x}] \exp(j2\pi \mathbf{u}_0\cdot\mathbf{x}) \stackrel{\text{DDFT}}{\longleftrightarrow} F(\mathbf{u}-\mathbf{u}_0)$	$\tilde{f}_c(\mathbf{x}) \exp(j2\pi \mathbf{u}_0\cdot\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{F}_c[\mathbf{u}-\mathbf{u}_0]$	$\tilde{f}[\mathbf{x}] \exp(j2\pi \mathbf{u}_0\cdot\mathbf{x}) \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{F}[\mathbf{u}-\mathbf{u}_0]$
Convolution	$\int_{\mathbb{R}^D} f_c(\mathbf{s}) g_c(\mathbf{x}-\mathbf{s})\, d\mathbf{s} \stackrel{\text{CDFT}}{\longleftrightarrow} F_c(\mathbf{u}) G_c(\mathbf{u})$	$\sum_{\mathbf{s}\in\Lambda} f[\mathbf{s}] g[\mathbf{x}-\mathbf{s}] \stackrel{\text{DDFT}}{\longleftrightarrow} F(\mathbf{u}) G(\mathbf{u})$	$\int_{\mathcal{P}_\Gamma} \tilde{f}_c(\mathbf{s}) \tilde{g}_c(\mathbf{x}-\mathbf{s})\, d\mathbf{s} \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{F}_c[\mathbf{u}] \tilde{G}_c[\mathbf{u}]$	$\sum_{\mathbf{s}\in\mathcal{B}} \tilde{f}[\mathbf{s}] \tilde{g}[\mathbf{x}-\mathbf{s}] \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{F}[\mathbf{u}] \tilde{G}[\mathbf{u}]$
Multiplication	$f_c(\mathbf{x}) g_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} \int_{\mathbb{R}^D} F_c(\mathbf{w}) G_c(\mathbf{u}-\mathbf{w})\, d\mathbf{w}$	$f[\mathbf{x}] g[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} d(\Lambda) \int_{\mathcal{P}_{\Lambda^*}} F(\mathbf{w}) G(\mathbf{u}-\mathbf{w})\, d\mathbf{w}$	$\tilde{f}_c(\mathbf{x}) \tilde{g}_c(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \frac{1}{d(\Gamma)} \sum_{\mathbf{w}\in\Gamma^*} \tilde{F}_c[\mathbf{w}] \tilde{G}_c[\mathbf{u}-\mathbf{w}]$	$\tilde{f}[\mathbf{x}] \tilde{g}[\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} \frac{1}{K} \sum_{\mathbf{w}\in\mathcal{D}} \tilde{F}[\mathbf{w}] \tilde{G}[\mathbf{u}-\mathbf{w}]$
Automorphism of domain	$f_c(\mathbf{A}\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} \frac{1}{\lvert\det \mathbf{A}\rvert} F_c(\mathbf{A}^{-T}\mathbf{u})$	$f[\mathbf{A}\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} F(\mathbf{A}^{-T}\mathbf{u})$	$\tilde{f}_c(\mathbf{A}\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{F}_c[\mathbf{A}^{-T}\mathbf{u}]$	$\tilde{f}[\mathbf{A}\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{F}[\mathbf{A}^{-T}\mathbf{u}]$
Differentiation	$\nabla_{\mathbf{x}} f_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} j2\pi \mathbf{u} F_c(\mathbf{u})$	N/A	$\nabla_{\mathbf{x}} \tilde{f}_c(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} j2\pi \mathbf{u} \tilde{F}_c[\mathbf{u}]$	N/A
Differentiation in frequency	$\mathbf{x} f_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} \frac{j}{2\pi} \nabla_{\mathbf{u}} F_c(\mathbf{u})$	$\mathbf{x} f[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} \frac{j}{2\pi} \nabla_{\mathbf{u}} F(\mathbf{u})$	N/A	N/A
Complex conjugation	$f_c^*(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} F_c^*(-\mathbf{u})$	$f^*[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} F^*(-\mathbf{u})$	$\tilde{f}_c^*(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \tilde{F}_c^*[-\mathbf{u}]$	$\tilde{f}^*[\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} \tilde{F}^*[-\mathbf{u}]$
Parseval	$\int_{\mathbb{R}^D} f_c(\mathbf{x}) g_c^*(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^D} F_c(\mathbf{u}) G_c^*(\mathbf{u})\, d\mathbf{u}$	$\sum_{\mathbf{x}\in\Lambda} f[\mathbf{x}] g^*[\mathbf{x}] = d(\Lambda) \int_{\mathcal{P}_{\Lambda^*}} F(\mathbf{u}) G^*(\mathbf{u})\, d\mathbf{u}$	$\int_{\mathcal{P}_\Gamma} \tilde{f}_c(\mathbf{x}) \tilde{g}_c^*(\mathbf{x})\, d\mathbf{x} = \frac{1}{d(\Gamma)} \sum_{\mathbf{u}\in\Gamma^*} \tilde{F}_c[\mathbf{u}] \tilde{G}_c^*[\mathbf{u}]$	$\sum_{\mathbf{x}\in\mathcal{B}} \tilde{f}[\mathbf{x}] \tilde{g}^*[\mathbf{x}] = \frac{1}{K} \sum_{\mathbf{w}\in\mathcal{D}} \tilde{F}[\mathbf{u}] \tilde{G}^*[\mathbf{u}]$
Duality	$F_c(\mathbf{x}) \stackrel{\text{CDFT}}{\longleftrightarrow} f_c(-\mathbf{u})$	$\tilde{F}_c[\mathbf{x}] \stackrel{\text{DDFT}}{\longleftrightarrow} d(\Gamma) \tilde{f}_c(-\mathbf{u})$	$F(\mathbf{x}) \stackrel{\text{CDFS}}{\longleftrightarrow} \frac{1}{d(\Lambda)} f[-\mathbf{u}]$	$\tilde{F}[\mathbf{C}\mathbf{x}] \stackrel{\text{DDFS}}{\longleftrightarrow} K \tilde{f}[-\mathbf{C}^{-1}\mathbf{u}]$