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ELG3175 Introduction to Communication Systems

Fourier transform, Parseval's theoren, Autocorrelation and Spectral Densities

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Fourier Transform of a Periodic Signal

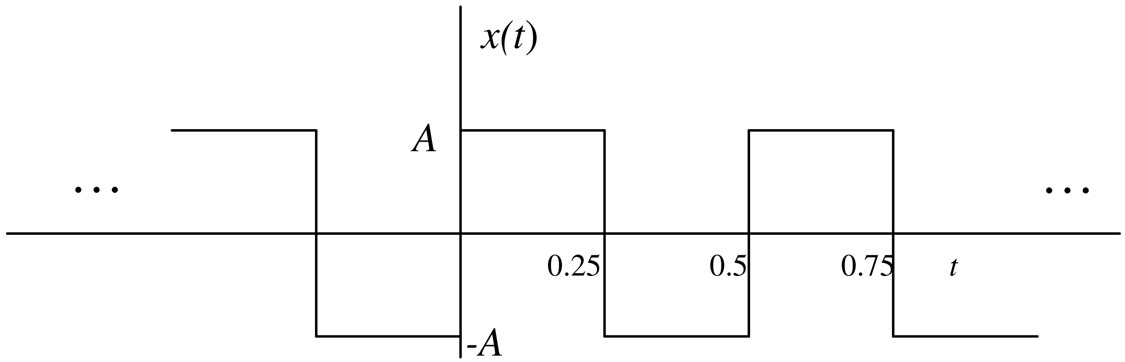
- A periodic signal can be expressed as a complex exponential Fourier series.
- If $x(t)$ is a periodic signal with period T, then :

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi \frac{n}{T} t}$$

- Its Fourier Transform is given by:

$$X(f) = \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} X_n e^{j2\pi \frac{n}{T} t} \right\} = \sum_{n=-\infty}^{\infty} X_n \mathcal{F}\left\{ e^{j2\pi \frac{n}{T} t} \right\} = \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_o)$$

