ELG3175 Introduction to Communication Systems

Hilbert Transform, Preenvelope and Quadrature Representation of Bandpass Signals

Phase shifting systems and the Hilbert Transform

- Let $x(t)$ be the input to a phse transformer system.
- Let y(t) be the output. Then y(t) is a phase-shfted version of x(t) where all frequency components are shifted by θ.
- Let$x_0(t) = A\cos(2\pi f_0 t)$ be the input.
- Then $y_0(t) = A\cos(2\pi f_0 t+\theta)$.
- Let's change the inpuit frequency. In other words, let $x_1(t) = A\cos(2\pi f_1 t)$.
- Its corresponding output is $y_1(t) = A\cos(2\pi f_1 t+\theta)$.
- The amount of phase shift is independent of input frequency.

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Frequency response of phase shifting systems

- For $x_0(t) = A\cos(2\pi f_0 t)$, $X_0(f) =$.
- The output $y_0(t) = A\cos(2\pi f_0 t+\theta)$ has Fourier transform $Y_0(f) = \frac{A}{2}e^{j\theta}\delta(f-f_0)+\frac{A}{2}e^{-j\theta}\delta(f+f_0)$
- Also $X_1(f) = \frac{A}{2}\delta(f-f_1)+\frac{A}{2}\delta(f+f_1)$
- While $Y_1(f) = \frac{A}{2}e^{j\theta}\delta(f-f_1)+\frac{A}{2}e^{-j\theta}\delta(f+f_1)$
- Since the same is true for all input frequencies it is obvious that the frequency response of the phase shifting system is :

$$H(f)=\begin{cases} e^{j\theta}, & f>0 \\ e^{-j\theta}, & f<0 \end{cases}$$

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Hilbert Transform

- The Hilbert Transform introduces a phase shift of θ = -90°.
- For $x(t)$, its Hilbert transform $x_h(t)$ is a -90° phase shifted version of $x(t)$ (-π/2 radians).
- The Fourier transform of $x_h(t)$ is $X_h(f)$ which is given by :

$$X_h(f) = \begin{cases} e^{-j\pi/2} X(f), & f > 0 \\ e^{j\pi/2} X(f), & f < 0 \end{cases}$$

$$= -j\,\mathrm{sgn}(f) X(f)$$

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Hilbert Transform

- The Hilbert transform is also given by :

- $x_h(t) = \mathcal{F}^{-1}\{-j\,\text{sgn}(f)X(f)\} = x(t) * 1/\pi t$

$$= \int_{-\infty}^{\infty} \frac{x(\lambda)}{\pi(t-\lambda)} d\lambda = \int_{-\infty}^{\infty} \frac{x(t-\lambda)}{\pi\lambda} d\lambda$$

Examples

- Find the Hilbert Transforms of
  - $x(t) = A\cos(2\pi f_o t)$ et
  - $y(t) = \text{sinc}(t)$

- SOLUTION (a)

$$X(f) = \frac{A}{2}\delta(f - f_o) + \frac{A}{2}\delta(f + f_o) \quad \text{therefore}$$

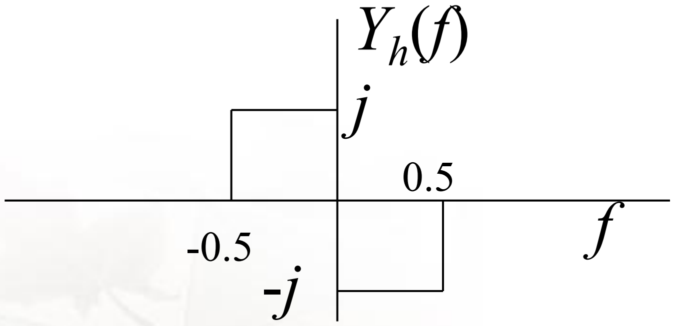
$$X_h(f) = \frac{-jA}{2}\delta(f - f_o) + \frac{jA}{2}\delta(f + f_o)$$

The Hilbert transform of $x(t)$ is $x_h(t) = \mathcal{F}^{-1}\{X_h(f)\}$

$= A\sin(2\pi f_o t)$.

Exemples

- SOLUTION (b)
  - $Y(f) = \Pi(f)$. The Fourier Transform of the Hilbert transform of $y(t)$ is $Y_h(f) = -j\mathrm{sgn}(f)\Pi(f)$.



