Université d'Ottawa | University of Ottawa

ELG3175 Introduction to Communication Systems

Frequency-Domain Representation: Fourier Series and Fourier Transform

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Generalized Fourier Series

- We are given a set of functions $\{\phi_n(t)\}_{n=1,2,\ldots,N}$ which are mutually orthogonal on the interval $t_o \le t \le t_o + T$.

$$\int_{t_o}^{t_o+T} \phi_i(t)\phi_j^*(t)dt = \begin{cases} 0, & i \neq j \\ c_n, & i = j = n \end{cases}$$

- If $c_n = 1$ for all n, we call this an orthonormal set of functions.

Generalized Fourier Series (2)

- We can find an approximation for any function $x(t)$, on the interval $(t_o, t_o + T)$ by $x_a(t)$ which is a weighted sum of the functions in the orthogonal set of functions :

$$x_a(t) = \sum_{n=1}^{N} X_n \phi_n(t)$$

- We wish to find the approximation that minimises the mean square error between $x(t)$ and $x_a(t)$:

$$\varepsilon_N = \int_{t_o}^{t_o+T} \left| x(t) - x_a(t) \right|^2 dt$$

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Generalized Fourier Series (3)

- We can show that $$X_n = \frac{1}{c_n} \int_{t_o}^{t_o+T} x(t)\phi_n^*(t)\,dt \qquad (1)$$
- And the best approximation is

$$x_a(t) = \sum_{n=1}^{N} X_n \phi_n(t)$$

- Where X_n is given by (1).

The complex exponential function

$$e^{j2\pi nf_o t} = \cos(2\pi nf_o t) + j\sin(2\pi nf_o t) \qquad n \text{ is an integer}$$

This is a periodic fiunction with period T_p.

$$e^{j2\pi nf_o t} = e^{j2\pi nf_o (t+T_p)} = e^{j2\pi nf_o t} e^{j2\pi nf_o T_p}$$

The second exponential equals 1 when $nf_o T_p$ is an integer.

Orthogonality and c_n

On the interval $t_o \le t \le t_o+T$, for $f_o = 1/T$.

$$\int_{t_o}^{t_o+T} \phi_n(t)\phi_m^*(t)dt = \int_{t_o}^{t_o+T} e^{j2\pi n f_o t} e^{-j2\pi m f_o t} dt = \int_{t_o}^{t_o+T} e^{j2\pi(n-m)f_o t} dt$$

For $m=n$

$$c_n = \left. \int_{t_o}^{t_o+T} e^{j2\pi(n-m)f_o t} dt \right|_{m=n}$$

$$= \int_{t_o}^{t_o+T} dt = T$$

For $m \neq n$

$$\int_{t_o}^{t_o+T} e^{j2\pi(n-m)f_o t} dt = \frac{e^{j2\pi(n-m)f_o(t_o+T)} - e^{j2\pi(n-m)f_o t_o}}{j2\pi(n-m)f_o}$$

$$= \frac{e^{j2\pi(n-m)f_o t_o} e^{j2\pi(n-m)f_o T} - e^{j2\pi(n-m)f_o t_o}}{j2\pi(n-m)f_o}$$

$$= 0$$

Complex exponential Fourier Series

On the interval $t_o \le t \le t_o + T$ the complex exponential Fourier Series of $x(t)$ is given by :

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t} = \sum_{n=-\infty}^{\infty} X_n e^{j\frac{2\pi n t}{T}}$$

where

$$X_n = \frac{1}{T} \int_{t_o}^{t_o+T} x(t) e^{-j2\pi n f_o t} \, dt$$

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Complex Exponential Fourier Series for periodic signals

- Consider the signal $\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t}$ on the interval -∞ ≤ *t* ≤ ∞.

- We know that complex exponential functions are periodic.
- The fundamental period of a complex exponential is *T*/|*n*|.
- The period of a sum of periodic signals is the lowest common multiple (LCM) of the individual periods
- In this case, the fundamental period is *T*.

Complex Exponential Fourier Series for periodic signals (2)

- $\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t}$ is periodic with period $T = 1/f_o$.

