On the Integrality Gap of the 2-Edge Connected Subgraph Problem *

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Abstract

Given a complete graph on n vertices with nonnegative edge costs, the 2-edge connected subgraph problem ($2EC$) is that of finding a 2-edge connected multi-subgraph of minimum cost. The linear programming relaxation of this problem ($2EC^{LP}$) provides a lower bound for $2EC$, and its study provides a promising direction for finding improved solutions for $2EC$. It has been conjectured that the integrality gap $\alpha 2EC$ between $2EC$ and $2EC^{LP}$ is $\frac{4}{3}$. Note that this is closely related to the well-known conjecture in combinatorial optimization that says that the integrality gap $\alpha TSP^{\triangle}$ of the linear programming relaxation of the metric Travelling Salesman problem is $\frac{4}{3}$. It can be shown that $\alpha 2EC \leq \alpha TSP^{\triangle}$, and thus if the conjecture for $\alpha 2EC$ is true it would give support to the conjecture for $\alpha TSP^{\triangle}$.

Currently not much is known about the integrality gap $\alpha 2EC$, except that it lies between $\frac{6}{5}$ and $\frac{3}{2}$. In this paper we strive to find the exact value for $\alpha 2EC$ for small values of n. This is difficult due to the exponential size of the data involved. In this paper we describe how we were able to overcome such difficulties and obtain the exact integrality gap for all values of n up to 10, and a tight lower bound for this gap for $11 \leq n \leq 14$.

1 Introduction

Let $c \in \mathbf{R}^E$, $c \neq 0$ be a set of costs assigned to the edges of the complete graph $K_n = (V, E)$ on n vertices. The *2-edge connected Subgraph Problem* ($2EC$) is that of finding a 2-edge connected spanning multi-subgraph of K_n which is of minimum cost with respect to c. This problem is an important network design problem with many applications, including the design of communication networks that can survive the loss of a link.

It is known that $2EC$ is NP-hard , even in the metric case, i.e. when the costs c satisfy the triangle inequalities $c_{ij} + c_{jk} \geq c_{ik}$ for all distinct triples $i, j, k \in V$. One method used for finding reasonably good solutions is to look for a k-approximation algorithm for the problem, i.e. try to find a heuristic method for $2EC$ such that the solution found is guaranteed to be of cost at most k times the optimal solution cost for some constant $k \geq 1$. Currently the best k-approximation algorithms known for the $2EC$ have $k = 2$ ([8], [7]), and if the costs c are metric then there is a $\frac{3}{2}$-approximation algorithm known ([5]).

We can formulate $2EC$ as an integer linear program (ILP). For any edge set $F \subseteq E$ and $x \in \mathbf{R}^E$, let $x(F)$ denote the sum $\sum_{e \in F} x_e$. For any vertex set $W \subset V$, let $\delta(W)$ denote $\{uv \in E : u \in W, v \notin W\}$. Then $2EC$ can be formulated as an ILP as follows:

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$$\begin{aligned}
&\text{minimize } cx && && (1)\\
&\text{subject to: } x(\delta(S)) \geq 2 && \text{for all } \emptyset \subset S \subset V, && (2)\\
&x_e \geq 0 && \text{for all } e \in E, && (3)\\
&x && \text{integer} && (4)
\end{aligned}$$

where x_e represents the number of copies of edge e in the $2EC$ solution. The constraints (2) are known as the *subtour elimination constraints* and the constraints (3) are known as the *non-negativity constraints*. The linear programming relaxation of $2EC$, which we will denote by $2EC^{LP}$, is obtained by dropping the integrality condition in the above ILP formulation. We use $opt(2EC)$ and $opt(2EC^{LP})$ to denote the optimal values of $2EC$ and $2EC^{LP}$ respectively.

We are interested in the *integrality gap* for $2EC^{LP}$, denoted by $\alpha 2EC$, which is the worst-case ratio between $2EC$ and $2EC^{LP}$, i.e.

$$\alpha 2EC = \max_{\substack{c \geq 0 \\ c \neq 0}} \frac{opt(2EC)}{opt(2EC^{LP})}.$$

This integrality gap gives one measure of the quality of the lower bound provided by $2EC^{LP}$ for $2EC$. Moreover, a constructive proof of $\alpha 2EC = k$ would provide a k-approximation algorithm for $2EC$.

