ELG3175 Introduction to Communication Systems

Angle Modulation Continued

Les caractéristiques des signaux modulés en angle

	PM Signal	FM Signal
Instantaneous phase $\phi_i(t)$	$k_p m(t)$	$2\pi k_f \int_0^t m(\tau)d\tau$
Instantaneous frequency	$f_c + \frac{k_p}{2\pi}\frac{dm(t)}{dt}$	$f_c + k_f m(t)$
Maximum phase deviation $\Delta\phi_{max}$	$k_p \mid m(t) \mid_{\max}$	$2\pi k_f \mid x(t) \mid_{\max}$ où $x(t) = \int_{-\infty}^t m(\tau)d\tau$
Maximum frequency deviation Δf_{max}	$\frac{k_p}{2\pi} \mid x(t) \mid_{\max}$ où $x(t) = \frac{dm(t)}{dt}$	$k_f \mid m(t) \mid_{\max}$
Power	$\frac{A_c^2}{2}$	$\frac{A_c^2}{2}$

uOttawa

Modulation index

- Assume that $m(t) = A_m \cos(2\pi f_m t)$. The resulting FM and PM signals are :

$$s_{PM}(t) = A_c \cos\left(2\pi f_c t + k_p A_m \cos(2\pi f_m t)\right)$$

$$s_{FM}(t) = A_c \cos\left(2\pi f_c t + \frac{A_m k_f}{f_m} \sin(2\pi f_m t)\right)$$

- For the PM signal, we define

$$\beta_p = k_p A_m = \Delta\phi_{\max}$$

- For the FM signal

$$\beta_F = \frac{k_f A_m}{f_m} = \frac{\Delta f_{\max}}{f_m}$$

uOttawa

Modulation indices

- For any $m(t)$ which has bandwidth B_m, we define the modulation indices as :

$$\beta_p = k_p \, |m(t)|_{\max} = \Delta\phi_{\max}$$

$$\beta_F = \frac{k_p \, |m(t)|_{\max}}{B_m} = \frac{\Delta f_{\max}}{B_m}$$

Example

- The signal $m(t) = 5\text{sinc}^2(10t)$. Find the modulation index for

1. PM modulation with $k_p = 0.3\pi$ rads/V.
2. FM modulation with $k_f = 20$ Hz/V.
   - SOLUTION

- $|m(t)|_{max} = 5$, therefore $\beta_p = 0.3\pi \times 5 = 1.5\pi$ rads.

- $B_m = 10\text{Hz}$, therefore $\beta_F = 20 \times 5/10 = 10$.

Narrowband FM

- Consider an FM signal :

$$s_{FM}(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)\,d\tau\right]$$

$$\text{where} \quad \left|2\pi k_f \int_{-\infty}^{t} m(\tau)\,d\tau\right| << 1$$

- We say that $s_{FM}(t)$ is a narrowband FM signal.
- For example, consider when $m(t) = A_m \cos(2\pi f_m t)$.

$$s_{FM}(t) = A_c \cos\left(2\pi f_c t + \frac{A_m k_f}{f_m}\sin(2\pi f_m t)\right)$$

