ELG 5372 Error Control Coding

Lecture 10: Performance Measures: BER after decoding

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Error Correction Performance Review

- The probability of incorrectly decoding a received word is the probability that the error pattern is not one of the coset leaders of the standard array.
  - For the Hamming (7,4) case (and for any Hamming Code), this is the probability that the error pattern has a weight of 2 or more.
  - For the (5,2) linear code example, it is the probability that the error pattern is not one of the 8 error patterns in the standard array (one of weight 0, 5 of weight 1 and 2 of weight 2).
- Conversely, we can state that the probability that the decoder correctly decodes received word is the probability that the error pattern is one of the coset leaders.
  - Denote this as P_c, then $P(E) = 1-P_c$.

Error Correction Performance Review

- Hamming (7,4), $P_c = (1\text{-}p)^7 + 7p(1\text{-}p)^6$.
- (5,2) block code: $Pc = (1\text{-}p)^5 + 5p(1\text{-}p)^4 + 2p^2(1\text{-}p)^3$ if the decoder is a complete decoder.
  - This means that it corrects all coset leaders in it standard array.
  - For imperfect codes, there are some cosets eladers with weight greater than $t = \left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor$

Bounded-Distance Decoder

- Decoder does not correct any error patterns of weight greater than $\left\lfloor \frac{d_{\min}-1}{2} \right\rfloor$
- In the event that the code is not perfect, this means that for coset leaders in the standard array that have weight greater than $\left\lfloor \frac{d_{\min}-1}{2} \right\rfloor$, then the decoder declares a decoder failure for any received words that fall in that coset.
- This is equivalent to saying that a decoder failure is declared for any received word that does not fall into any Hamming sphere of radius $\left\lfloor \frac{d_{\min}-1}{2} \right\rfloor$.

Examples

Coset leaders ↓

codewords →

00000	01011	10110	11101
00001	01010	10111	11100
00010	01001	10100	11111
00100	01111	10010	11001
01000	00011	11110	10101
10000	11011	00110	01101
01100	00111	11010	10001
11000	10011	01110	00101

Correct error

Declare failure

Example 2

t=1

Correct error

Declare failure

000000	011011	101101	110110
000001	011010	101100	110111
000010	011001	101111	110100
000100	011111	101001	110010
001000	010011	100101	111110
010000	001011	111101	100110
100000	111011	001101	010110
000011	**011000**	101110	110101
000101	011110	**101000**	110011
000110	011101	101011	**110000**
001001	**010010**	**100100**	111111
001010	**010001**	100111	111101
001100	010111	**100001**	111010
010100	001111	111001	**100010**
000111	011100	101010	110001
001110	010101	100011	111000

Probability of Decoder Failure

- We declare decoder failure for bounded-distance decoders if the received word is in a row of the standard array that corresponds to a coset that is outside all Hamming spheres of radius *t*, where *t*= $\left\lfloor \frac{d_{\min}-1}{2} \right\rfloor$.
- The probability of decoder failure, P(*F*), is the probability that the error pattern is among the cosets found below the "correct error line" of the standard array.

$$P(F) = 1 - \sum_{i=0}^{\left\lfloor d_{\min}-1 / 2 \right\rfloor} \binom{n}{i} p^{i} (1-p)^{n-1} - P(E)$$

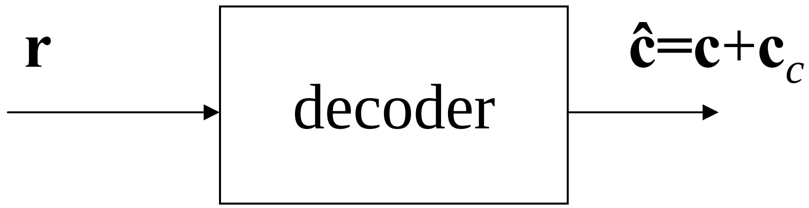
- Where *P*(*E*) is now the probability that the error pattern is not a coset leader among all of the n-tuples above the "correct error line".

Example

- For the (5,2) code, a decoder failure is declared if the error pattern is one of the following:
  - 01100, 00111, 11010, 10001, 11000, 10011, 01110 and 00101.
  - Therefore $P(F) = 4p^2(1-p)^3+4p^3(1-p)^2$.
  - $P_c = (1-p)^5+5p(1-p)^4$.
  - Therefore $P(E) = 1-P_c-P(F) = 1-(1-p)^5+5p(1-p)^4+ 4p^2(1-p)^3+4p^3(1-p)^2$.

Probability of bit error (after decoding)

- When the decoder corrects an error, there are two scenarios
  - Decoder correctly "corrects" a codeword ($\hat{\mathbf{c}} = \mathbf{c}$).
  - Decoder incorrectly "corrects" a codeword ($\hat{c}$ = another codeword = $\mathbf{c}+\mathbf{c}_c$).