It can be shown that $\alpha 2EC \leq 3/2$ (see (9) in Section 2). However no examples are known for which this ratio comes close to this value. In [3], Carr and Ravi study $\alpha 2EC$. They state the following conjecture, and give a result that gives some support for it:

Conjecture 1 *The integrality gap $\alpha 2EC$ for $2EC^{LP}$ is $\frac{4}{3}$.*

Note that this conjecture is related to a well-known conjecture in combinatorial optimization for another closely related problem. Given the complete graph $K_n = (V, E)$ on n vertices with nonnegative edge costs $c \in \mathbf{R}^E$, $c \neq 0$, the *Symmetric Travelling Salesman Problem* (henceforth $STSP$) is to find a Hamiltonian cycle (or *tour*) in K_n of minimum cost. When the costs satisfy the triangle inequality the problem is called the *metric $STSP$*.

An ILP formulation for the $STSP$ is obtained from the ILP formulation (1) for $2EC$ by setting the subtour elimination constraints (2) to equality for all $S \subset V$ consisting of a single vertex, i.e.

$$x(\delta(v)) = 2 \qquad \text{for all } v \in V. \tag{5}$$

These constraints are called the *degree constraints* for the $STSP$. If we drop the integer requirement in this ILP we obtain the linear programming relaxation of the $STSP$ called the *Subtour Elimination Problem* (SEP). We use $opt(TSP)$ and $opt(SEP)$ to denote the optimal solution values for the $STSP$ and SEP respectively. The associated *SEP polytope* is the set of all vectors x satisfying the constraints of the SEP, i.e. is $\{x \in \mathbf{R}^E : x \text{ satisfies } (2), (3), (5)\}$.

We denote the integrality gap for between $STSP$ and SEP in the metric case by $\alpha TSP^{\triangle}$. It is known that for the metric $STSP$, $\alpha TSP^{\triangle}$ is at most $\frac{3}{2}$ (see Shmoys and Williamson [10], Wolsey [12]), however no example for which this ratio comes close to $\frac{3}{2}$ has yet been found. In fact, a well-known conjecture states the following:

Conjecture 2 *For the metric STSP, the integrality gap $\alpha TSP^{\triangle}$ for SEP is $\frac{4}{3}$.*

It can be shown (see (10) in Section (2)) that $\alpha 2EC \leq \alpha TSP^{\triangle}$. Thus Conjecture 1 and Conjecture 2 are closely related, and if Conjecture 1 is indeed true, it would provide some evidence for the truth of the more famous Conjecture 2, and this is part of our motivation for studying $\alpha 2EC$.

Note that if Conjecture 2 is true, then it is best possible, as there exists a family of cost functions for which the asymptotic ratio between $opt(TSP)$ and $opt(SEP)$ is $\frac{4}{3}$. However, Conjecture 1 for $\alpha 2EC$ is not known to be best possible. In [3], Carr and Ravi mention the existence of a family of cost functions for which the integrality gap ratio reaches $\frac{6}{5}$ asymptotically. Currently no example for which the integrality gap ratio is larger than $\frac{6}{5}$ is known. Below we describe a family of cost functions different from that of Ravi and Carr for which the ratio also reaches $\frac{6}{5}$ asymptotically. However this family has the advantage over their family in that the ratio for the integrality gap is bigger for any particular value of n, and this will be useful for discussions later in the paper.

Consider the family of cost functions shown in Figure 1, where the numbers shown are the edge costs, and edges with cost 0 are not shown, and the "gadget" pattern is repeated k times. For this family, it can be shown that

$$\frac{opt(2EC)}{opt(2EC^{LP})} = \frac{6\text{k} + 1}{5\text{k} + 1},$$

which tends to $\frac{6}{5}$ as $k \to \infty$.

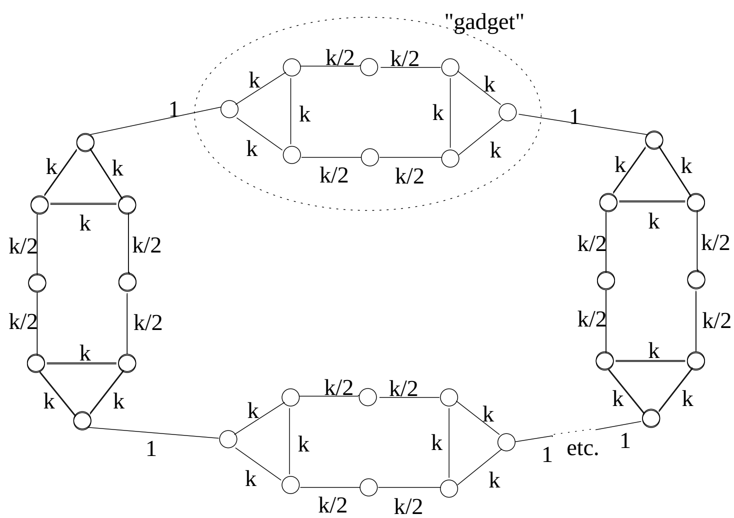

Figure 1: An example for which $\alpha 2EC = \frac{6}{5}$.