$$s_{FM}(t) = A_c \cos\left(2\pi f_c t + \beta_F \sin(2\pi f_m t)\right)$$

uOttawa

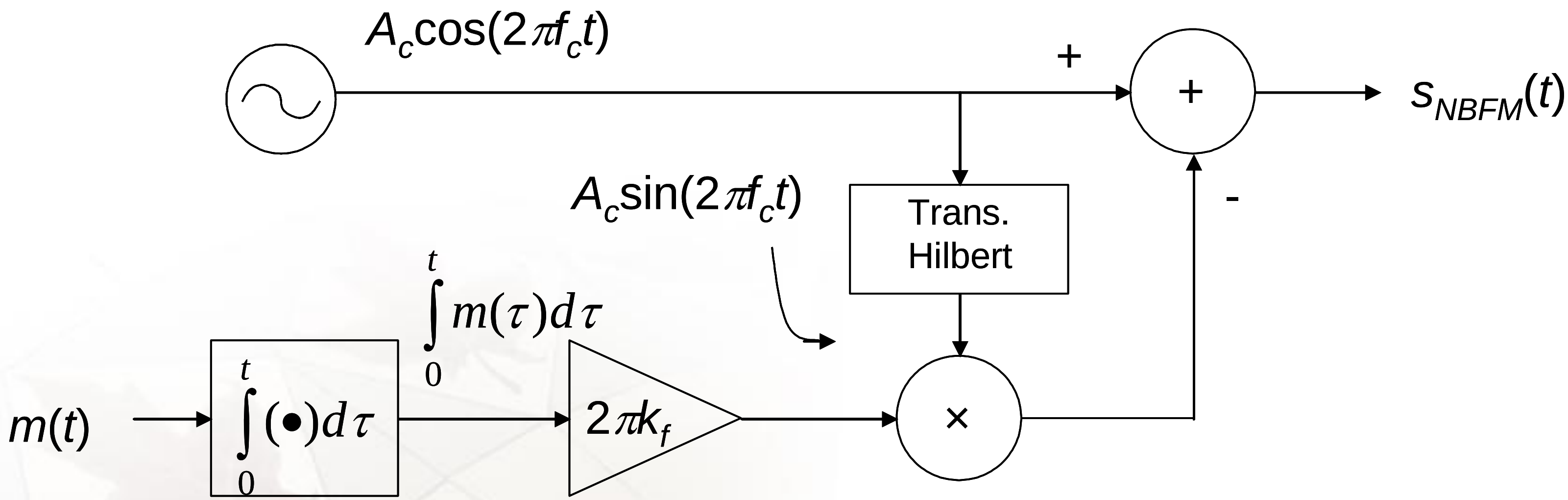
Narrowband FM

- When $\beta_F << 1$, the FM signal is NBFM.
- cos(*A*+*B*) = cos(*A*)cos(*B*)-sin(*A*)sin(*B*). Therefore

$$
\begin{aligned}
s_{FM}(t) &= A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right] \\
&= A_c \cos(2\pi f_c t)\cos\left(2\pi k_f \int_0^t m(\tau)d\tau\right) - A_c \sin(2\pi f_c t)\sin\left(2\pi k_f \int_0^t m(\tau)d\tau\right) \\
&\approx A_c \cos(2\pi f_c t) - A_c\left(2\pi k_f \int_0^t m(\tau)d\tau\right)\sin(2\pi f_c t)
\end{aligned}
$$

(if $A << 1$, $\cos(A) \approx 1$ and $\sin(A) \approx A$.)

NBFM Modulator



NBFM Spectrum

- The spectrum of an NBFM signal is given by:

$$S_{NBFM}(f) = \frac{A_c}{2}\delta(f - f_c) + \frac{A_c}{2}\delta(f + f_c) + \frac{A_c k_f}{f - f_c} M(f - f_c)$$

$$- \frac{A_c k_f}{f + f_c} M(f + f_c)$$

- Assuming $M(0) = 0$, then $M(f\text{-}f_c) = 0$ at $f{=}f_c$ and $M(f{+}f_c) = 0$ at $f{=}\text{-}f_c$.
- The bandwidth of $s_{NBFM}(t)$ is therefore $2B_m$ where B_m is the bandwidth of $m(t)$.

Wideband FM - WBFM

- For an FM signal to be NBFM, $\beta_F << 1$.
- Any signal that is not narrowband is therefore wideband.
- However, typically $\beta_F > 1$ for an FM signal to be considered wideband.
- The bandwidth of WBFM signals is larger than NBFM since Δf_{max} is increased.

WBFM signal for $m(t) = A_m \cos 2\pi f_m t$ and its complex envelope.

