CEG4316 Image Processing

Mid-term exam

Date: Nov. 2, 2011

Time: 13:00-14:20

Professor: E. Dubois

This exam has four pages and two questions. Answer all questions. Closed-book exam: you may not use any books, notes or calculator. Explain all answers; I am more interested in the reasoning than in precise numerical answers. Unless otherwise specified, you may use the results provided on pages 3 and 4 without proof, but state which ones you use. Vous pouvez répondre en anglais ou en français.

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1. A two-dimensional continuous-space linear shift invariant system has impulse response $h(x, y)$ given by (a is an arbitrary distance, measured in ph):

$$h(x,y) = \begin{cases} 1 & 0 \le x \le 2a \text{ and } 0 \le y \le a \\ -1 & -2a \le x \le 0 \text{ and } -a \le y \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

3 (a) Sketch this impulse response in the XY plane, showing the region of support and the values of the function. Use our standard convention for orientation of axes and label your axes.

3 (b) Express the impulse response in terms of rect functions. (See page 4 for the definition of the rect function.)

3 (c) Compute the frequency response $H(u, v)$ of this system. What is the DC gain of this filter?

3 (d) What is the output $g(x, y)$ of the filter if the input is the signal

$$f(x,y) = \delta(x + a, y + \frac{a}{2}) + \delta(x - a, y - \frac{a}{2})?$$

Make sure the solution is as simplified as possible, expressed in terms of the rect function. Sketch $g(x, y)$ in the same way as requested in (a).

2. A signal $f[x,y]$ is defined on the lattice $\Lambda = \mathrm{LAT}(\mathbf{V}_\Lambda)$, where

$$\mathbf{V}_\Lambda = \begin{bmatrix} 2X & X \\ 0 & X \end{bmatrix}.$$

We want to up-sample $f[x,y]$ to the rectangular lattice Γ with sampling matrix $\mathbf{V}_\Gamma$, where

$$\mathbf{V}_\Gamma = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}.$$

This is accomplished using the system shown in Fig. 1, where the linear shift-invariant filter $\mathcal{H}$ has unit sample response

$$h[x,y] = \begin{cases} 1 & (x,y) = (0,0) \\ 0.3 & (x,y) = (\pm X, 0) \\ 0.2 & (x,y) = (0, \pm X) \\ 0 & \text{elsewhere in } \Gamma \end{cases}.$$

There are five non-zero values.

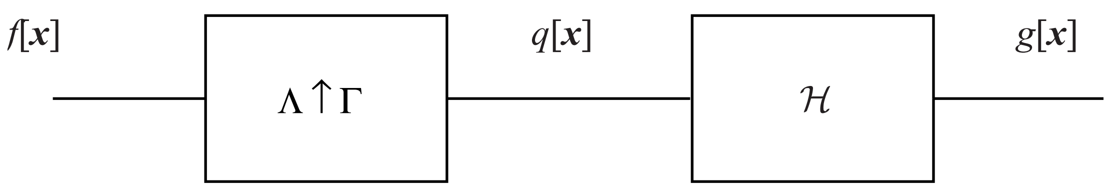

Figure 1: Upsampling system.

2 (a) What is the upsampling factor (or interpolation ratio)?

3 (b) Determine the frequency response $H(u,v)$ of the LSI filter $\mathcal{H}$. Express it as a real function of u and v.

3 (c) Suppose that the input signal is $f[x,y] = \delta[x,y] + \delta[x-X, y-X] + \delta[x+X, y-X]$, $(x,y) \in \Lambda$. Sketch this signal in the space domain. Determine the output signal $g[x,y]$ for all $(x,y) \in \Gamma$ and sketch this signal on a separate set of axes.

	$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi\boldsymbol{u}\cdot\boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi\boldsymbol{u}\cdot\boldsymbol{x})\, d\boldsymbol{x}$
(i)	$af_1(\boldsymbol{x}) + bf_2(\boldsymbol{x})$	$aF_1(\boldsymbol{u}) + bF_2(\boldsymbol{u})$
(ii)	$f(\boldsymbol{x} - \boldsymbol{x}_0)$	$F(\boldsymbol{u}) \exp(-j2\pi\boldsymbol{u}\cdot\boldsymbol{x}_0)$
(iii)	$f(\boldsymbol{x}) \exp(j2\pi\boldsymbol{u}_0\cdot\boldsymbol{x})$	$F(\boldsymbol{u} - \boldsymbol{u}_0)$
(iv)	$f(\mathbf{A}\boldsymbol{x})$	$\frac{1}{\lvert\det\mathbf{A}\rvert} F(\mathbf{A}^{-T}\boldsymbol{u})$
(v)	$f_1(\boldsymbol{x}) * f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u})F_2(\boldsymbol{u})$
(vi)	$f_1(\boldsymbol{x})f_2(\boldsymbol{x})$	$F_1(\boldsymbol{u}) * F_2(\boldsymbol{u})$
(vii)	$\nabla_{\boldsymbol{x}} f(\boldsymbol{x})$	$j2\pi\boldsymbol{u}F(\boldsymbol{u})$
(viii)	$\boldsymbol{x}f(\boldsymbol{x})$	$\frac{j}{2\pi}\nabla_{\boldsymbol{u}} F(\boldsymbol{u})$
(ix)	$F(\boldsymbol{x})$	$f(-\boldsymbol{u})$
(x)	$f^*(\boldsymbol{x})$	$F^*(-\boldsymbol{u})$
(xi)	$f_1(x)f_2(y)$	$F_1(u)F_2(v)$
(xii)	$\int_{\mathbb{R}^D} \lvert f(\boldsymbol{x})\rvert^2\, d\boldsymbol{x} = \int_{\mathbb{R}^D} \lvert F(\boldsymbol{u})\rvert^2\, d\boldsymbol{u}$	

Multidimensional Fourier transform properties.

