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Towards a Typed Geometry of Interaction

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Girard's Geometry of Interaction (GoI) develops a mathematical framework for modelling the dynamics of cut-elimination. We introduce a typed version of GoI, called Multiobject GoI for both multiplicative linear logic (**MLL**) and multiplicative exponential linear logic (**MELL**) with units. We present a categorical setting which includes our previous (untyped) GoI models, as well as more general models based on monoidal $*$-categories. Our development of multiobject GoI depends on a new theory of partial traces and trace classes which we believe is of independent interest, as well as an abstract notion of orthogonality (related to work of Hyland and Schalk.) We develop Girard's original theory of types, data and algorithms in our setting, and show his execution formula to be an invariant of Cut Elimination (under some restrictions). We prove Soundness Theorems of the MGoI interpretation (for Multiplicative and Multiplicative Exponential Linear Logic) in partially traced $*$-categories with an orthogonality. Finally, we briefly discuss relating our GoI interpretation to other categorical interpretations of GoI.

1. Introduction

The Geometry of Interaction (GoI) is a highly original interpretation of linear logic, introduced by Girard in a fundamental series of papers beginning in the late 80's (Gir89a; Gir88; Gir95a) and continued recently in (Gir07; Gir08). One striking feature of this work is that it provides a mathematical framework for modelling cut-elimination (normalization) of proofs as a dynamical process of information flow, independent of logical syntax. Girard introduces methods from functional analysis and operator algebras to model proofs and their dynamical behaviour.

Girard's original framework (in GoI I and II), based on C^*-algebras, was studied in

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detail in several works of Danos and Regnier (for example in (DR95)) and by Malacaria and Regnier (MR91). The GoI program itself has been applied to the analysis of optimal reduction by Gonthier, Abadi, and Lévy (GAL92), to complexity theory (BP01; Sch07), to game semantics and token machines (B95; Lau01), etc. For further history see (AHS02) and our recent (HS10).

Let us briefly recall some aspects of Girard's original GoI. Traditional denotational semantics models normalization of proofs (or lambda terms) by static equalities: if Π, Π' are proofs and if Π reduces to Π' by cut-elimination, then in any appropriate model, $[\![\Pi]\!] = [\![\Pi']\!]$. Instead, in his GoI program, Girard considers proofs as operators, pictured as I/O boxes: a proof of a sequent $\vdash \Gamma$ is interpreted as a box with input and output wires labelled by Γ. The formulas or types in Γ form the I/O-*interface* of the proof box and the rules of logic modify this interface. A graphical representation of this process (in the language of combinatory algebras) is described in (AHS02). However Girard works in an *untyped* setting, so in fact the labels of the wires range over a single space U satisfying various domain equations (see below).

Now consider a proof Π of a sequent $\vdash [\Delta], \Gamma$, where Δ is a list of all the cut-formulas used. Girard associates to such a proof a pair of partial symmetries (u, σ), where u is of norm at most 1, and σ represents the cuts Δ. The dynamics of cut-elimination may now be captured in a solution of a system of *feedback equations*, summarized in an operator $Ex(u, \sigma)$ (the *Execution Formula*). It can be shown (Gir89a; HS04a) that for denotations of proofs ($u = [\![\Pi]\!]$) of appropriately restricted types in System F, $Ex([\![\Pi]\!], \sigma)$ is an invariant of cut-elimination. We feel that the general categorical framework described below (based on partial traces) permits a structured approach to solving these general feedback equations and deriving properties of the Execution formula.

Categorical foundations of GoI were initiated in the 90's in lectures by M. Hyland and by S. Abramsky. An early categorical framework was given in Abramsky-Jagadeesan (AJ94). Recent work has stressed the role of Joyal-Street-Verity's *traced monoidal categories* (JSV96) (with additional structure) as a unifying framework for the different approaches. For example, Abramsky's *GoI situations* (Abr96; Hagh00; AHS02) provide a basic algebraic foundation for GoI for multiplicative, exponential linear logic (**MELL**). In our paper (HS04a), we studied a particular class of GoI situations (using traced unique decomposition categories) to axiomatize the details of Girard's original GoI I paper.