- Let us consider $m(t) = A_m \cos 2\pi f_m t$.
- The resulting FM signal is :

$$s_{FM}(t) = A_c \cos\left[2\pi f_c t + \frac{Ak_f}{f_m}\sin(2\pi f_m t)\right] = A_c \cos[2\pi f_c t + \beta_F \sin(2\pi f_m t)]$$

$$s_{FM}(t) = A_c \,\mathrm{Re}\left\{e^{j(2\pi f_c t + \beta_F \sin(2\pi f_m t))}\right\}$$

$$s_{FM}(t) = \mathrm{Re}\{\tilde{s}_{FM}(t)e^{j2\pi f_c t}\}$$

$$\tilde{s}_{FM}(t) = A_c e^{j\beta_F \sin(2\pi f_m t)}$$

The Fourier series of the WBFM signal when $m(t) = A_m \cos 2\pi f_m t$.

- The preceeding complex envelope is periodic with fundamental frequency f_m.

$$\tilde{s}_{FM}(t) = \sum_{n=-\infty}^{\infty} \tilde{S}_n e^{j2\pi n f_m t}$$

where

$$\begin{aligned}
\tilde{S}_n &= f_m \int_{-1/2f_m}^{1/2f_m} A_c e^{j\beta_F \sin(2\pi f_m t)} e^{-j2\pi n f_m t} dt \\
&= f_m \int_{-1/2f_m}^{1/2f_m} A_c e^{j(\beta_F \sin(2\pi f_m t) - 2\pi n f_m t)} dt
\end{aligned}$$

The Fourier series of the WBFM signal when $m(t) = A_m \cos 2\pi f_m t$.

- Replacing $2\pi f_m t$ by x, $\tilde{S}_n$ becomes

$$\tilde{S}_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta_F \sin x - nx)} dx$$

- The nth order Bessel function of the first kind, $J_n(\beta)$ is given by:

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx$$

- Therefore $\tilde{S}_n = A_c J_n(\beta_F)$

uOttawa

The Fourier series of the WBFM signal when $m(t) = A_m \cos 2\pi f_m t$.

- Therefore we can express the complex envelope of the WBFM signal as

$$\tilde{s}_{FM}(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta_F) e^{j2\pi n f_m t}$$

- And the WBFM signal itself becomes:

$$\begin{aligned} s_{FM}(t) &= \mathrm{Re}\{\tilde{s}_{FM}(t) e^{j2\pi f_c t}\} \\ &= \mathrm{Re}\left\{ \sum_{n=-\infty}^{\infty} A_c J_n(\beta_F) e^{j(2\pi f_c t + 2\pi n f_m t)} \right\} \\ &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta_F) \cos(2\pi (f_c + n f_m) t) \end{aligned}$$