Probability of bit error 2

- Consider the all 0 word of Hamming (7,4) systematic code:
  - There are 7 other codewords that are a distance of 3 from this codeword
  - There are 7 other codewords that are a distance 4 from this codeword.
  - There is 1 codeword that is a distance 7 from this codeword.
- Since the code is linear, it is a subset of V_2^7.
- If we select any codeword as the coset leader, and add it to all of the vectors in the code, the result is the code itself.
- This means that all codewords in the code have the same distance profile as the all 0 codeword.

Probability of bit error 3

- Assuming that the all 0 codeword is transmitted, then the decoder is successful if **ĉ** = 0000000.
- If a decoder error occurs, then **ĉ** will be another codeword in the code.
- If the error pattern has weight 0 or 1, ĉ = 0000000.
- If the error pattern has weight 2 an error will occur
  - Since the code is a one error correcting code, it will change one bit from a 1 to a 0 (will change received word from weight 2 to weight 1) or from a 0 to a 1 (will change received word from weight 2 to weight 3).
  - **ĉ** must have weight 3.

Probability of bit error 4

- If the error pattern has weight 3
  - If **e** = a codeword of weight 3, then the decoder will not change any bits and **ĉ** will have weight 3.
  - If **e** ≠ a codeword, then the decoder will attempt to correct by inverting a bit. Since there are no codewords of weight 2, the result here is that **ĉ** will have weight 4.
  - There are 7 codewords of weight 3 and 7!/(3!4!)-7 = 28 weight 3 error patterns that are not codewords. All error patterns of weight 3 are equally likely. Therefore if the error pattern has weight 3, there is a probability of 0.2 that **ĉ** will have weight 3 and 0.8 probability that it will have weight 4.

Probability of bit error 5

- If error pattern has weight 4:
  - $\hat{\mathbf{c}}$ will have weight 4 with probability 0.2 and weight 3 with probability 0.8.
- If error pattern has weight 5
  - $\hat{\mathbf{c}}$ will have weight 4.
- If error pattern has weight 6
  - $\hat{\mathbf{c}}$ will have weight 7
- If error pattern has weight 7
  - $\hat{\mathbf{c}}$ will have weight 7.
- Since the transmitted codeword is the all 0 codeword, then the number of bit errors in the decoded codeword is the weight of the decoded codeword.

Probability of bit error 6

- For block codes, how does bit error rate in decoded codeword translate to bit error rate in decoded message?
- Let us consider systematic codes for ease of illustration.
- Assume that the all 0 codeword is transmitted. The first 4 bits are the message.
- Let us assume that the decoded codeword has weight 3. (therefore 3/7 of the code bits are in error).

Probability of bit error 7

codeword	HW(**c**)	codeword	HW(**c**)
0000000	0	1000110	3
0001101	3	1001011	4
0010111	4	1010001	3
0011010	3	1011100	4
0100011	3	1100101	4
0101110	4	1101000	3
0110100	3	1110010	4
0111001	4	1111111	7

Probability of bit error 8

- 0001101, 0011010, 0100011, 0110100, 1000110, 1010001, 1101000.
- We can show that if we transmit the all 0 codeword and the decoded codeword has 3 errors in it, it is equally likely that any one of the above if the erroneous codeword.
- Therefore probability that a message bit is in error when the decoded codeword has 3/7 bits in error is (1/4 + 2/4 +1/4 + 2/4 + 1/4 + 2/4 + 3/4)/7 = 12/28 = 3/7.
- It is equal to the code bit error rate. We can show the same thing for when the decoded codeword has 4/7 bits in error or when the decoded codeword has 7/7 bits in error.
- This is because errors are uniformly distributed among the info and parity bits.

Probability of bit error 9

- Therefore P_b for Hamming (7,4) is given by:

$$P_b = \frac{3}{7}\binom{7}{2}p^2(1-p)^5 + \left(0.2\frac{3}{7}+0.8\frac{4}{7}\right)\binom{7}{3}p^3(1-p)^4 + \left(0.8\frac{3}{7}+0.2\frac{4}{7}\right)\binom{7}{4}p^4(1-p)^3$$

$$+\frac{4}{7}\binom{7}{5}p^5(1-p)^2 + \binom{7}{6}p^6(1-p) + \binom{7}{7}p^7$$

$$P_b = 9p^2(1-p)^5 + 19p^3(1-p)^4 + 16p^4(1-p)^3 + 12p^5(1-p)^2 + 7p^6(1-p) + p^7$$