In this paper we attempt to shed some light on the value of $\alpha 2EC$ by examining the problem of finding the exact integrality gap for fixed values of n when n is small. Let $\alpha 2EC_n$, $2EC_n$ and $2EC_n^{LP}$ represent the $\alpha 2EC$, $2EC$ and $2EC^{LP}$ for problems on n vertices (and similiarly for $STSP$). Then we find the value of $\alpha 2EC_n$ exactly for all values n up to 10, and a tight lower bound for this gap for $11 \leq n \leq 14$. These results, which are reported in Section 3, support

Conjecture 1, and in fact lead us to propose a new conjecture in Section 3. Note that many models for the problem of finding the exact integrality gap for 2EC are too complex and too large to be practical to solve. However, in [1], Benoit and Boyd describe a way to overcome this for finding the exact integrality gap $\alpha TSP_n^{\triangle}$ on problems with n vertices, and we will attempt to use a simliar method here. In the next section we will show how to reformulate the problem of finding $\alpha 2EC_n$ in order to make it possible to use the method described in [1].

2 Reformulating the Integrality Gap

Given a nonnegative cost function $c \in \mathbf{R}^E$ for the complete graph $K_n = (V, E)$ on n vertices, the *metric completion* c' of c is the metric cost function obtained from c by replacing c_{ij} by the cost of a shortest path connecting vertices i and j for every edge $ij \in E$. Then as noted by Carr and Ravi in [3], without loss of generality we can replace c by its metric completion c' for $2EC_n$. This follows from the fact that if any optimal solution for $2EC_n$ uses an edge ij for which c_{ij} is greater than the cost of of a shortest path connecting vertices i and j, then since we allow multiple edges in our $2EC$ solutions, we could simply replace edge ij with the edges of that shortest path in the solution without increasing the solution value. A similar result holds for $2EC^{LP}$. Hence without loss of generality, in our study of $\alpha 2EC_n$ we can restrict our attention to only those sets of nonnegative costs that are metric. We use the notation $\alpha 2EC^{\triangle}$ for this integrality gap when restricted to metric cost functions, i.e.

$$\alpha 2EC_n^{\triangle} = \max_{\substack{c \geq 0 \\ c \text{ is metric} \\ c \neq 0}} \frac{opt(2EC_n)}{opt(2EC_n^{LP})}, \tag{6}$$

we thus have from the above discussion that

$$\alpha 2EC_n = \alpha 2EC_n^{\triangle}. \tag{7}$$

Another very useful result, also mentioned in [3] and shown in [11], is that for metric costs, there exists an optimal solution for $2EC^{LP}$ which is also feasible and therefore optimal for SEP. This also follows from a more general result of Goemans and Bertsimas in [6]. Thus we have

$$\alpha 2EC_n^{\triangle} = \max_{\substack{c \geq 0 \\ c \text{ is metric} \\ c \neq 0}} \frac{opt(2EC_n)}{opt(SEP_n)}. \tag{8}$$

Note that this implies

$$\alpha 2EC \leq \frac{3}{2}, \tag{9}$$

since

$$\alpha 2EC = \alpha 2EC^{\triangle} = \max_{\substack{c \geq 0 \\ c \text{ is metric} \\ c \neq 0}} \frac{opt(2EC)}{opt(SEP)} \leq \max_{\substack{c \geq 0 \\ c \text{ is metric} \\ c \neq 0}} \frac{opt(TSP)}{opt(SEP)} = \alpha TSP^{\triangle} \leq \frac{3}{2}. \tag{10}$$

The reformulation (8) is extremely useful since the SEP has been well-studied. Benoit and Boyd [1] generated all of the non-isomorphic extreme points of the SEP polytope on n vertices for all $3 \leq n \leq 10$, and were able to find the exact integrality gap for $\alpha TSP_n^{\triangle}$ for $n \leq 10$ using these extreme points. The above reformulation for $\alpha 2EC_n$ allows us to make use of these results and to use a similar method to the one outlined in [1] to compute $\alpha 2EC_n$ for small values of n.