In our previous articles, we emphasized several important aspects of Girard's seminal work (at least in GoI I and II).

1 The original Girard framework is essentially *untyped*: there is a reflexive object U in the underlying model (with various retractions and/or domain isomorphisms, e.g. $U \otimes U \lhd U$).

2 Cut-elimination is interpreted by *feedback*, naturally represented in traced monoidal categories. The execution formula, defined via trace, provides an invariant for cut-elimination (for certain restricted sequents).

3 Girard introduced an *orthogonality* operation $\perp$ on endomaps of U together with the notion of *types* (as sets of endomaps equal to their biorthogonal).

4 Following the original Girard papers, there are notions of *data* and *algorithm* encoded

into this dynamical setting, with fundamental theorems connecting types, algorithms, and the convergence of the execution formula.

Points (1) and (2) above were already emphasized in the Abramsky program, as well as in work of Danos and Regnier (Abr96; AHS02; HS04a; DR95). Orthogonalities have been studied abstractly by Hyland and Schalk (HylSch03). The points (1)–(4) are critical to our view of GoI in (HS04a; HS04b) and to the technical developments in this paper. We have recently given a general survey of categorical foundations of GoI in (HS10).

As mentioned, Girard's GoI (both the original, as well as the more recent versions (Gir08)) is essentially untyped: there are domain isomorphisms of a reflexive object U and an associated $*$-algebra of codings and uncodings. Categorically, proofs are interpreted in the monoid $Hom(U, U)$ using this $*$-algebra) (see (Gir88; AHS02; HS04a; HS04b)) and the Execution formula is used to model the dynamics of cut-elimination. In an important series of works, Danos and Regnier (see (DR95) and the references there) studied this $*$-algebra in detail in concrete models, leading to their extensive analysis of reduction paths in untyped lambda calculus.

Our aim in this paper is to move away from "uni-object GoI" to a typed version. This permits us to both generalize GoI and axiomatize its essential features. For example, by removing reflexive objects U, we also unlock the possibilities of generalizing Girard-style GoI to more general tensor categories including cases where the tensor is "product-like" in addition to "sum-like", in the sense of (Abr96; AHS02). We shall illustrate both of these styles in the examples below.

This paper combines and details the treatments in (Hagh06) and (HS05a). The contributions of this paper can be summarized as follows:

— We introduce an axiomatization for partially traced symmetric monoidal categories (and $*$-categories) and provide concrete examples based on $\mathbf{Vec}_{fd}$, finite dimensional vector spaces, and **CMet**, complete metric spaces, as well as **Hilb**, the category of Hilbert spaces, among others. This axiomatization is different from that in (ABP99), although related in spirit.

— We introduce an abstract orthogonality relation (see (HylSch03)), appropriate for GoI, on our models.

— We present a multiobject version of Girard's GoI semantics (MGoI) in partially traced $*$-categories with orthogonality. We define versions of Girard's *types, data, algorithms* in our setting, as well as a categorical version of the *execution formula*. We give an MGoI interpretation for the multiplicative as well as multiplicative-exponential fragments of linear logic with units (**MLL** and **MELL**) in appropriate partially traced $*$-categories, and show that the execution formula is an invariant of cut-elimination, in an appropriate sense (see Section 4.3 below).

— We end by briefly discussing how the MGoI interpretation compares to categorical interpretations in various Int-categories (cf. (JSV96; AJ94)) as well as a brief description of the intrinsic "paracategory" of types and terms in which MGoI lives.

Finally, we should remark that even in the simple case of multiplicative linear logic (**MLL**), the differences between typed and untyped GoI are evident: recall that Girard's original GoI (as presented in (AHS02)) requires a reflexive object $U \neq \{\mathbf{0}\}$, with a retraction $U \oplus U \lhd U$, which is impossible in say finite dimensional vector spaces $\mathbf{Vec}_{fd}$, although we can nevertheless give a typed GoI interpretation. On the other hand, in the case of exponentials in **MELL**, infinity forces itself into the framework: it is no longer possible to carry out the MGoI interpretation in finite dimensions. This is discussed further in Section 5 below, and contrasts with the collapsing of types in untyped GoI based on a reflexive object.

