Université d'Ottawa | University of Ottawa

ELG3175 Introduction to Communication Systems

Frequency Response, Energy and Power

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Frequency Response of Linear Time Invariant Systems

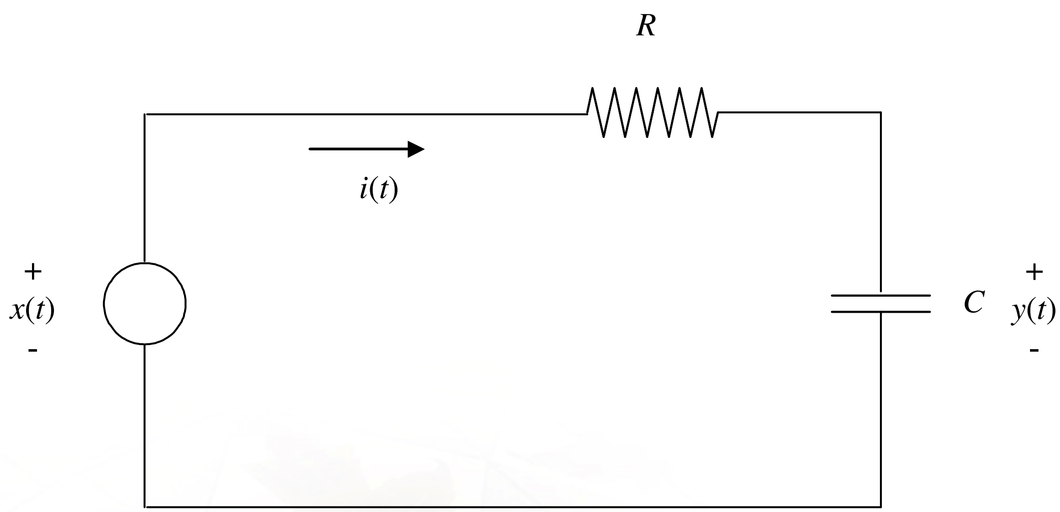
- An LTI system is described by its impulse response, $h(t)$.
- For an input $x(t)$, the output is $y(t)$ which is given by

$$y(t) = x(t) * h(t)$$

- The Fourier Transform of the output is $Y(f) = \mathcal{F}\{y(t)\}$ which is given by: $Y(f) = X(f)H(f)$
- where $X(f) = \mathcal{F}\{x(t)\}$ is the FT of the imput and $H(f) = \mathcal{F}\{h(t)\}$ is called the frequency response of the LTI system.
- Therefore the frequency response can also be found by:

$$H(f) = \frac{Y(f)}{X(f)}$$

Example

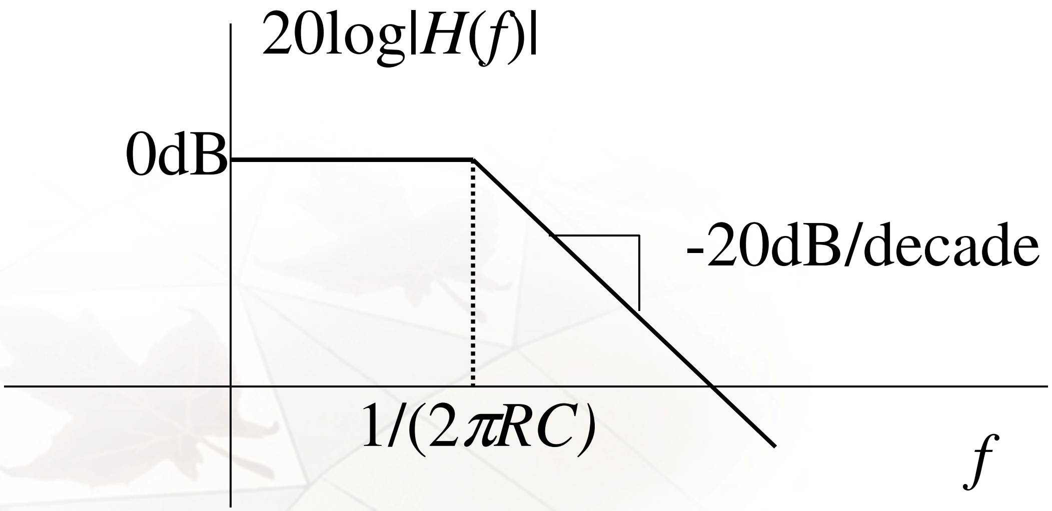


$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$H(f) = ?$$

Example continued

- $H(f) = \mathcal{F}\{h(t)\} = \mathcal{F}\left\{\frac{1}{RC}e^{-t/RC}u(t)\right\} = \frac{1}{RC}\frac{1}{j2\pi f + \frac{1}{RC}} = \frac{1}{j2\pi fRC + 1}$