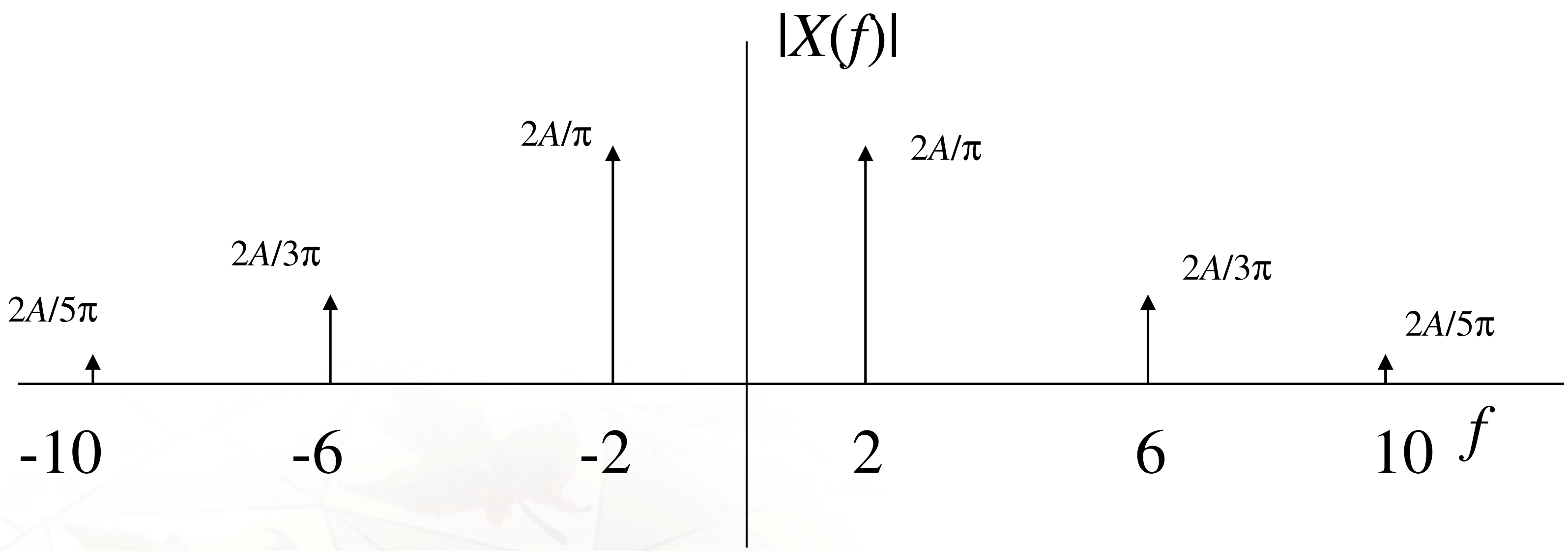
Example



$$x(t) = \sum_{\substack{n=-\infty \\ n \text{ is odd}}}^{\infty} \frac{2A}{j\pi n} e^{j4\pi n t} = \sum_{\substack{n=-\infty \\ n \text{ is odd}}}^{\infty} \left(-j\frac{2A}{\pi n} \right) e^{j4\pi n t}$$

$$X(f) = \sum_{\substack{n=-\infty \\ n \text{ is odd}}}^{\infty} \left(-j\frac{2A}{\pi n} \right) \delta(f - 2n)$$

Example Continued



Energy of a periodic signal

- If $x(t)$ is periodic with period T, the energy on one period is:

$$E_p = \int_T |x(t)|^2 \, dt$$

- The energy on N periods is $E_N = NE_p$.
- The average normalized energy is $E = \lim_{N \to \infty} NE_p = \infty$

- Therefore periodic signals are never energy signals.

Average normalized power of periodic signals

- The power of $x(t)$ on one period is :

$$P_p = \frac{1}{T} \int_T |x(t)|^2 \, dt$$

- And it's power on N periods is :

$$P_{Np} = \frac{1}{NT} \int_{NT} |x(t)|^2 \, dt = \frac{1}{N} \left(\sum_{i=1}^{N} \frac{1}{T} \int_{(i-1)T}^{iT} |x(t)|^2 \, dt \right) = \frac{1}{N} \times N P_p = P_p$$

- It's average normalized power is therefore $P = \lim_{N \to \infty} P_{Np} = P_p$
- Therefore, for a periodic signal, its average normalized power is equal to the power over one period.

Hermetian symmetry of Fourier Transform when x(t) is real

$$
\begin{aligned}
X^*(f) &= \left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt\right]^* \\
&= \int_{-\infty}^{\infty} x^*(t)e^{j2\pi ft}\,dt \\
&= \int_{-\infty}^{\infty} x(t)e^{j2\pi ft}\,dt \\
&= \int_{-\infty}^{\infty} x(t)e^{-j2\pi(-f)t}\,dt \\
&= X(-f)
\end{aligned}
$$

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Parseval's Theorem

- Let us assume that $x(t)$ is an energy signal.
- Its average normalized energy is :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} x(t) x^*(t) dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \right] x^*(t) dt$$

$$= \int_{-\infty}^{\infty} X(f) \int_{-\infty}^{\infty} x^*(t) e^{-j2\pi(-f)t} dt df$$

$$= \int_{-\infty}^{\infty} X(f) X^*(f) df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Example

$$\int_{-\infty}^{\infty} \operatorname{sinc}^2(t)dt = ?$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \operatorname{sinc}^2(t)dt &= \int_{-\infty}^{\infty} |\operatorname{sinc}(t)|^2 \, dt \\
&= \int_{-\infty}^{\infty} |\Pi(f)|^2 \, df \\
&= \int_{-1/2}^{1/2} df \\
&= 1
\end{aligned}$$

Autocorrelation function of energy signals

- The autocorrelation function is a measure of similarity between a signal and itself delayed by τ.
- The function is given by :

$$\varphi_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

- We note that

$$\varphi_x(0) = \int_{-\infty}^{\infty} x(t)x^*(t)dt = \int_{-\infty}^{\infty} |x(t)|^2 dt = E$$

- We also note that $\varphi_x(\tau) = x(\tau) * x^*(-\tau)$

Energy spectral density

- Let $x(t)$ be an energy signal and let us define $G(f) = |X(f)|^2$
- The inverse Fourier transform of $G(f)$ is

$$\begin{aligned}
\int_{-\infty}^{\infty} G(f)e^{j2\pi f\tau}df &= \int_{-\infty}^{\infty} X(f)X^*(f)e^{j2\pi f\tau}df \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dtX^*(f)e^{j2\pi f\tau}df \\
&= \int_{-\infty}^{\infty} x(t)\int_{-\infty}^{\infty} X^*(f)e^{j2\pi f\tau}e^{-j2\pi ft}dfdt \\
&= \int_{-\infty}^{\infty} x(t)\int_{-\infty}^{\infty} X^*(f)e^{j2\pi f\tau}e^{j2\pi f(-t)}dfdt \\
&= \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt
\end{aligned}$$