2. Partially Traced Categories

2.1. *Parametric Trace Class*

The notion of categorical trace was introduced by Joyal, Street and Verity in an influential paper (JSV96). The motivation for their work arose in algebraic topology and knot theory, although the authors were aware that such traces also have many applications in Computer Science, where they include such notions as feedback, fixedpoints, iteration theories, etc. For references and history, see (Abr96; AHS02; HS04a).

In this paper we go one step further and look at partial traces. The idea of generalizing the abstract trace of (JSV96) to the partial setting is not new. For example, partial traces were already studied in work of Abramsky, Blute, and Panangaden (ABP99), in unpublished lecture notes of Gordon Plotkin (plot03), in Blute, Cockett, and Seely (BCS00), as well as in (KSW02), (Jeff98) and others (see the discussion in Remark 2.2 below). Unfortunately none of these extant theories is appropriate for our treatment of Girard's GoI. So we present a suitable axiomatization for partial traces which we believe is of independent mathematical interest.

Recall, following Joyal, Street, and Verity (JSV96), a (parametric) trace in a symmetric monoidal category $(\mathbb{C}, \otimes, I, s)$ is a family of maps

$$Tr^U_{X,Y} : \mathbb{C}(X \otimes U, Y \otimes U) \longrightarrow \mathbb{C}(X, Y),$$

satisfying various naturality equations. A *partial* (parametric) trace requires instead that each $Tr^U_{X,Y}$ be a partial map (with domain denoted $\mathbb{T}^U_{X,Y}$) satisfying various closure conditions.

Definition 2.1 (Trace Class). Let $(\mathbb{C}, \otimes, I, s)$ be a symmetric monoidal category. A *(parametric) trace class* in $\mathbb{C}$ is a choice of a family of subsets, for each object U of $\mathbb{C}$, of the form

$$\mathbb{T}^U_{X,Y} \subseteq \mathbb{C}(X \otimes U, Y \otimes U) \text{ for all objects } X,\, Y \text{ of } \mathbb{C}$$

together with a family of functions, called a *(parametric) partial trace*, of the form

$$Tr^U_{X,Y} : \mathbb{T}^U_{X,Y} \longrightarrow \mathbb{C}(X, Y)$$

subject to the following axioms. Here the parameters are X and Y and a morphism $f \in \mathbb{T}^U_{X,Y}$, by abuse of terminology, is said to be *trace class*.

— **Naturality** in X and Y: For any $f \in \mathbb{T}^U_{X,Y}$ and $g : X' \longrightarrow X$ and $h : Y \longrightarrow Y'$,

$$(h \otimes 1_U) f (g \otimes 1_U) \in \mathbb{T}^U_{X',Y'},$$

$$\text{and} \qquad Tr^U_{X',Y'}((h \otimes 1_U)f(g \otimes 1_U)) = h\, Tr^U_{X,Y}(f)\, g.$$

— **Dinaturality** in U: For any $f : X \otimes U \longrightarrow Y \otimes U'$, $g : U' \longrightarrow U$,

$$(1_Y \otimes g)f \in \mathbb{T}^U_{X,Y} \text{ iff } f(1_X \otimes g) \in \mathbb{T}^{U'}_{X,Y},$$

$$\text{and} \qquad Tr^U_{X,Y}((1_Y \otimes g)f) = Tr^{U'}_{X,Y}(f(1_X \otimes g)).$$

— **Vanishing I**: $\mathbb{T}^I_{X,Y} = \mathbb{C}(X \otimes I, Y \otimes I)$, and for $f \in \mathbb{T}^I_{X,Y}$

$$Tr^I_{X,Y}(f) = \rho_Y f \rho_X^{-1}.$$

Here $\rho_A : A \otimes I \longrightarrow A$ is the right unit isomorphism of the monoidal category.