Let $c \neq 0$ be some nonnegative metric cost vectorand let x an optimal solution of $2EC_n$, i.e. $cx = opt(2EC_n)$. It is easy to show that if $cx = 0$ then $c = 0$, hence $cx > 0$. If we divide all the edges costs c_e, $e \in E$ by cx then the new costs are also metric, and the ratio $opt(2EC_n)/opt(SEP_n)$ is still the same. Moreover, the new value of $opt(2EC_n)$ will be 1 with this new cost function. So to find $\alpha 2EC_n^{\triangle}$, it is enough to only consider metric cost functions c for which $opt(2EC_n) = 1$. Thus

$$\alpha 2EC_n^{\triangle} = \max_{\substack{c \geq 0 \\ c \text{ is metric} \\ opt(2EC_n)=1}} \frac{1}{opt(SEP_n)},$$

or

$$\frac{1}{\alpha 2EC_n^{\triangle}} = \min_{\substack{c \geq 0 \\ c \text{ is metric} \\ opt(2EC_n)=1}} opt(SEP_n).$$

Now suppose we have an exhaustive list, $\{x_1, \ldots, x_k\}$, of the extreme points of the SEP polytope on n vertices. Then we know that for any cost vextor, c, there exists some $1 \leq i \leq k$ such that x_i is an optimal solution of SEP_n. Thus we can compute

$$opt(SEP_n) = min_{1 \leq i \leq k} c x_i.$$

Hence

$$\frac{1}{\alpha 2EC_n^{\triangle}} = \min_{1 \leq i \leq k} \min_{\substack{c \geq 0 \\ c \text{ is metric} \\ opt(2EC_n)=1 \\ c \text{ is optimal for SEP w.r.t. } x_i}} c x_i.$$

The condition that c is optimal for SEP with respect to a given extreme point x_i can be easily written as a set of linear constraints - namely the complementary slackness constraints. Similarly, $c \geq 0$ and c is metric are linear constraints. Since the x_i are given, we almost have a finite list of linear program whose optimal value is the reciprocal of the integrality gap. We just need to look a bit closer at the constraint $opt(2EC_n) = 1$. Thus we will explore some of the structural properties of the optimal solutions to $2EC_n$.

To begin, let x be a set of nonnegative integer values assigned to the edges of the complete graph on n vertices. We construct the *support multigraph*, G_x of x by starting with n vertices and for each pair of vertices, u and v, G_x contains exactly x_{uv} copies of the edge uv. Notice too, that given a loopless multigraph G, we can construct a unique set of nonnegative integer values, call it x, where x_e is defined to be the number of copies of the edge e present in G such that $G = G_x$.

Lemma 3 *For every $n \geq 3$ and every nonnegative metric cost vector c there is an optimal solution of $2EC_n$ whose support multigraph is simple and minimial with respect to the number of edges.*

Proof Let x be an optimal solution of $2EC_n$ with support multigraph G_x. If G_x is not minimal with respect to the number of edges then let $H \subset G$ such that H is two-edge-connected and is also minimal with respect to the number of edges. Let y be such that $G_y = H$ so y is a feasible solution to $2EC_n$ and $y \leq x$. Thus, since c is nonnegative, $cy \leq cx$. However, x is an optimal solution to $2EC_n$ so it must be that $cy = cx$ so y is also an optimal solution to $2EC_n$.

Hence we may suppose that G_x is minimal with respect to the number of edges. Suppose that G_x is not simple however. Thus G_x has multiple edges and so let e be an edge such that $x_e \geq 2$. If, for every cut $\delta(S)$ which contains e, we have that $x(\delta(S)) \geq 3$ then we can remove one of the copies of e from G_x and remain two-edge-connected. This contradicts the fact that G_x is edge-minimal. Thus there exists some $\emptyset \subset S \subset V$ such that $e \in \delta(S)$ and $x(\delta(S)) = 2$.

As a consequence, we have that $x_e \leq 2$ and hence $x_e = 2$ so $\delta(S) = \{e\}$. Let $e = uv$ where $u \in S$ and $v \in \overline{S}$. If we remove the two copies of e from G_x we obtain a graph with exactly two components, with vertex sets S and $\overline{S}$ respectively. Furthermore, consider any $\emptyset \subset A \subset S$. If $u \notin A$ then $\delta_{G_x[S]}(A) = \delta_{G_x}(A)$. If $u \in A$ then $\delta_{G_x[S]}(A) = \delta_{G_x}(A \cup \overline{S})$. Thus in either case, if we restrict x in a natural way to $G_x[S]$ then $x(\delta_{G_x[S]}(A)) \geq 2$ and hence $G_x[S]$ is two-edge-connected. By symmetry, $G_x[\overline{S}]$ is also two-edge-connected.