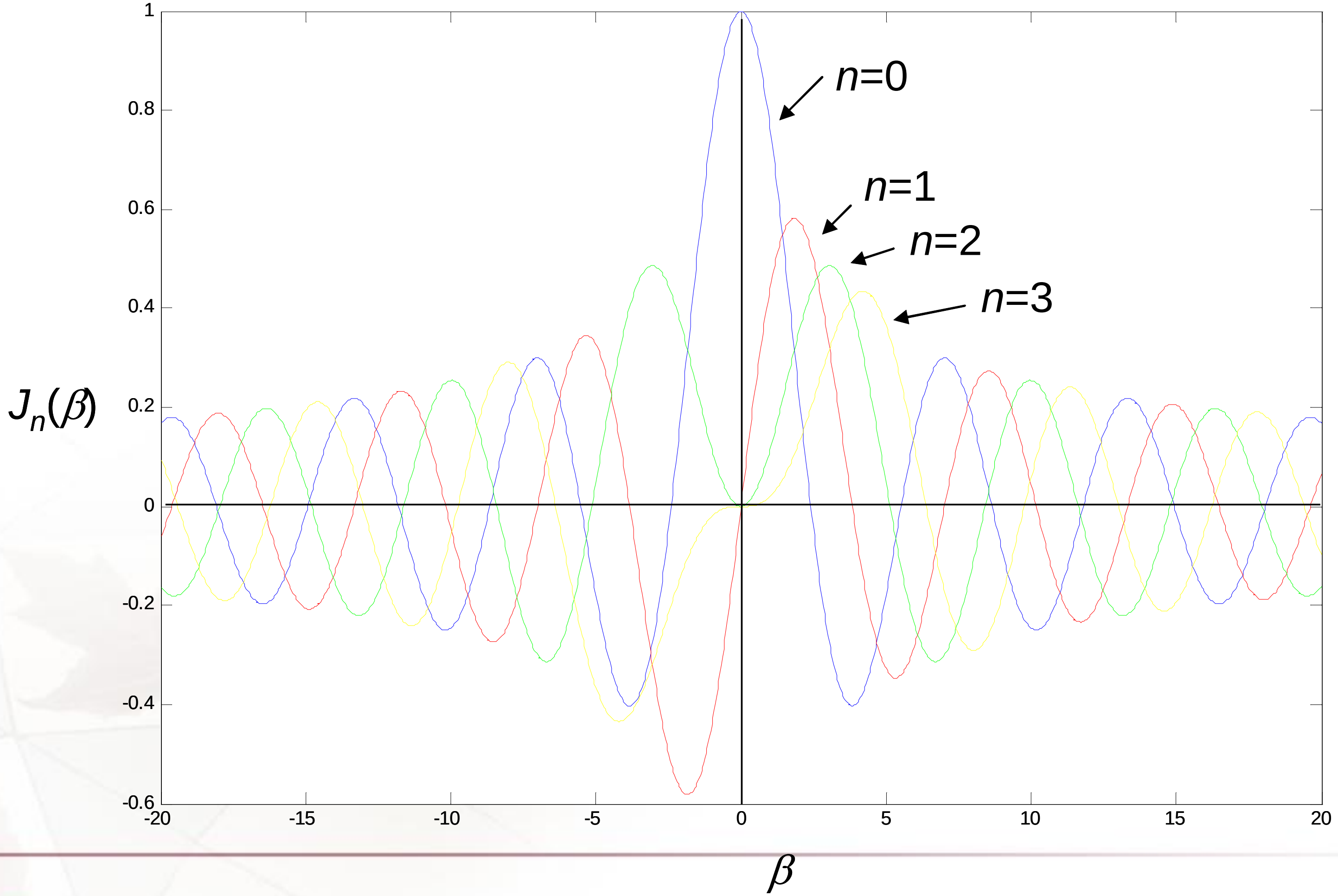
Spectrum of the WBFM signal when $m(t) = A_m \cos 2\pi f_m t$.

- The spectrum of this signal is:

$$S_{FM}(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta_F)\left[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right]$$

- This expression shows that the FM signal's spectrum is made up of an infinite number of impulses at frequencies $f = f_c + nf_m$.
- Therefore, theoretically, this WBFM signal has infinite bandwidth.
- However, the properties of the Bessel function show that most of these impulses contribute little to the overall power of the signal and are negilgible.
  - We define the practical bandwidth as the range of frequencies which contains at least 99% of the total power of the WBFM signal.

The function $J_n(\beta)$



Properties of $J_n(\beta)$

1) If n is an integer :

$J_n(\beta) = J_{-n}(\beta)$ for even n

and

$J_n(\beta) = -J_{-n}(\beta)$ for odd n

2) when $\beta << 1$

$J_0(\beta) \approx 1$

$J_1(\beta) \approx \beta/2$

and

$J_n(\beta) \approx 0,\ n > 1$

3) $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$

4) $Im\{J_n(\beta)\}=0$

uOttawa

Power of the FM signal

- The power of an FM signal is:

$$P_{FM} = \frac{A_c^2}{2}$$

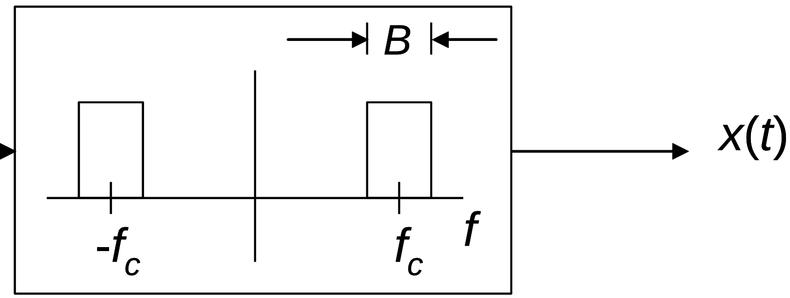
$$s_{FM}(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta_F)\cos(2\pi(f_c + nf_m)t)$$