— **Vanishing II**: For any $g : X \otimes U \otimes V \longrightarrow Y \otimes U \otimes V$, if $g \in \mathbb{T}^V_{X\otimes U, Y \otimes U}$, then

$$g \in \mathbb{T}^{U\otimes V}_{X,Y} \text{ iff } Tr^V_{X\otimes U,Y\otimes U}(g) \in \mathbb{T}^U_{X,Y},$$

$$\text{and} \qquad Tr^{U\otimes V}_{X,Y}(g) = Tr^U_{X,Y}(Tr^V_{X\otimes U,Y\otimes U}(g)).$$

— **Superposing**: For any $f \in \mathbb{T}^U_{X,Y}$ and $g : W \longrightarrow Z$,

$$g \otimes f \in \mathbb{T}^U_{W\otimes X, Z\otimes Y},$$

$$\text{and} \qquad Tr^U_{W\otimes X,Z\otimes Y}(g \otimes f) = g \otimes Tr^U_{X,Y}(f).$$

— **Yanking**: $\quad s_{UU} \in \mathbb{T}^U_{U,U}, \quad$ and $\quad Tr^U_{U,U}(s_{U,U}) = 1_U.$

A symmetric monoidal category $(\mathbb{C}, \otimes, I, s)$ with such a trace class is called a *partially traced category*, or a *category with a trace class*. If we let X and Y be I (the unit of the tensor), we get a family of operations $Tr^U_{I,I} : \mathbb{T}^U_{I,I} \longrightarrow \mathbb{C}(I,I)$ defining what we call a *non-parametric* (or *scalar-valued*) trace.

Remark 2.2. An early definition of a partial parametric trace is due to Abramsky, Blute and Panangaden in (ABP99). The guiding example in (ABP99) is the relationship between trace class operators on a Hilbert space and Hilbert-Schmidt operators. This allows the authors to establish a close correspondence between trace and nuclear ideals in a tensor $*$-category. Our definition is different but related to theirs. First, we have used the Yanking axiom in Joyal, Street and Verity (JSV96), whereas in (ABP99) they use a conditional version of the so-called "generalized yanking"; that is, for $f : X \longrightarrow U$ and $g : U \longrightarrow Y$, $Tr^U_{X,Y}(s_{U,Y}(f \otimes g)) = gf$ *whenever* $s_{U,Y}(f \otimes g)$ is trace class. It was shown in (Hagh00) that for traced monoidal categories the two axioms of yanking and generalized yanking are equivalent in the presence of all the other axioms. This equivalence remains valid for the partially traced categories introduced here. In our theory s_{UU} is traceable for all U; on the other hand, many examples in (ABP99) do not have this property. Our Vanishing II axiom differs from and is weaker than the one proposed in (ABP99): it is a "conditional" equivalence. More importantly, we do not require one of the ideal axioms in (ABP99). Namely, we do **not** ask that for $f \in \mathbb{T}^U_{X,Y}$ and any $h : U \longrightarrow U$, $(1_Y \otimes h)f$ and $f(1_X \otimes h)$ be in $\mathbb{T}^U_{X,Y}$. Indeed in the next section we prove that the categories $\mathbf{Vec}_{fd}$ of finite dimensional vector spaces, and $(\mathbf{CMet}, \times)$ of complete metric spaces are partially traced. It can be shown that in both categories the above ideal axiom and Vanishing II of (ABP99) fail and hence they are not traced in the sense of ABP. In defense of not enforcing this ideal axiom, we observe that it is not required for any of the trace axioms. Any partially traced category in the sense of ABP for which the yanking axiom holds will be partially traced according to our definition. Finally, we observe that the nonparametric version of our partial trace is also different from the one in (ABP99).

Plotkin's work develops a theory of Conway ideals on biproduct categories, and an associated categorical trace theory.