  - $-j\mathrm{sgn}(f)\Pi(f) = -j\Pi(2(f-¼)) + j\Pi(2(f+¼))$,

Therefore $y_h(t) = \dfrac{-j}{2}\mathrm{sinc}(t/2)e^{j\frac{\pi}{2}t} + \dfrac{j}{2}\mathrm{sinc}(t/2)e^{-j\frac{\pi}{2}t}$

$$= \mathrm{sinc}\left(\tfrac{t}{2}\right)\sin\left(\tfrac{\pi}{2}t\right)$$

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Positive Pre-enveloppe

- Let $x(t)$ be a real signal with Fourier Transform $X(f)$.
- Let us define $x_+(t)$ as the positive pre-envelope of $x(t)$.
- The spectrum of $x_+(t)$ is 0 for negative frequencies and proportional to the spectrum of $x(t)$ for positive frequencies.
- The spectrum of the positive pre-envelope is:

$$X_+(f) = \begin{cases} 2X(f), & f > 0 \\ X(0), & f = 0 \\ 0, & f < 0 \end{cases}$$

Positive Pre-envelope

- We can show that $X_+(f) = X(f) + \text{sgn}(f)X(f) = X(f) + j(-j\text{sgn}(f)X(f)) = X(f) + jX_h(f)$ where $X_h(f) = \mathcal{F}\{x_h(t)\}$.
- Therefore

$$x_+(t) = x(t) + jx_h(t)$$

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Examples

- Find the positive pre-envelope of $x(t) = \cos(2\pi f_c t)$.
- Find te pre-envelope of $y(t) = \text{sinc}(t)$.
  - SOLUTION
- We know that$x_h(t) = \sin(2\pi f_c t)$, therefore $x_+(t) = \cos(2\pi f_c t)+j\sin(2\pi f_c t)$.
- Therefore $x_+(t) = e^{j2\pi fct}$.
- For $y_+(t)$ we should find $Y_+(f)$.
- $Y(f) = \Pi(f)$, therefore $Y_+(f) = 2\Pi(2(f-¼))$. Therefore $y_+(t) = \mathcal{F}^{-1}\{Y_+(f)\} = \text{sinc}(t/2)e^{j(\pi/2)t}$.

Negative pre-envelope

- The negative pre-envelope of $x(t)$ is a signal that has only the negative spectrum of $x(t)$.

$$X_-(f) = \begin{cases} 2X(f), & f < 0 \\ X(0), & f = 0 \\ 0, & f > 0 \end{cases}$$

- We see that $X_+(f)+X_-(f) = 2X(f)$, therefore $x_+(t)+x_-(t) = 2x(t)$.
- Therefore $x_-(t) = x(t)-jx_h(t)$.

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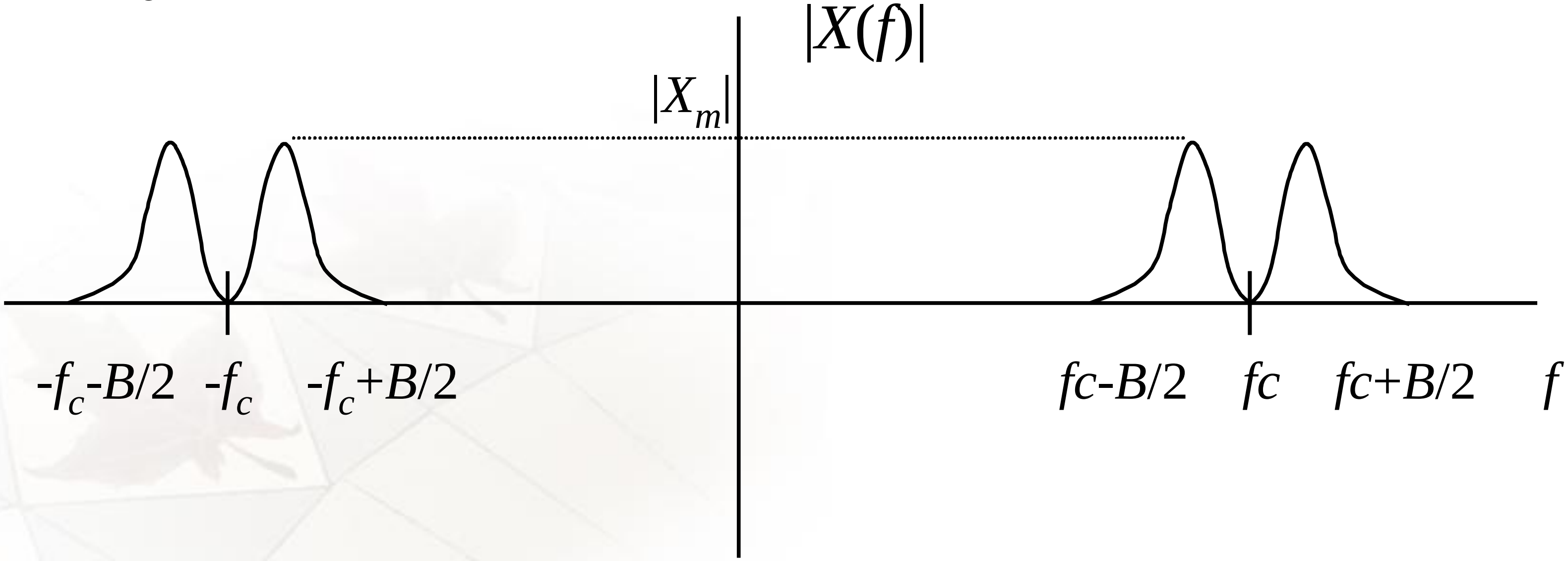
Bandpass Signals