- The fundamental frequency f_o is the inverse of the period.
- Therefore if $x(t)$ is periodic with period T, then

$$\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t} = x(t) \text{ pour } -\infty < t < \infty$$

- Therefore if $x(t)$ is periodic with period T then its Fourier series is an exact representation . In other words,

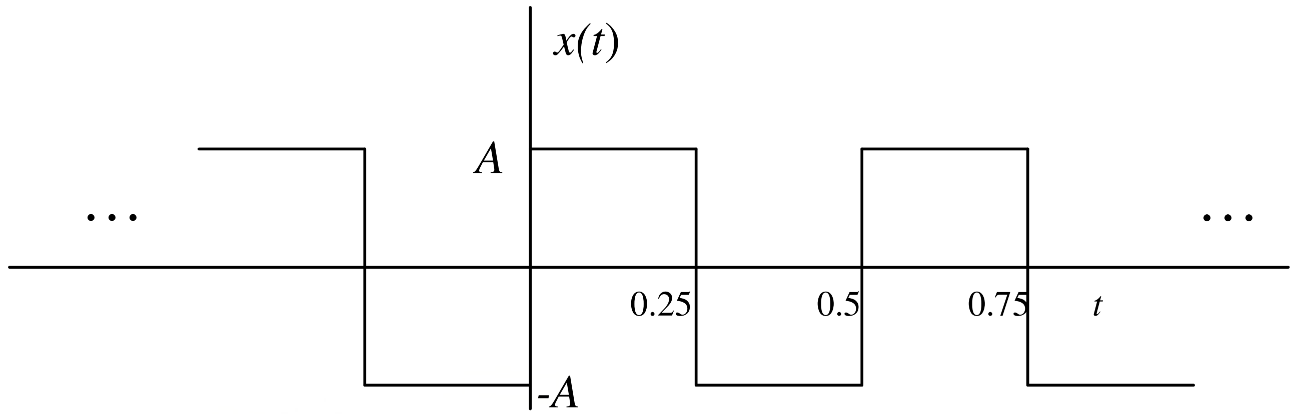
$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t}$$

Complex Exponential Fourier Series for periodic signals (3)

- We can use any period to calculate the Fourier coefficients.
  - Therefore we write

$$X_n = \frac{1}{T} \int_T x(t) e^{-j2\pi n f_o t} dt$$

Example



Trouvez la série de Fourier exponentielle complexe du signal périodique $x(t)$

Find the complex exponential Fourier Series of the periodic signal $x(t)$.

Solution

- We need to find
  - The period of $x(t)$ as well as f_o.
  - The coefficients X_n
  - Then the Fourier series itself $\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t}$

Solution (2)

- In our example, the period is 0.5, therefore $f_o = 2$.
- The set of functions is therefore $e^{j4\pi nt}$.
- Therefore

$$
\begin{aligned}
X_n &= 2\int_0^{0.5} x(t)e^{-j4\pi nt}\,dt \\
&= 2\left(\int_0^{0.25} Ae^{-j4\pi nt}\,dt - \int_{0.25}^{0.5} Ae^{-j4\pi nt}\,dt\right) \\
&= 2A\left(-\frac{1}{j4\pi n}e^{-j4\pi nt}\Big|_0^{0.25} + \frac{1}{j4\pi n}e^{-j4\pi nt}\Big|_{0.25}^{0.5}\right) \\
&= 2A\left(-\frac{1}{j4\pi n}e^{-j\pi n} + \frac{1}{j4\pi n} + \frac{1}{j4\pi n}e^{-j2\pi n} - \frac{1}{j4\pi n}e^{-j\pi nt}\right) \\
&= 2A\left(\frac{1}{j2\pi n} - \frac{1}{j2\pi n}e^{-j\pi n}\right) = \frac{A}{j\pi n}\left[1-(-1)^n\right]
\end{aligned}
$$

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Solution continued

- For n = 0, we have X_0 = 0/0.

$$X_0 = 2\int_0^{0.5} x(t)dt = 0$$

$$x(t) = \sum_{\substack{n=-\infty \\ n \text{ is odd}}}^{\infty} \frac{2A}{j\pi n} e^{j4\pi nt} = \sum_{i=-\infty}^{\infty} \frac{2A}{j\pi(2i-1)} e^{j4\pi(2i-1)t}$$

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Hermetian symmetry for real valued signals