Other notions of categorical partial trace have been examined by Alan Jeffrey (Jeff98) and also by various category theorists. One may ask: why did we not use those? For example, Jeffrey cuts down the domain of the trace operator to admissible (traceable) objects U which form a full subcategory of the original category. This is not possible for us: our trace classes do not form subcategories. For example, in keeping with functional analysis on infinite dimensional spaces, the ABP theory of traced ideals (ABP99), and with Girard's papers on GoI, we do not wish (in general) for the identity map to be traced; nor are our trace classes necessarily closed under all possible compositions.

P. Katis, N. Sabadini, R.F.C. Walters (KSW02) give an interesting theory of categories with partial feedback. Although their theory permits treating feedback with delay in a natural way, their approach will not work for us, since dinaturality for them is restricted to isomorphisms, which is unsuitable for giving a GoI interpretation for linear logics.

In (BCS00), Blute, Cockett, and Seely develop an interesting and detailed theory of trace (and fixpoint) combinators in a linearly distributive category, including an appropriate version of the *Int* construction of (JSV96) in that setting. The notion of trace is somewhat similar to Jeffrey's, in that trace is defined for objects and one talks about the traceability of objects in a category. The authors take a local view of the trace combinator: rather than assuming that a trace is available at every object, they consider the effect of particular objects having a trace (partiality of trace), as well as restricting to "compatible classes" of trace operators (which guarantees that an object may have at most one trace structure.)

In the case when (in their notation) tr_U is a partial operator, the authors modify the trace axioms in line with the definitions in (ABP99), in particular Yanking is replaced by Generalized Yanking etc. Thus *for our purposes*, the notion of partial trace in (BCS00) suffers the same issues as the previously-discussed ABP ideal-structure axiomatization (save for the difference in formulation on objects instead of arrows). Also, as in the case of ABP mentioned above, our examples $(\mathbf{Vec}_{fd}, \oplus)$ and $(\mathbf{CMet}, \times)$ will not be partially traced in the sense of (BCS00).

One is obliged to say that there are many different approaches to partial categorical traces and ideals; ours is geared to the details of Girard's GoI. We believe our traceability conditions are most naturally formulated as we did above, as properties of morphisms rather than objects, but this may be a matter of taste.

2.2. *Examples of Partially Traced Categories*

(a) Finite Dimensional Vector Spaces

The category $\mathbf{Vec}_{fd}$ of finite dimensional vector spaces and linear transformations is a symmetric monoidal, indeed an additive, category (see (Mac98)), with monoidal product taken to be $\oplus$, the direct sum (biproduct). Hence, given $f : \oplus_I X_i \longrightarrow \oplus_J Y_j$ with $|I| = n$ and $|J| = m$, we can write f as an $m \times n$ matrix $f = [f_{ij}]$ of its components, where $f_{ij} : X_j \longrightarrow Y_i$ (notice the switch in the indices i and j).

We give a trace class structure on the category $(\mathbf{Vec}_{fd}, \oplus, 0)$ as follows. We shall say an $f : X \oplus U \longrightarrow Y \oplus U$ is *trace class* iff $(I - f_{22})$ is invertible, where I is the identity matrix, and I and f_{22} have size $dim(U)$. In that case, we write

$$Tr^U_{X,Y}(f) = f_{11} + f_{12}(I - f_{22})^{-1} f_{21} \tag{1}$$

This definition is motivated by a generalization of the fact that for a matrix A, $(I-A)^{-1} = \sum_i A^i$, whenever the infinite sum converges. Clearly this sum converges when the matrix norm of A is strictly less than 1, or when A is nilpotent, but in both cases the general idea is the desire to have $(I - A)$ invertible. If the infinite sum for $(I - f_{22})^{-1}$ exists, the above formula for $Tr^U_{X,Y}(f)$ becomes the usual "particle-style" trace in (Abr96; AHS02; HS04a). One advantage of formula (1) is that it does not *a priori* assume the convergence of the sum, nor even that $(I - f_{22})^{-1}$ be computable by iterative methods.

