Basic Modal Logic

Definition 5.4. Let $\mathcal{M} = (W, R, L)$ be a model of basic modal logic. Let x be a world in W and ϕ a formula. We define ϕ *is true in the world* x or x *satisfies* ϕ (written $x \Vdash \phi$) by structural induction on formulas.

1. $x \Vdash \top$
2. $x \nVdash \bot$
3. $x \Vdash p$ iff $p \in L(x)$
4. $x \Vdash \neg\phi$ iff $x \nVdash \phi$
5. $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$.
6. $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$.
7. $x \Vdash \phi \to \psi$ iff whenever we have $x \Vdash \phi$, we also have $x \Vdash \psi$.
8. $x \Vdash \phi \leftrightarrow \psi$ iff $(x \Vdash \phi$ iff $x \Vdash \psi)$.
9. $x \Vdash \Box\psi$ iff for all $y \in W$ with $R(x, y)$, we have $y \Vdash \psi$.
10. $x \Vdash \Diamond\psi$ iff there is some $y \in W$ such that $R(x, y)$ and $y \Vdash \psi$.

Definition 5.5. $\mathcal{M} = (W, R, L)$ *satisfies* a formula ϕ if for every world $x \in W$, x satisfies ϕ (written $\mathcal{M} \models \phi$).

Definition 5.8. A formula ϕ of basic modal logic is said to be *valid* if it is true in **every** world of **every** model (written $\models \phi$).
Note: If a formula ϕ holds in propositional logic (i.e. $\models \phi$ or equivalently $\vdash \phi$), then also $\models \phi$ in basic modal logic.

Definition 5.7. A set of formulas Γ of basic modal logic *semantically entails* a formula ϕ of basic modal logic if, in any world x of any model $\mathcal{M} = (W, R, L)$, whenever $x \Vdash \psi$ for all $\psi \in \Gamma$, we have $x \Vdash \phi$.
In this case, we write $\Gamma \models \phi$.

Definition

- Let $\mathbb{L}$ be a set of formula schemes.
- Let $\mathbb{L}_c$ be the smallest set containing all instances of all formulas in $\mathbb{L}$.
- Let Γ be a set of formulas of basic modal logic.
- Let ϕ be a formula of basic modal logic.

We say that Γ *semantically entails* ϕ *in* $\mathbb{L}$ and write

$$\Gamma \models_{\mathbb{L}} \phi$$

iff $\Gamma \cup \mathbb{L}_c \models \phi$ in basic modal logic.