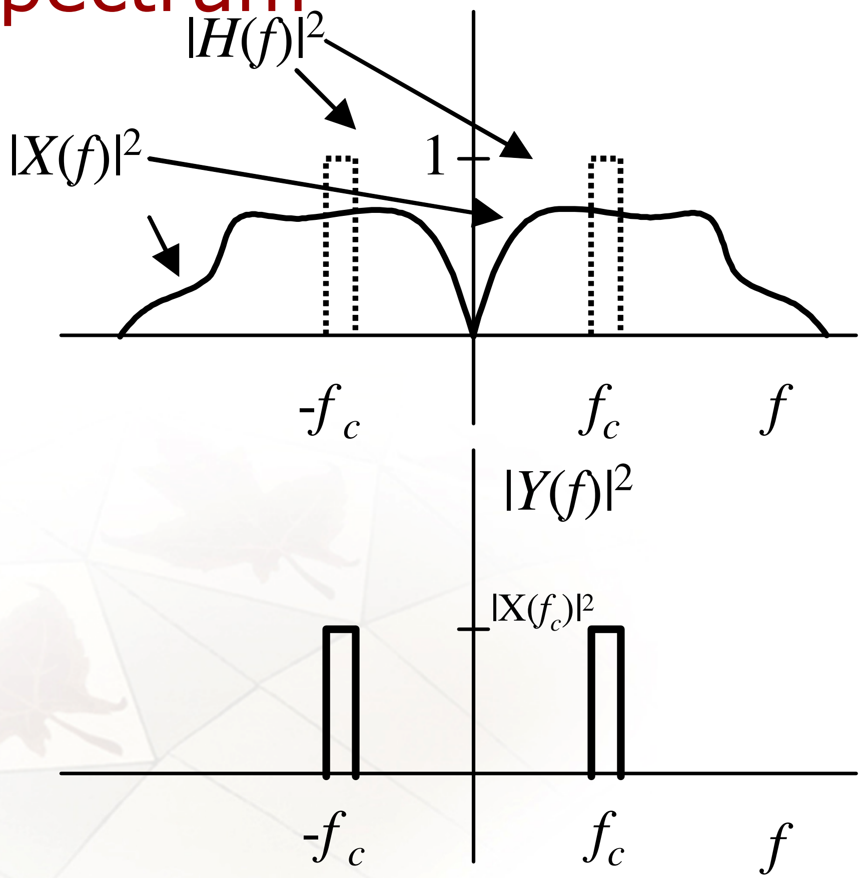
Energy spectral density

- The energy spectral density $G(f) = \mathcal{F}\{\varphi(\tau)\}$.
- Example
  - Find the autocorrelation function of $\Pi(t)$.

Energy spectral density of output of LTI system

- If an energy signal with ESD $G_x(f)$ is input to an LTI system with impulse response $H(f)$, then the output, $y(t)$, is also an energy signal with ESD $G_y(f) = G_x(f)|H(f)|^2$.

ESD describes how the energy of the signal is distributed throughout the spectrum



$E_y = 2|X(f)|^2 \Delta f$

Δf is Hz, therefore

$|X(f)|^2$ is in J/Hz

Autocorrelation Function of Power Signals

- We denote the autocorrelation of power signals as $R_x(\tau)$.
- Keeping in mind that for the energy signal autocorrelation, $\varphi(0)=E_x$, then for the power signal autocorrelation function $R_x(0)$ should equal P_x.
- Therefore

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x^*(t-\tau)dt$$

Power Spectral Density

- Power signals can be described by their PSD.
- If $x(t)$ is a power signal, then its PSD is denoted as $S_x(f)$, where $S_x(f) = \mathscr{F}\{R_x(\tau)\}$.
- The PSD describes how the signal's power is distributed throughout its spectrum.
- If $x(t)$ is a power signal and is input to an LTI system, then the output, $y(t)$, is also a power signal with PSD $S_y(f) = S_x(f)|H(f)|^2$.