Proposition 2.3. $(\mathbf{Vec}_{fd}, \oplus, 0)$ *is partially traced, with trace class as above.*

The proof of Proposition 2.3 uses the following standard facts from linear algebra:

Lemma 2.4. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be a partitioned matrix with blocks* A $(m \times m)$, B $(m \times n)$, C $(n \times m)$ *and* D $(n \times n)$*. If* D *is invertible, then* M *is invertible iff* $A - BD^{-1}C$ *(the* Schur Complement *of* D*) is invertible.*

Proof. We write

$$M = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}.$$

Clearly M is invertible iff $A - BD^{-1}C$ is invertible, and in that case

$$M^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}.$$

□

Lemma 2.5. *Given* A $(m \times n)$ *and* B $(n \times m)$, $(I_m - AB)$ *is invertible iff* $(I_n - BA)$ *is invertible. Moreover* $(I_m - AB)^{-1}A = A(I_n - BA)^{-1}$.

Proof. Let $K = (I_m - AB)^{-1}$. One can check that $(I_n - BA)^{-1} = (I_n + BKA)$ and conversely if $L = (I_n - BA)^{-1}$, then $(I_m - AB)^{-1} = (I_m + ALB)$. The second identity follows easily. □

Proof. **(Proposition 2.3)** We shall verify the axioms.

— **Naturality** in X and Y: Suppose $f \in \mathbb{T}^U_{X,Y}$ and $g : X' \longrightarrow X$ and $h : Y \longrightarrow Y'$, $(h \oplus 1_U) f (g \oplus 1_U)$ can be represented by its matrix $\begin{bmatrix} hf_{11}g & hf_{12} \\ f_{21}g & f_{22} \end{bmatrix}$ whose component from U to itself is f_{22} and hence $(h \oplus 1_U)f(g \oplus 1_U) \in \mathbb{T}^U_{X',Y'}$ and it is easy to see that $hTr^U_{X,Y}(f)g = Tr^U_{X',Y'}((h \oplus 1_U)f(g \oplus 1_U))$.

— **Dinaturality** in U: Let $f : X \oplus U \longrightarrow Y \oplus U'$, $g : U' \longrightarrow U$. $(1_Y \oplus g)f \in \mathbb{T}^U_{X,Y}$ iff $I - gf_{22}$ is invertible iff $I - f_{22}g$ is invertible by Lemma 2.5 and thus iff $f(1_X \oplus g) \in \mathbb{T}^{U'}_{X,Y}$.

$$\begin{aligned} Tr^U_{X,Y}((1_Y \oplus g)f) &= f_{11} + f_{12}(I - gf_{22})^{-1}gf_{21} \\ &= f_{11} + f_{12}g(I - f_{22}g)^{-1}f_{21} \text{ by Lemma 2.5.} \\ &= Tr^{U'}_{X,Y}(f(1_X \oplus g)). \end{aligned}$$

— **Vanishing I**: Follows from the fact that I (the unit of the monoidal product) is the zero object in $\mathbf{Vec}_{fd}$.

— **Vanishing II**: Let $g : X \oplus U \oplus V \longrightarrow Y \oplus U \oplus V$ be given by $g = \begin{bmatrix} a & b & c \\ d & e & f \\ m & n & p \end{bmatrix}$.

And suppose $g \in \mathbb{T}^V$, then $(I - p)$ is invertible. $g \in \mathbb{T}^{U \oplus V}$ iff $\begin{bmatrix} I-e & -f \\ -n & I-p \end{bmatrix}$ is invertible, iff $I - e - f(I-p)^{-1}n$ is invertible by Lemma 2.4. Thus, iff $Tr^V(g) \in \mathbb{T}^U$. Finally, $Tr^{U \oplus V}_{X,Y}(g) = Tr^U_{X,Y}(Tr^V_{X \oplus U, Y \oplus U}(g))$ follows from the expression for the inverse of $\begin{bmatrix} I-e & -f \\ -n & I-p \end{bmatrix}$ as in Lemma 2.4.