- The signal $x(t)$ is a bandpass signal if its spectrum is non zero in a range of frequencies f_c - $(B/2) \le |f| \le f_c + B/2$, and 0 elsewhere where B is its bandwidth and $B < f_c$.



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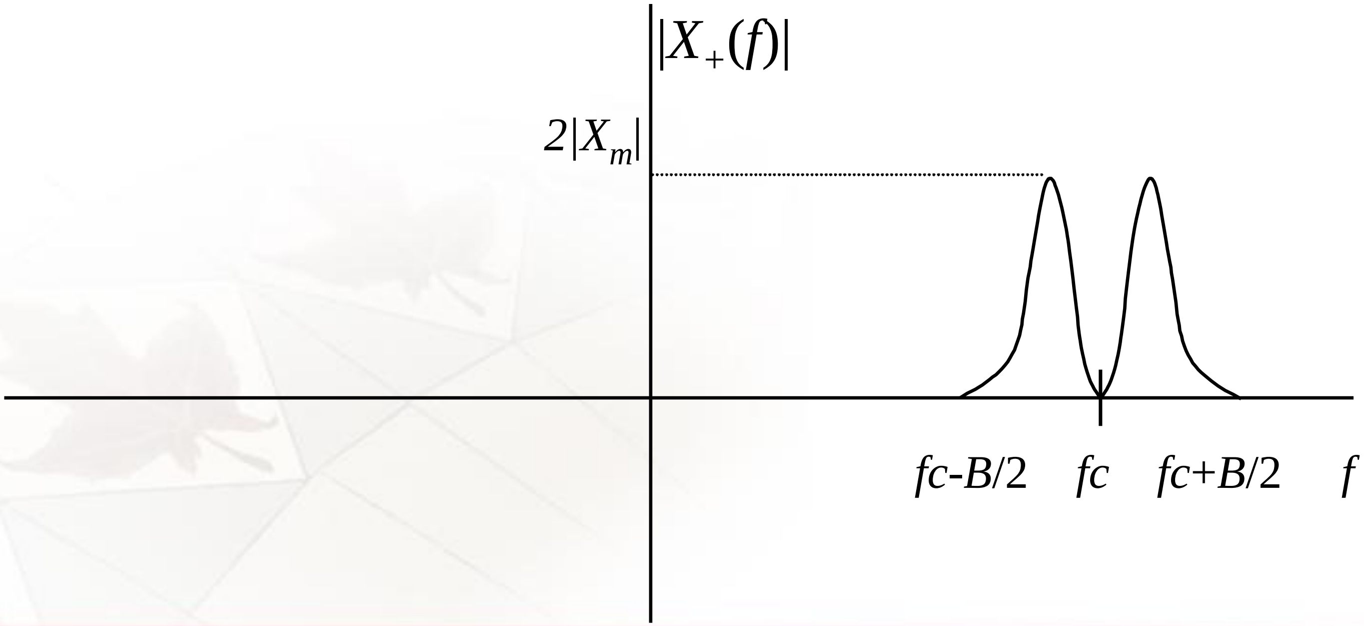
Pre-envelope of a bandpass signal

- Let's find the pre-envelope of a bandpass signal, $x(t)$.
- $|X_+(f)|$ is shown below.
- The pre-envelope is $x_+(t) = x(t)+jx_h(t)$.



