1. A real-valued, zero-mean, wide-sense-stationary white noise process \( v_1(n) \) with variance \( \sigma_1^2 \) is passed through a linear shift-invariant filter with transfer function \( H_1(z) = 1/(1 + 0.5z^{-1}) \), producing the process \( u(n) \). A second zero-mean white-noise process \( v_2(n) \), uncorrelated with \( v_1(n) \), of variance \( \sigma_2^2 \) is added to \( u(n) \) to produce \( x(n) \). The situation is illustrated below. We suppose that \( \sigma_1^2 = 2.5 \) and \( \sigma_2^2 = 6.0 \).

(a) Determine the autocorrelation functions \( r(k) \), the complex spectral density functions \( S(z) \) and the power spectral density \( S(e^{j\omega}) \) for the four processes \( v_1(n) \), \( v_2(n) \), \( u(n) \) and \( x(n) \).

(b) Show that \( x(n) \) can be considered to be an ARMA(1,1) process generated as shown below, where \( e(n) \) is a zero-mean white noise process with variance \( \sigma_e^2 = 1.0 \) and \( H_2(z) \) is a minimum-phase filter. Specifically, show that \( H_2(z) = 3(1 + \frac{1}{5}z^{-1})/(1 + \frac{1}{5}z^{-1}) \).
What is the difference equation relating \( x(n) \) and \( e(n) \).
2. Let \( \{U(n)\} \) be a zero-mean, wide-sense-stationary, complex-valued discrete-time random process. Use the Cauchy-Schwartz inequality to prove that \( |r_u(k)| \leq r_u(0) \). Carefully justify all steps.

3. Two sensors measure two related complex signals \( u_1(n) \) and \( u_2(n) \). The two signals are realizations of two jointly wide-sense-stationary, zero-mean random processes with autocorrelation functions \( r_{u_1}(k) \) and \( r_{u_2}(k) \) and cross-correlation function \( r_{u_1u_2}(k) \). We wish to estimate \( u_1(n) \) by \( \hat{u}_1(n) = w u_2(n) \), where the complex parameter \( w \) is chosen to minimize \( E[|U_1(n) - \hat{U}_1(n)|^2] \).

(a) Show how the problem can be formulated as the projection of a vector on a subspace of a Hilbert space. Clearly specify the definitions of the Hilbert space, the subspace, the inner product and the norm.

(b) Apply the projection theorem to obtain the optimal coefficient \( w_0 \). What is the minimum mean-squared error? Express your result in terms of the correlation functions \( r_{.}(k) \). Justify all steps completely.
Cauchy-Schwartz inequality: $|\langle f \mid g \rangle|^2 \leq \langle f \mid f \rangle \langle g \mid g \rangle$ with equality iff $f = \alpha g$ for some $\alpha \in \mathbb{C}$.

Projection: If $v_1, \ldots, v_M$ are linearly independent, then the projection of $x$ on $\text{span}(v_1, \ldots, v_M)$ is $\hat{x} = \sum_{i=1}^{M} \alpha_i u_i$, where $\mathcal{U} \alpha^* = p$, $[\mathcal{U}]_{ij} = \langle u_i \mid u_j \rangle$ and $[p]_i = \langle u_i \mid x \rangle$. $\|x - \hat{x}\|^2 = \|x\|^2 - \alpha^T p$.

Linear shift-invariant filtering of a wide-sense stationary random process $u$:

$S_y(z) = H(z)H^*(1/z^*)S_u(z)$, and $S_y(e^{j\omega}) = |H(e^{j\omega})|^2 S_u(e^{j\omega})$.

Inverse Z-transform by residues:

$$x(n) = \frac{1}{2\pi j} \oint_{C} X(z)z^{n-1} \, dz = \sum \text{residues of } X(z)z^{n-1} \text{ at the poles inside } C.$$ 

For a simple pole at $z = p$ inside $C$, the residue of $X(z)z^{n-1}$ at $p$ is $X(z)z^{[n-1]}(z-p)|_{z=p}$. 