Now, since $n \geq 3$, either u must have a neighbour other than v in G_x or v has a neighbour other than u in G_x. Assume, without loss of generality, that u has a neighbour $w \neq v$ in G_x. Hence $w \in S$ and let y be the nonnegative integer vector such that G_y is obtained by removing a copy of uv and a copy of uw from G_x and adding the edge vw. Since vw is not an edge of G_x, vw is a single edge of G_y and hence G_y has fewer parallel edges than G_x. Furthermore, since c is metric,

$$\begin{aligned} cy &= cx - c_{uv} - c_{uw} + c_{vw} \\ &\leq cx. \end{aligned}$$

Choose any $\emptyset \subset A \subset V$. If $uv, uw \notin \delta_{G_x}(A)$ then $y(\delta_{G_y}(A)) \geq x(\delta_{G_x}(A))$. If $uv \in \delta_{G_x}(A)$ and $uw \notin \delta_{G_x}(A)$ then $vw \in \delta_{G_y}(A)$ and hence $y(\delta_{G_y}(A)) = x(\delta_{G_x}(A))$. Similarly, if $uw \in \delta_{G_x}(A)$ and $uv \notin \delta_{G_x}(A)$ then $vw \in \delta_{G_y}(A)$ and hence $y(\delta_{G_y}(A)) = x(\delta_{G_x}(A))$. As a result, in the above three cases we have that $y(\delta_{G_y}(A)) \geq 2$. The only remaining possibility is that $uv, uw \in \delta_{G_x}(A)$ and hence $vw \notin \delta_{G_y}(A)$. In this case, $y(\delta_{G_y}(A)) = x(\delta_{G_x}(A)) - 2$. Assume that $u \in A$ (since otherwise, we can replace A with $\overline{A}$) so $v, w \notin A$. Thus $\emptyset \subset A \cap S \subset S$ and hence $x(\delta_{G_x[S]}(A \cap S)) \geq 2$. However, $\{uv\} \cup \delta_{G_x[S]}(A \cap S) \subseteq \delta_{G_x}(A)$, and hence

$$\begin{aligned} y(\delta_{G_y}(A)) &= x(\delta_{G_x}(A)) - 2 \\ &\geq x_{uv} + x(\delta_{G_x[S]}(A \cap S)) - 2 \\ &\geq 2 + 2 - 2 \\ &= 2. \end{aligned}$$

Therefore, G_y is two-edge-connected and as a result, y is a feasible solution to $2EC_n$. However, as noted above, $cy \leq cx$ so it must be that $cy = cx$ and so y is an optimal solution to $2EC_n$.

If G_y is not edge-minimal, we have seen above how to generate a new optimal solution which is. Furthermore, this new edge-minimal solution will not introduce any new parallel edges. If parallel edges remain, we now have a way of generating a new solution without them. Thus by repeated application of the above arguments, we can obtain an optimal solution of $2EC_n$ whose support multigraph is edge-minimal and simple.

□

Since Lemma 3 tells us that there are always optimal solutions which are simple, the support multigraph of any such solution is in fact a graph. Thus, we will refer to the *support graph* of such solutions. It turns out that we are guaranteed the existence of an even smaller class of optimal solutions. We describe this in the next lemma.

Lemma 4 *For every $n \geq 3$ and every nonnegative metric cost vector c there is an optimal solution of $2EC_n$ whose support graph is simple, minimal with respect to the number of edges, and two-vertex-connected.*

Proof From Lemma 3, we know that $2EC_n$ has optimal solutions whose support multigraphs are simple. Among all such solutions, let x be one which has a support graph which has the minimum number of edges. As a consequence, such a support graph is also edge-minimal.

Suppose, for a contradiction, that G_x has a cut-vertex, v. Thus $G_x - v$ has at least two components so let $A, B \subset V$ be the vertex-sets of distinct components of $G_x - v$. Since G_x is two-edge-connected, v is adjacent in G_x to some vertex $u \in A$ and some vertex $w \in B$. Let y be the nonnegative integer vector associated with the support graph G_y obtained from G_x by removing the edges uv and vw and adding the edge uw.

Consider any $\emptyset \subset S \subset A$. Then $\delta_{G_x[A]}(S) = \delta_{G_x}(V \backslash \overline{S})$. However G_x is two-edge-connected and simple so $|\delta_{G_x}(V \backslash \overline{S})| \geq 2$. Thus $|\delta_{G_x[A]}(S)| \geq 2$ so $G_x[A]$ is two-edge-connected. By symmetry, $G_x[B]$ is also two-edge-connected.

