CEG4316 Tutorial
Nov. 15, 2013

Image representation

1(a) Consider the following four one-dimensional vectors in $\mathbb{R}^4$:

$$\mathbf{b_1} = [0\ \ 1\ \ 1\ \ 1];\ \mathbf{b}_2 = [0\ \ -2\ \ 1\ \ 1];\ \mathbf{b_3} = [1\ \ 0\ \ 1-\ 1];\ \mathbf{b_4} = [-2\ \ 0\ \ 1\ \ -1]$$

Verify explicitly and completely that these form an orthogonal set, but not an orthonormal set. Normalize them to form an orthonormal set $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, $\mathbf{c}_4$ and display them in the same format as $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, $\mathbf{b}_4$ above.

(b) Consider the following orthonormal basis for $\mathbb{R}^4$:

$$\mathbf{b_1} = \left[\frac{1}{\sqrt{10}} \ \frac{2}{\sqrt{10}} \ \frac{2}{\sqrt{10}} \ \frac{1}{\sqrt{10}}\right]; \ \mathbf{b}_2 = \left[\frac{2}{\sqrt{10}} \ \frac{1}{\sqrt{10}} \ -\frac{1}{\sqrt{10}} \ -\frac{2}{\sqrt{10}}\right];$$

$$\mathbf{b_3} = \left[\frac{2}{\sqrt{10}} \ -\frac{1}{\sqrt{10}} \ -\frac{1}{\sqrt{10}} \ \frac{2}{\sqrt{10}}\right];$$

$$\mathbf{b_4} = \left[\frac{1}{\sqrt{10}} \ -\frac{2}{\sqrt{10}} \ \frac{2}{\sqrt{10}} \ -\frac{1}{\sqrt{10}}\right]$$

Let $\{\mathbf{d}_{ij}$, i=1,4; j=1,4$\}$ be a separable basis for $\mathbb{R}^{4\times4}$, where

$$\mathbf{d}_{ij} = \mathbf{b}_j^T\mathbf{b}_i$$

Let $\mathbf{f} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$ be a 4×4 image with the same orientation as the $\mathbf{d}_{ij}$ above. Let $\mathbf{f}_b$ be the matrix of coefficients, with the same orientation: $\mathbf{f} = \sum_{i=1}^{4}\sum_{j=1}^{4} f_b[i,j]\mathbf{d}_{ij}$. **Determine $f_b[2,3]$ using the appropriate inner product formula. Find the entire matrix $\mathbf{f}_b$ using matrix operations.**

2(a) As presented in the course, the DCT basis vectors are given by

$$^{M}\psi_i[m] = \cos\left(\frac{(2m+1)i\pi}{2M}\right) \; m = 0, \ldots, M-1; i = 0, \ldots, M-1, \text{ with}$$

$$\langle {}^{M}\vec{\psi}_k, {}^{M}\vec{\psi}_l \rangle = \frac{M}{\beta_k} \delta[k-l] \text{ and}$$

$$\beta_k = \begin{cases} 1 & k = 0 \\ 2 & k = 1, 2, \ldots, M-1 \end{cases}$$

Let us consider $M = 4$. The Matlab function T=dctmtx(4) produces a 4 by 4 matrix whose rows are the orthonormal DCT basis vectors.

Compute the four by four matrix T returned by dctmtx. Note that only two distinct numbers have to be calculated, if you use the properties $\cos(\pi - a) = -\cos(a)$ and $\cos(2\pi - a) = \cos(a)$. **Verify that vectors defined by the four rows of the matrix T do form an orthogonal set.** Note that no multiplications are required to do this.

(b) A two-dimensional separable 4×4 DCT basis is defined using the formula $b_{ij}[m,n] = b_{hi}[m]b_{vj}[n]$. **Illustrate the two basis vectors $\vec{b}_{03}$ and $\vec{b}_{21}$ as 4 by 4 image blocks with the correct spatial orientation. Carefully explain your reasoning.**

3. Consider the following four 2×2 image blocks:

$$\vec{b}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \vec{b}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \vec{b}_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(a) Prove that these four blocks form an *orthogonal*, but *not* an orthonormal, *basis* for the vector space of all 2×2 real image blocks. Is this basis separable?

(b) **Normalize these basis vectors to obtain an *orthonormal* basis $\{\vec{c}_i\}$ for the vector space of all 2×2 real image blocks.**

(c) Determine the representation of the 4×4 image

$$\mathbf{f} = \begin{bmatrix} 80 & 85 & 190 & 195 \\ 75 & 80 & 175 & 200 \\ 100 & 125 & 120 & 125 \\ 105 & 125 & 120 & 120 \end{bmatrix}$$

using the basis of *part (a)* and the method of disjoint-block bases. You need to identify 16 basis vectors and 16 coefficients.

(d) Suppose that only the basis vectors $\mathbf{b}_1$ and $\mathbf{b}_3$ are used in each sub-image to represent the image of part (c). Determine the approximate image $\widehat{\mathbf{f}}$ in this case.