- The power of the above expression is:

$$P = \frac{A_c^2}{2}\sum_{n=-\infty}^{\infty} J_n^2(\beta_F)$$

Filtering a WBFM signal to limit its bandwidth.

$$s_{FM}(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta_F)\cos(2\pi(f_c + nf_m)t)$$



We want to choose B so that the power of $x(t)$
Is at least 0.99× the power of $s_{FM}(t)$.

$$x(t) = \sum_{n=-X}^{X} A_c J_n(\beta_F)\cos(2\pi(f_c + nf_m)t)$$

where X is the greatest integer that satisfies :

$$f_c + Xf_m \le f_c + \frac{B}{2} \quad \text{and} \quad f_c - Xf_m \ge f_c - \frac{B}{2}$$

- The power of $x(t)$ is:

$$P_x = \frac{A_c^2}{2} \sum_{n=-X}^{X} J_n^2(\beta_F)$$

- Therefore we must choose X so that:

$$\sum_{n=-X}^{X} J_n^2(\beta_F) \geq 0.99$$

- We know that $J_n^2(\beta_F) = J_{-n}^2(\beta_F)$. Therefore

$$J_0^2(\beta_F) + 2\sum_{n=1}^{X} J_n^2(\beta_F) \geq 0.99$$

Values of $J_n(\beta)$.

n	β=0.1	β=0.2	β=0.5	β=1	β=2	β=3	β=5	β=10
0	0.997	0.99	0.938	0.765	0.224	-0.2601	-0.178	-0.246
1	0.05	0.1	0.242	0.44	0.577	0.3391	-0.323	0.043
2	0.001	0.005	0.031	0.115	0.353	0.4861	0.047	0.255
3	2 10^{-5}≈0	1.6 10^{-4}	0.0026	0.02	0.129	0.3091	0.365	0.058
4				0.002	0.034	0.1320	0.391	-0.220
5					0.007	0.0430	0.261	-0.234
6					0.001	0.0114	0.131	-0.014
7						0.0025	0.053	0.217
8							0.018	0.318
9							0.006	0.292
10							0.001	0.207
11								0.123
12								0.063
13								0.029

Example

- The signal $m(t) = A_m \cos(2\pi f_m t)$ is to be transmitted using FM techniques. Find the practical bandwidth if

  (a) A_m = 5V, f_m = 20 Hz and k_f = 4 Hz/V

  (b) A_m = 10V, f_m = 400 Hz and k_f = 200 Hz/V.
- SOLUTION

  (a) IN this example, β_F = (5)(4)/(20) = 1. We need to find X so that $S = J_0^2(\beta_F) + 2\sum_{n=1}^{X} J_n^2(\beta_F) \geq 0.99$.
- From the table, if X = 1, $S = (0.765^2+2\times0.44^2)$ = 0.9648. If X = 2, S = $0.9648+2\times0.115^2$ = 0.9912. Therefore X = 2 and $B = 4f_m$.

  (b) Here, β_F = (10)(200)/(400) = 5. We can show that X = 6 yields S = 0.994. Therefore $B = 12f_m$.

Carson's Rule

- For $m(t) = A_m\cos(2\pi f_m t)$, When β is an integer, we always find that $X = \beta+1$.
- Therefore we can estimate that the practical bandwidth of an FM signal is $B = 2(\beta_F+1)f_m$.
- For any random $m(t)$ with maximum value A_m and bandwidth B_m, the true bandwidth is difficult to find.
- According to Carson, the worst case is when the spectrum of $m(t)$ is concentrated around $f = B_m$ (such as a sinusoid).
- Based on experiments by Carson, the bandwidth of a WBFM signal, B_{FM}, can be estimated by

$$B_{FM} = 2(\beta_F + 1)B_m \quad (*****)$$

uOttawa