Now let $\emptyset \subset S \subset V$. If $uv \notin \delta_{G_x}(S)$, or $vw \notin \delta_{G_x}(S)$, or $uw \in \delta_{G_y}(S)$ then we have that $y(\delta_{G_y}(S)) \geq x(\delta_{G_x}(S)) \geq 2$. Hence assume that $uv, vw \in \delta_{G_x}(S)$ and $uw \notin \delta_{G_y}(S)$. Without loss of generality, we may assume that $v \in S$ and $u, w \in \overline{S}$. Thus $S \cap A \subset A$ and $S \cap B \subset B$. If $S \cap A \neq \emptyset$ then $\{uv, vw\} \cup \delta_{G_x[A]}(S \cap A) \subseteq \delta_{G_x}(S)$ and thus $x(\delta_{G_x}(S)) \geq 4$. Similarly, if $S \cap B \neq \emptyset$ then we also have that $x(\delta_{G_x}(S)) \geq 4$. Lastly if $S \cap (A \cup B) = \emptyset$ then $\delta_{G_x}(A) \cup \delta_{G_x}(B) \subseteq \delta_{G_x}(S)$ where $\delta_{G_x}(A) \cap \delta_{G_x}(B) = \emptyset$. Hence $x(\delta_{G_x}(S)) \geq 4$. Now, $y(\delta_{G_y}(S)) = x(\delta_{G_x}(S)) - 2 \geq 2$. Therefore, in all cases, G_y is two edge-connected.

However, since $G_x[A]$ and $G_x[B]$ are components of $G_x - v$, it must be that uw is not an edge of G_x. Hence G_y is also simple. Furthermore, since c is metric,

$$\begin{aligned} cy &= cx - c_{uv} - c_{vw} + c_{uw} \\ &\leq cx. \end{aligned}$$

since x is an optimal solution so it must be that $cy = cx$ and so y is an optimal solution of $2EC_n$. Note though that G_y has strictly fewer edges than G_x. This is a contradiction since G_x was assumed to have the minimum number of edges among all solutions with simple support multigraphs. Therefore G_x cannot have any cut-vertices and hence G_x is two-vertex-connected.

□

Now every two-vertex-connected graph is also two-edge-connected, so Lemma 4 shows that if $n \geq 3$ and c is a nonnegative metric cost vector then any minimum cost two-vertex-connected subgraph of K_n with respect to c has a cost of $2EC_n$. A theorem by Monma, Munson, and Pulleyblank [9] further elaborates on the structure of these optimal solutions.

Theorem 5 *For any nonnegative metric cost vector c there exists a minimum cost two-vertex-connected graph G with respect to c such that*

1. *every vertex of G has degree two or three,*
2. *G is minimally two-edge-connected, and*
3. *deleting any pair of edges in G leaves a bridge in one of the resulting connected components of G.*

Hence we know that for every $n \geq 3$ and any nonnegative metric cost vector c, that there exists an optimal solution to $2EC_n$ whose support multigraph is simple, two-vertex-connected, and has the properties described in Theorem 5. But the authors of [9] have more to say - namely that any such optimal solution is either a cycle or contains the graph shown in Figure 2 as a vertex-induced subgraph.

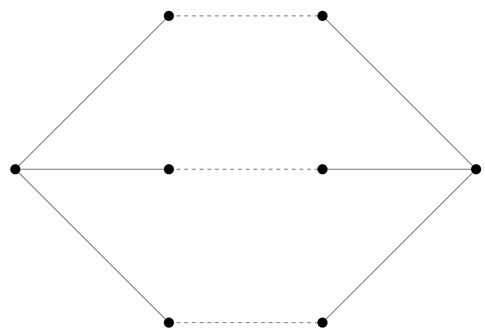

Figure 2: Graph where dashed lines represent paths containing at least one edge each.

Let $\mathcal{M}_n$ denote the set of all spanning subgraphs, M, of K_n for which

- M is two vertex-connected,
- every vertex of M has degree two or three,
- M is minimally two-edge-connected, and
- deleting any pair of edges in M leaves a bridge in one of the resulting connected components of M

according to the conditions of Theorem 5. For our current definition of $\alpha_n^{\triangle}$ we have the condition that $opt(2EC_n) = 1$. However, from Theorem 5 we know that there is some $M \in \mathcal{M}_n$ which is optimal. Suppose then that we consider the constraint