$f(\boldsymbol{x}) = \int_{\mathbb{R}^D} F(\boldsymbol{u}) \exp(j2\pi\boldsymbol{u}\cdot\boldsymbol{x})\, d\boldsymbol{u}$	$F(\boldsymbol{u}) = \int_{\mathbb{R}^D} f(\boldsymbol{x}) \exp(-j2\pi\boldsymbol{u}\cdot\boldsymbol{x})\, d\boldsymbol{x}$
$\mathrm{rect}(x, y)$	$\frac{\sin\pi u}{\pi u}\frac{\sin\pi v}{\pi v}$
$\mathrm{circ}(x, y)$	$\frac{1}{\sqrt{u^2+v^2}} J_1(2\pi\sqrt{u^2+v^2})$
$\exp(-(x^2+y^2)/2r^2)$	$2\pi r^2 \exp(-2\pi^2(u^2+v^2)r^2)$
$\cos(\pi(x^2+y^2)/r^2)$	$r^2 \sin(\pi(u^2+v^2)r^2)$
$\exp(j\pi(x^2+y^2)/r^2)$	$jr^2 \exp(-j\pi(u^2+v^2)r^2)$
$\delta(\boldsymbol{x})$	1
1	$\delta(\boldsymbol{u})$

Multidimensional Fourier transform of selected functions.

	$f[\boldsymbol{x}] = d(\Lambda)\int_{\mathcal{P}^*} F(\boldsymbol{u})\exp(j2\pi\boldsymbol{u}\cdot\boldsymbol{x})\,d\boldsymbol{u}$	$F(\boldsymbol{u}) = \sum_{\boldsymbol{x}\in\Lambda} f[\boldsymbol{x}]\exp(-j2\pi\boldsymbol{u}\cdot\boldsymbol{x})$
(i)	$af_1[\boldsymbol{x}] + bf_2[\boldsymbol{x}]$	$aF_1(\boldsymbol{u}) + bF_2(\boldsymbol{u})$
(ii)	$f[\boldsymbol{x} - \boldsymbol{x}_0]$	$F(\boldsymbol{u})\exp(-j2\pi\boldsymbol{u}\cdot\boldsymbol{x}_0)$
(iii)	$f[\boldsymbol{x}]\exp(j2\pi\boldsymbol{u}_0\cdot\boldsymbol{x})$	$F(\boldsymbol{u} - \boldsymbol{u}_0)$
(iv)	$f_1[\boldsymbol{x}] * f_2[\boldsymbol{x}]$	$F_1(\boldsymbol{u})F_2(\boldsymbol{u})$
(v)	$f_1[\boldsymbol{x}]f_2[\boldsymbol{x}]$	$d(\Lambda)\int_{\mathcal{P}^*} F_1(\boldsymbol{r})F_2(\boldsymbol{u}-\boldsymbol{r})\,d\boldsymbol{r}$
(vi)	$\boldsymbol{x}f[\boldsymbol{x}]$	$\frac{j}{2\pi}\nabla_{\boldsymbol{u}}F(\boldsymbol{u})$
(vii)	$f^*[\boldsymbol{x}]$	$F^*(-\boldsymbol{u})$
(viii)	$\sum_{\boldsymbol{x}\in\Lambda} \lvert f[\boldsymbol{x}]\rvert^2 = d(\Lambda)\int_{\mathcal{P}^*} \lvert F(\boldsymbol{u})\rvert^2\,d\boldsymbol{u}$	

Properties of the multidimensional Fourier transform over a lattice Λ.

Formulas

$$\exp(jX) = \cos(X) + j\sin(X), \qquad j = \sqrt{-1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\mathrm{rect}(x, y) = \begin{cases} 1, & \text{if } |x| \le 0.5 \text{ and } |y| \le 0.5; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathrm{circ}(x, y) = \begin{cases} 1, & \text{if } x^2 + y^2 \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

Affine transformation: $\mathcal{Q}_{\mathbf{A},\boldsymbol{d}} : \vec{\boldsymbol{g}} = \mathcal{Q}_{\mathbf{A},\boldsymbol{d}}\vec{\boldsymbol{f}} : g(\boldsymbol{x}) = f(\mathbf{A}(\boldsymbol{x} - \boldsymbol{d}))$

If $\Lambda = \mathrm{LAT}(\mathbf{V})$, then $d(\Lambda) = |\det(\mathbf{V})|$, and $\Lambda^* = \mathrm{LAT}(\mathbf{V}^{-T})$.

Sampling

If $f[\boldsymbol{x}] = f_c(\boldsymbol{x}), \quad \boldsymbol{x} \in \Lambda$ then

$$F(\boldsymbol{u}) = \frac{1}{d(\Lambda)}\sum_{r\in\Lambda^*} F_c(\boldsymbol{u} + \boldsymbol{r})$$