- Let us assume that $x(t)$ is a real-valued signal.
- In other words $\mathrm{Im}\{x(t)\} = 0$.
- Then X_n^* is given by :

$$
\begin{aligned}
X_n^* &= \left[\frac{1}{T}\int_T x(t)e^{-j2\pi n f_o t}\,dt\right]^* \\
&= \frac{1}{T}\int_T \left(x(t)e^{-j2\pi n f_o t}\right)^* dt \\
&= \frac{1}{T}\int_T x^*(t)e^{j2\pi n f_o t}\,dt \\
&= \frac{1}{T}\int_T x(t)e^{-j2\pi(-n) f_o t}\,dt = X_{-n}
\end{aligned}
$$

Trigonometric Fourier Series

- Assuming x(t) is real, the real part of X_n is :

$$
\begin{aligned}
\mathrm{Re}\{X_n\} &= \mathrm{Re}\left\{\frac{1}{T}\int_T x(t)e^{-j2\pi n f_o t}\,dt\right\} \\
&= \mathrm{Re}\left\{\frac{1}{T}\int_T x(t)(\cos 2\pi n f_o t - j\sin 2\pi n f_o t)dt\right\} \\
&= \mathrm{Re}\left\{\frac{1}{T}\int_T x(t)\cos 2\pi n f_o t dt - j\frac{1}{T}\int_T x(t)\sin 2\pi n f_o t dt\right\} \\
&= \frac{1}{T}\int_T x(t)\cos 2\pi n f_o t dt
\end{aligned}
$$

Trigonometric Fourier Series (2)

- Therefore the imaginary part of X_n is:

$$\operatorname{Im}\{X_n\} = -\frac{1}{T}\int_T x(t)\sin 2\pi n f_o t\,dt$$

- The complex exponential Fourier series can be written as :

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t}$$
$$= X_0 + \sum_{n=1}^{\infty}\left(X_n e^{j2\pi n f_o t} + X_{-n} e^{-j2\pi n f_o t}\right)$$

Trigonometric Fourier Series (3)

- Since $x(t)$ is real, $X_{-n} = X_n^*$,

$$x(t) = X_0 + \sum_{n=1}^{\infty} \{ (\mathrm{Re}\{X_n\} + j\,\mathrm{Im}\{X_n\})(\cos(2\pi n f_o t) + j\sin(2\pi n f_o t))$$

$$+ (\mathrm{Re}\{X_n\} - j\,\mathrm{Im}\{X_n\})(\cos(2\pi n f_o t) - j\sin(2\pi n f_o t)) \}$$

$$= X_0 + \sum_{n=1}^{\infty} (2\,\mathrm{Re}\{X_n\}\cos(2\pi n f_o t) - 2\,\mathrm{Im}\{X_n\}\sin(2\pi n f_o t))$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_o t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_o t)$$

$$a_0 = X_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$a_n = 2\,\mathrm{Re}\{X_n\} = \frac{2}{T}\int_T x(t)\cos 2\pi n f_o t\,dt$$

$$b_n = -2\,\mathrm{Im}\{X_n\} = \frac{2}{T}\int_T x(t)\sin 2\pi n f_o t\,dt$$

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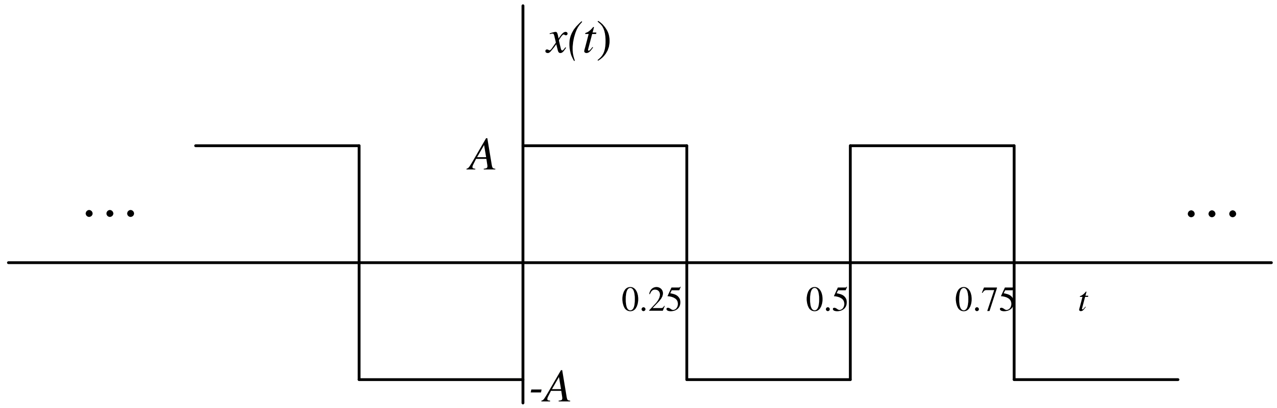
Example