$$cM \geq 1 \text{ for all } M \in \mathcal{M}_n.$$

Clearly this constraint is implied by the former. However, suppose that $cM > 1$ for all $M \in \mathcal{M}_n$. Since $\mathcal{M}_n$ is finite, there must be some $M^\star \in \mathcal{M}_n$ such that $1 < cM^\star \leq cM$ for every $M \in \mathcal{M}_n$. Let $c' = \frac{1}{cM^\star}c$. Since $cM^\star > 1$, for each $1 \leq i \leq k$, $c'x_i = \frac{1}{cM^\star}cx_i < cx_i$. Furthermore, c' is nonnegative, metric, and is optimal with respect to x_i. This contradicts the minimality of cx_i. Therefore the constraints $cM \geq 1$ for all $M \in \mathcal{M}_n$ are equivalent to the constraint $opt(2EC_n) = 1$ in our definition of $\alpha 2EC_n^{\triangle}$. Moreover, we can generate all the graphs in $\mathcal{M}_n$ using methods developed by Boyd and Elliott-Magwood in [2]. Hence we can reformulate our integrality gap as

$$\frac{1}{\alpha 2EC_n^{\triangle}} = \min_{1\leq i\leq k} \min_{\substack{c\geq 0 \\ c \text{ is metric} \\ cM\geq 1 \text{ for all } M\in\mathcal{M}_n \\ c \text{ is optimal for SEP w.r.t. } x_i}} cx_i.$$

Now the constraints $c \geq 0$, c is metric, $cM \geq 1$ for all $M \in \mathcal{M}_n$, and c is optimal with respect to x_i are all linear constraints. Furthermore, there are finitely many of them. Thus,

$$\min_{\substack{c\geq 0 \\ c \text{ is metric} \\ cM\geq 1 \text{ for all } M\in\mathcal{M}_n \\ c \text{ is optimal for SEP w.r.t. } x_i}} cx_i \tag{11}$$

is a linear program. Therefore, in order to compute $\alpha 2EC_n^{\triangle}$ we simply need to solve k linear programs, one for each extreme point, x_i of SEP_n.

Let $\mathcal{T}_n$ represent all incidence vectors of tours for $STSP_n$. In [1], Benoit and Boyd solved the following linear programs very similar to (11) for each extreme point, x_i of the SEP_n polytope:

$$\min_{\substack{c\geq 0 \\ c \text{ is metric} \\ cT\geq 1 \text{ for all } T\in\mathcal{T}_n \\ c \text{ is optimal for SEP w.r.t. } x_i}} cx_i. \tag{12}$$

They then used their results to find the exact integrality gap $\alpha TSP_n^{\triangle}$ for $6 \leq n \leq 10$ (note that for $n \leq 5$, $\alpha 2EC_n = \alpha TSP_n^{\triangle} = 1$). Surprisingly, for each value of n, there was a unique extreme point of the SEP polytope that gave the maximum ratio $\alpha TSP_n^{\triangle}$.

The data for the optimal solution values for the linear programs (12) was extremely useful to us in the following way. Since $\mathcal{T}_n \subset \mathcal{M}_n$ and all other constraints for (12) and (11) are the same, it follows that for each SEP polytope extreme point x_i, the optimal solution value, call it $opt2EC_i$, for (11) is greater than or equal to the optimal solution value, call it $optTSP_i$, for (12). So instead of solving the linear program (11) for every extreme point x_i for the SEP_n polytope, we first solve it for the unique extreme point, say x_k, that gave the integrality gap $\alpha TSP_n^{\triangle}$. We then only need to solve linear program (11) for extreme points x_i for which $optTSP_i < opt2EC_k$ when looking for the minimum $opt2EC_i$ value, since for all other extreme points x_i we have $opt2EC_i \geq optTSP_i \geq opt2EC_k$. Using these ideas, we only had to solve linear program (11) for one extreme point for $6 \leq n \leq 9$, and for two extreme points for $n = 10$.

3 Results

Using the method described in Section (2), we were able to find the exact value of $\alpha 2EC_n = \alpha 2EC_n^{\triangle}$ for $6 \leq n \leq 10$. The results are shown in Table 1. Column 2 contains the value of $\alpha 2EC_n$, which is clearly less than 4/3 for all $n \leq 10$, supporting Conjecture 1. In Column 3 we give the value of $\alpha TSP_n^{\triangle}$ for comparison.

Table 1: Integrality gap results.

n	$\boldsymbol{\alpha 2EC_n}$	$\boldsymbol{\alpha TSP_n^{\triangle}}$
6	10/9	10/9
7	9/8	9/8
8	8/7	8/7
9	7/6	7/6
10	7/6	20/17

It is interesting to note that we have $\alpha 2EC_n = \alpha TSP_n^{\triangle}$ for $6 \leq n \leq 9$, and there was the same unique extreme point of the SEP_n polytope that gave the integrality gap for both. However, for $n = 10$, $\alpha 2EC_n$ was the same as for $n = 9$ and was bigger than $\alpha TSP_n^{\triangle}$. Moreover, there were two extreme points that gave this gap of $\frac{7}{6}$.