— **Superposing**: Suppose $f \in \mathbb{T}^U_{X,Y}$ and $g : W \longrightarrow Z$, then $I - f_{22}$ is invertible and so $g \oplus f \in \mathbb{T}^U_{W \oplus X, Z \oplus Y}$. Moreover,

$$\begin{aligned} Tr^U_{W \oplus X, Z \oplus Y}(g \oplus f) &= \begin{bmatrix} g & 0 \\ 0 & f_{11} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & f_{12}(I - f_{22})^{-1}f_{21} \end{bmatrix} \\ &= g \oplus Tr^U_{X,Y}(f). \end{aligned}$$

— **Yanking**: $s_{UU} \in \mathbb{T}^U_{U,U}$ as its U to U component is 0_{UU} and so $I - 0_{UU} = I$ which is invertible. Also $Tr^U_{U,U}(s_{U,U}) = 0_{UU} + 1_U(I - 0_{UU})^{-1}1_U = 1_U$.

□

As discussed in Remark 2.2, the category $(\mathbf{Vec}_{fd}\ , \oplus)$ is *not* partially traced in the sense of ABP; nor is it traced in the sense of A. Jeffrey, since (for example) the identity is not trace class.

(b) Metric Spaces

Consider the category **CMet** of complete metric spaces with non-expansive maps, where $f : (M, d_M) \longrightarrow (N, d_N)$ is said to be *non-expansive* iff $d_N(f(x), f(y)) \leq d_M(x, y)$, for all $x, y \in M$. Note that the tempting collection of complete metric spaces and contractions $(d_N(f(x), f(y)) < d_M(x, y))$ is not a category: there are no identity morphisms! **CMet** has products, namely given (M, d_M) and (N, d_N) we define $(M \times N, d_{M \times N})$ with $d_{M \times N}((m, n), (m', n')) = max\{d_M(m, m'), d_N(n, n')\}$.

We define the trace class structure on **CMet** (where $\otimes = \times$) as follows. We say that a morphism $f : X \times U \longrightarrow Y \times U$ is in $\mathbb{T}^U_{X,Y}$ iff for every $x \in X$ the induced map $\pi_2 \lambda u.f(x,u) : U \longrightarrow U$ has a unique fixed point; in other words, iff for every $x \in X$, there is a unique u, and a y, such that $f(x,u) = (y,u)$. Note that in this case y is necessarily unique. Also, note that contractions have unique fixed points, by the Banach fixed point theorem.

Suppose $f \in \mathbb{T}^U_{X,Y}$. We define $Tr^U_{X,Y}(f) : X \longrightarrow Y$ by $Tr^U_{X,Y}(f)(x) = y$, where $f(x,u) = (y,u)$ for the unique u. Equivalently, $Tr^U_{X,Y}(f)(x) = \pi_1 f(x,u)$ where u is the unique fixed point of $\pi_2 \lambda t.f(x,t)$.

Proposition 2.6. $(\mathbf{CMet}, \times, \{*\})$ *is a partially traced category with trace class as above.*

Lemma 2.7. *Let A and B be sets, $f : A \longrightarrow B$ and $g : B \longrightarrow A$. Then, gf has a unique fixed point if and only if fg does. Moreover, let $a \in A$ be the unique fixed point of $gf : A \longrightarrow A$ and $b \in B$ be the unique fixed point of $fg : B \longrightarrow B$. Then $f(a) = b$ and $g(b) = a$.*

Proof. Suppose gf has a unique fixed point a, then $fgf(a) = f(a)$ and so fg has a fixed point. Now suppose b is another fixed point of fg, then $gfg(b) = g(b)$, so $g(b) = a$ and $b = fg(b) = f(a)$. Similarly for the converse direction. □

Proof. **(Proposition 2.6)** We shall verify the axioms. For $f : X \times U \longrightarrow Y \times U$ and $x \in X$, we will use f_x to denote the map $\lambda u.f(x,u) : U \longrightarrow Y \times U$.