Example 2

- Find the circuit's output when $x(t) = A\cos 2\pi f_o t$.
  - Solution
- The output's spectrum is: $Y(f) = X(f)H(f)$

$$
\begin{aligned}
Y(f) &= \left(\tfrac{1}{2}\delta(f-f_o)+\tfrac{1}{2}\delta(f+f_o)\right)\left(\tfrac{1}{1+j2\pi fRC}\right) \\
&= \frac{1}{2(1+j2\pi f_o RC)}\delta(f-f_o)+\frac{1}{2(1-j2\pi f_o RC)}\delta(f+f_o) \\
&= 1/\sqrt{1+(2\pi f_o RC)^2}\left[e^{-j\tan^{-1}2\pi f_o RC}\times\tfrac{1}{2}\delta(f-f_o)+e^{j\tan^{-1}2\pi f_o RC}\times\tfrac{1}{2}\delta(f+f_o)\right]
\end{aligned}
$$

Frequency and phase response

$$y(t) = 1/\sqrt{1+(2\pi f_o RC)^2}\left[e^{-j\tan^{-1}2\pi f_o RC}\times\tfrac{1}{2}e^{j2\pi f_o t} + e^{j\tan^{-1}2\pi f_o RC}\times\tfrac{1}{2}e^{-j2\pi f_o t}\right]$$

$$= 1/\sqrt{1+(2\pi f_o RC)^2}\left[\tfrac{1}{2}e^{j(2\pi f_o t-\tan^{-1}2\pi f_o RC)} + \tfrac{1}{2}e^{-j(2\pi f_o t-\tan^{-1}2\pi f_o RC)}\right]$$

$$= 1/\sqrt{1+(2\pi f_o RC)^2}\cos\left(2\pi f_o t - \tan^{-1}2\pi f_o RC\right)$$

The term $1/\sqrt{1+(2\pi f_o RC)^2} = |H(f_o)|$ is the amplitude response of the circuit at frequency f_o and $-\tan^{-1}2\pi f_o RC = \langle H(f_o))$ is its phase response at frequency f_o.

Frequency and phase response

$$|H(f)| = \sqrt{\mathrm{Re}\{H(f)\}^2 + \mathrm{Im}\{H(f)\}^2}$$

$$\angle H(f) = \tan^{-1}\left(\frac{\mathrm{Im}\{H(f)\}}{\mathrm{Re}\{H(f)\}}\right)$$

Energy and Power

- The root mean square (RMS) of a signal on interval $t_o \leq t \leq t_o + T$ is :

$$X_{RMS} = \sqrt{\frac{1}{T} \int_{t_o}^{t_o+T} |x(t)|^2 \, dt}$$

- The instantaneous power $P(t) = v(t)i(t)$, where $v(t)$ is the votltage drop and $i(t)$ is the current producing the voltage drop.
- For a voltage drop across a resistor, $P(t) = v^2(t)/R$ where R is the resistance. On the interval $t_o \leq t \leq t_o + T$, the average power is:

$$\overline{P(t)} = \frac{1}{T} \int_{t_o}^{t_o+T} P(t)dt = \frac{1}{T} \int_{t_o}^{t_o+T} \frac{v^2(t)}{R} dt = \frac{V_{RMS}^2}{R}$$

Normalized power

- The normalized power (R = 1) is :

$$P = \frac{1}{T} \int_{t_o}^{t_o+T} |\, v(t) \,|^2 \; dt$$

- If we take an infinite time window, then the expression becomes

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\, v(t) \,|^2 \; dt$$

Power Signal

- The signal x(t) is a power signal if the value of P calculated below is greater than 0 and finite ($0 < P < \infty$).

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

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Normalized Energy

- Power is energy per unit time. Therefore

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Energy Signal

- The signal x(t) is an energy signal if the energy computed below is finite (E<∞).

$$E = \int_{-\infty}^{\infty} | x(t) |^2 \, dt$$

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Examples

- Determine whether te following signals are energy or power signal (or neither).
  - $x(t) = A\cos(2\pi f_o t)$
  - $y(t) = \Pi(t)$
  - $z(t) = tu(t)$.