The extreme points x^* of the SEP polytope which gave the maximum ratio for each value of n for $2EC$ are shown in Figure 3, where $x_e^* = 1$ for solid edges e, $x_e^* = \frac{1}{2}$ for dashed edges e, and $x_e^* = 0$ for all the other edges. Note that the costs c which gave $\alpha 2EC_n$ are shown on the edges, and for edges not shown the cost c_{ij} was the cost of a minimum cost i to j path using the costs shown.

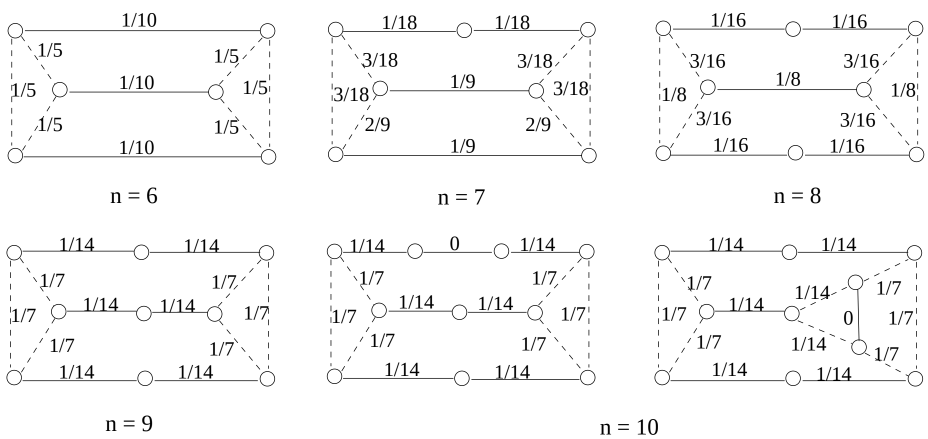

Figure 3: Extreme points and costs which give $\alpha 2EC_n$.

Although Benoit and Boyd [1] were not able to find all the extreme points for SEP_n for $n \geq 11$, they were able to find all the half-integer extreme points for $11 \leq n \leq 14$, i.e. all those extreme points x for which x_e is 0, 1 or $\frac{1}{2}$. Note that the linear programs (11) that needed to be solved for $n \geq 11$ had millions of $\mathcal{M}_n$ constraints, and were too large to be solved directly using CPLEX. This difficulty was easily overcome by adapting a cutting plane approach to handle these constraints. Thus using the same method on the half-integer extreme points of the SEP polytope, we were able to find the value of $\alpha 2EC_n$ for $11 \leq n \leq 14$ when restricted to cost functions c which are optimized at half-integer vertices. (Note that Carr and Ravi proved in [3] that $\alpha 2EC \leq \frac{4}{3}$ under this restriction.) The results are shown in Table 2. Column 2 contains the value of $\alpha 2EC_n$. In Column 3 we give the value of $\alpha TSP_n^{\triangle}$ for the half integer extreme points, once again for comparison.

Table 2: Integrality gap results for half-integer extreme points.

n	$\boldsymbol{\alpha 2EC_n}$	$\boldsymbol{\alpha TSP_n^{\triangle}}$
11	7/6	19/16
12	7/6	6/5
13	7/6	35/29
14	7/6	17/14

Note that although $\alpha TSP_n^{\triangle}$ continued to grow for the half-integer extreme points, $\alpha 2EC_n$ remained stalled at $\frac{7}{6}$ up to $n = 14$. However, we know it grows higher than $\frac{7}{6}$ for $n = 16$, since for $k = 2$ for the family of cost functions shown in Figure 1, we have $n = 16$, and $opt(2EC)/opt(SEP)$ has value $13/11 > 7/6$.

In Benoit and Boyd [1] there was a distinct pattern for the extreme points that gave $\alpha TSP_n^{\triangle}$ and formed a family of cost functions for which the integrality gap ratio approached $\frac{4}{3}$ asymptotically. We tried to find such a pattern for the extreme points that gave $\alpha 2EC_n$, but all the patterns we found led to a gap of at most $\frac{6}{5}$. Thus we would like to conclude this paper by making the following strengthened conjecture for $\alpha 2EC$:

Conjecture 6 *The integrality gap $\alpha 2EC$ for $2EC^{LP}$ is $\frac{6}{5}$.*

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