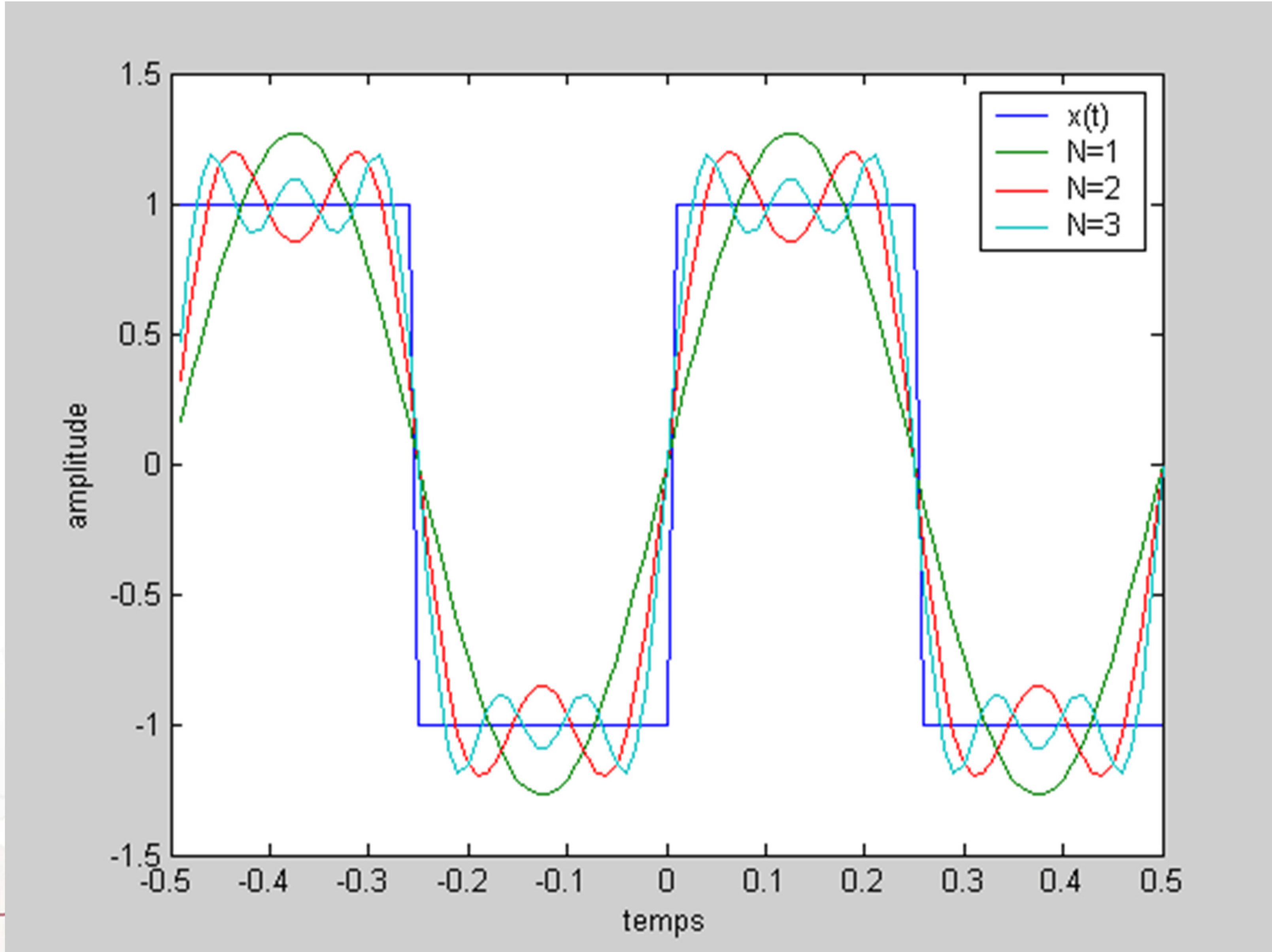
$$x(t) = \sum_{\substack{n=-\infty \\ n \text{ impaire}}}^{\infty} \frac{2A}{j\pi n} e^{j4\pi n t} = \sum_{\substack{n=-\infty \\ n \text{ impaire}}}^{\infty} \left(-j\frac{2A}{\pi n}\right) e^{j4\pi n t}$$

$X_0 = 0$, $\mathrm{Re}\{X_n\} = 0$ and $\mathrm{Im}\{X_n\} = -2A/\pi n$ for odd values of n.
Therefore $b_n = 4A/\pi n$ for odd values of n.