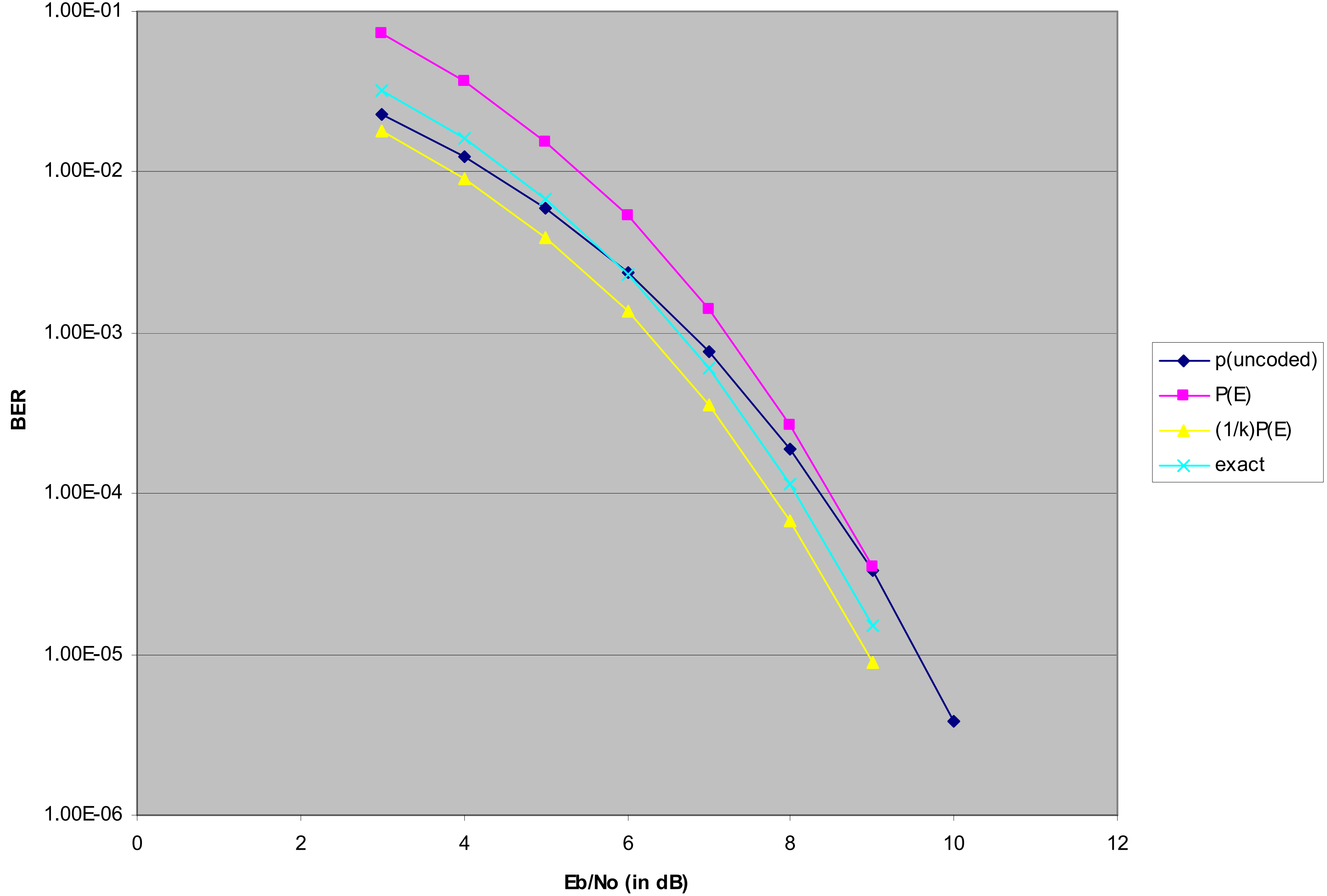
- Of course, the above equation is for the case when the all 0 codeword is transmitted. By using distance rather than weight arguments, we can show that this is the probability of bit error for any transmitted codeword in the code.

Probability of bit error 10

- The equation derived for the bit error of probability for Hamming (7,4) code is long and requires knowledge of the weight distribution of the code.
- In many cases we don't know the exact weight distribution of the code.
- Therefore, P_b is estimated using bounds.

Probability of bit error bounds

- Given the probability of decoder error $P(E)$.
  - If a decoder error has occurred, at least one of the message bits must be in error.
  - If a decoder error has occurred, at most all of the message bits are in error.
- $(1/k)P(E) \leq P_b \leq P(E)$.
- Let us consider the AWGN channel with BPSK modulation
  - $$p = Q\left(\sqrt{\frac{2E_b}{N_0}R}\right).$$


1.00E-01
1.00E-02
1.00E-03
1.00E-04
1.00E-05
1.00E-06
BER
0
2
4
6
8
10
12
Eb/No (in dB)
p(uncoded)
P(E)
(1/k)P(E)
exact

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