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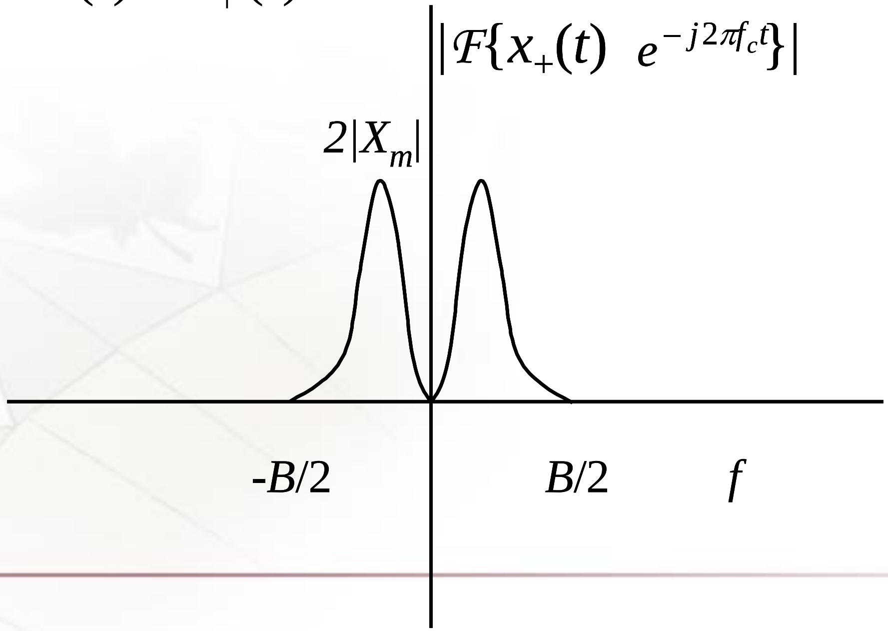
Complex envelope (lowpass equivalent)

- The complex envelope of $x(t)$, $\tilde{x}(t)$, is its lowpass equivalent
- The spectrum of $\tilde{x}(t)$ has the same form as $x_+(t)$, but it is centred at f = 0.
- Therefore, $\tilde{x}(t) = x_+(t)e^{-j2\pi f_c t}$



- Alors, nous définissons $\tilde{x}(t) = x_+(t)e^{-j2\pi f_c t}$ comme l'enveloppe complexe du signal en bande passante *x*(*t*).
- Nous voyons du spectre de $\tilde{x}(t)$ que la largeur de bande de l'enveloppe complexe est *B*/2 pour un signal en bande passante avec largeur de bande *B*.

Quadrature representation of bandpass signals

- If $\tilde{x}(t) = x_+(t)e^{-j2\pi f_c t}$
- Then $x_+(t) = \tilde{x}(t)e^{j2\pi f_c t}$
- Also $x(t) = \mathrm{Re}\{x_+(t)\}$
- Then

$$x(t) = \mathrm{Re}\{\tilde{x}(t)e^{j2\pi f_c t}\} = \mathrm{Re}\{\tilde{x}(t)\}\cos 2\pi f_c t - \mathrm{Im}\{\tilde{x}(t)\}\sin 2\pi f_c t$$

- All bandpass signals can be written in the form

$$x(t) = x_I(t)\cos 2\pi f_c t - x_Q(t)\sin 2\pi f_c t$$

- where $x_I(t) = \mathrm{Re}\{\tilde{x}(t)\}$ and $x_Q(t) = \mathrm{Im}\{\tilde{x}(t)\}$

Example 1

- $x(t)$ = $A\cos(2\pi f_c t+\phi)$.
- Find it's complex envelope and its quadrature representation.
  - SOLUTION
- $X(f) = (1/2)e^{j\phi}\delta(f-f_c)+(1/2)e^{-j\phi}\delta(f+f_c)$
- $X_+(f) = e^{j\phi}\delta(f-f_c)$

$$\tilde{X}(f) = X_+(f+f_c) = e^{j\phi}\delta(f)$$

$$\tilde{x}(t) = e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

- Therefore $x(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t) = \cos(\phi)\cos(2\pi f_c t) - \sin(\phi)\sin(2\pi f_c t)$.

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Example 2

- $y(t)$ = $100\sin(2\pi(f_c f_1)t) + 500\cos 2\pi f_c t + 100\sin(2\pi(f_c+f_1)t)$.
- $Y(f) = -j50\delta(f-f_c+f_1)+j50\delta(f+f_c-f_1) + 250\delta(f-f_c) + 250\delta(f+f_c) - j50\delta(f-f_c-f_1) + j50\delta(f+f_c+f_1)$.
- $Y_+(f) = -j100\delta(f-f_c+f_1) +500\delta(f-f_c)-j100\delta(f-f_c-f_1)$

$$\tilde{Y}(f) = Y_+(f+f_c) = -j100\delta(f+f_1) + 500\delta(f) - j100\delta(f-f_1)$$

$$\tilde{y}(t) = 500 - j200\cos(2\pi f_1 t)$$

- $y(t) = 500\cos(2\pi f_c t)+200\cos(2\pi f_1 t)\sin(2\pi f_c t)$