Example continued

$$x(t) = \sum_{\substack{n=1 \\ n \text{ impaire}}}^{\infty} \frac{4A}{\pi n} \sin 4\pi n t = \sum_{i=1}^{\infty} \frac{4A}{\pi(2i-1)} \sin 4\pi(2i-1)t$$

Let $x_N(t) = \sum_{i=1}^{N} \frac{4A}{\pi(2i-1)} \sin 4\pi(2i-1)t$ represent the first N terms of $x(t)$.


1.5
1
0.5
0
-0.5
-1
-1.5
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
amplitude
temps
x(t)
N=1
N=2
N=3

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Fourier Transform

- The Fourier transform is a frequency-dependent function that is an extension of the Fourier series to non periodic functions.
- It describes the spectral content of a signal
  - In other words it is the frequency domain representation of a signal.

Fourier Transform (2)

- La fonction $X(f)$ est la transformée de Fourier de $x(t)$.
- $X(f)$ décrit le contenu spectral de $x(t)$.
- $X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$
- $x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df$

Example

- Find the Fourier Transform of $x(t) = \Pi(t)$.

- **Solution**
- The Fourier Transform of $x(t)$ is :

$$X(f) = \int_{-\infty}^{\infty} \Pi(t) e^{-j2\pi ft} dt = \int_{-1/2}^{1/2} e^{-j2\pi ft} dt$$

$$= -\frac{1}{j2\pi f} e^{-j2\pi ft} \Big|_{-1/2}^{1/2} = -\frac{1}{j2\pi f}(e^{-j\pi f} - e^{j\pi f})$$

$$= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} = \frac{\sin \pi f}{\pi f} = \mathrm{sinc}(f)$$

Example 2

- Trouvez la transformée de Fourier de $x(t) = \delta(t)$.
- **Solution**

$$X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = e^{-j2\pi ft}\Big|_{t=0} = 1$$

Fourier Transform Properties

1. Linearity

- The Fourier Transform is a linear function. In other words, if $X_1(f) = \mathcal{F}\{x_1(t)\}$ et $X_2(f) = \mathcal{F}\{x_2(t)\}$, then for $x_3(t) = ax_1(t) + bx_2(t)$, $X_3(f) = \mathcal{F}\{x_3(t)\} = aX_1(f) + bX_2(f)$.

2. Time Delay

- If the Fourier Transform of $x_1(t)$ is $X_1(f)$ then the Fourier transform of $x_2(t) = x_1(t-t_o)$ is $X_2(f) = X_1(f)e^{-j2\pi f t_o}$

3. Time Scaling

- If $\mathcal{F}\{x(t)\} = X(f)$, then $\mathcal{F}\{x(at)\} = (1/|a|)X(f/a)$.

4. Duality

- If $\mathcal{F}\{x(t)\} = X(f)$, then $\mathcal{F}\{X(t)\} = x(-f)$.

Properties (2)

5. Frequency shift

- If $X(f) = \mathcal{F}\{x(t)\}$, then $X(f-f_o) = \mathcal{F}\{x(t)\, e^{j2\pi f_o t}\}$

6. Convolution

- If $z(t) = x(t)*y(t)$, then $Z(f) = X(f)Y(f)$.

7. Multiplication

- If $z(t) = x(t)y(t)$, then $Z(f) = X(f)*Y(f)$.

8. Derivative

- $\mathcal{F}\{\frac{dx(t)}{dt}\} = j2\pi f X(f)$

9. Integration

- $\mathcal{F}\left\{\int_{-\infty}^{t} x(\lambda)d\lambda\right\} = \frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$

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Transform Properties (3)

10. Complex conjugate

- $\mathcal{F}\{x^*(t)\} = X^*(-f)$