— **Naturality** in X and Y: Suppose $f \in \mathbb{T}^U_{X,Y}$ and $g : X' \longrightarrow X$ and $h : Y \longrightarrow Y'$, for any $x' \in X'$, $\pi_2((h \times 1)f(g \times 1))_{x'} = \pi_2 f_{g(x')}$ and hence $(h \times 1)f(g \times 1) \in \mathbb{T}^U_{X',Y'}$. Moreover, $Tr^U((h \times 1)f(g \times 1))(x') = \pi_1(h \times 1)f(g \times 1)(x',u)$ where u is the unique fixed point of $\pi_2((h \times 1)f(g \times 1))_{x'}$. Observe that

$$\begin{array}{rcl} Tr^U((h \times 1)f(g \times 1))(x') & = & \pi_1(h \times 1)f(g \times 1)(x',u) \\ & = & h\pi_1 f(g(x'),u) \\ & = & hTr^U(f)(g(x')) \\ & = & hTr^U(f)g(x'). \end{array}$$

— **Dinaturality** in U: Let $f : X \times U \longrightarrow Y \times U'$, $g : U' \longrightarrow U$. Note that for any $x \in X$, $\pi_2((1_Y \times g)f)_x = g(\pi_2 f_x)$ and $\pi_2(f(1_X \times g))_x = (\pi_2 f_x)g$ and $g(\pi_2 f_x)$ has a unique fixed point iff $(\pi_2 f_x)g$ has a unique fixed point, by Lemma 2.7. Thus $(1_Y \times g)f \in \mathbb{T}^U_{X,Y}$ iff $f(1_X \times g) \in \mathbb{T}^{U'}_{X,Y}$.

$$\begin{array}{rcl} Tr^U_{X,Y}((1_Y \times g)f)(x) & = & \pi_1(1 \times g)f(x,u) \text{ where } u \text{ is the unique} \\ & & \quad \text{fixed point of } g(\pi_2 f_x) \\ & = & \pi_1 f(x,u) \\ & = & \pi_1 f(x, g(u')) \text{ by Lemma 2.7} \\ & & \text{where } u' \text{ is the unique fixed point of } (\pi_2 f_x)g \\ & = & Tr^{U'}_{X,Y}(f(1_X \times g))(x). \end{array}$$

— **Vanishing I**: Follows from the fact that I (the unit of monoidal product) is $\{*\}$.

— **Vanishing II**: Let $g : X \times U \times V \longrightarrow Y \times U \times V$ and suppose $g \in \mathbb{T}^V_{X\times U, Y\times U}$, then $Tr^V(g) : X \times U \longrightarrow Y \times U$ is well-defined.
$Tr^V(g) \in \mathbb{T}^U_{X,Y}$ iff for every x, there is a unique u such that $Tr^V(g)(x,u) = (y,u)$ for some y, iff for every x, there is a unique u and a unique v such that $g(x,u,v) = (y,u,v)$ for some y, iff $g \in \mathbb{T}^{U\times V}_{X,Y}$.
$Tr^U_{X,Y}(Tr^V_{X\times U,Y\times U}(g))(x) = y$ iff there is a unique u such that $Tr^V(g)(x,u) = (y,u)$ iff there is a unique u and a unique v such that $g(x,u,v) = (y,u,v)$ iff $Tr^{U\times V}_{X,Y}(g)(x) = y$.

— **Superposing**: Suppose $f \in \mathbb{T}^U_{X,Y}$ and $g : W \longrightarrow Z$, note that $\pi_2^{Z\times Y,U}(g \times f)_{(w,x)} = \pi_2^{Y,U} f_x$, for all $w \in W$ and $x \in X$, so $g \times f \in \mathbb{T}^U_{W\times X, Z\times Y}$. Moreover, $Tr^U(g \times f)(w,x) = (z,y)$ iff there is a unique u such that $(g \times f)(w,x,u) = (z,y,u)$ iff there is unique u such that $f(x,u) = (y,u)$ and $g(w) = z$ iff $(g \times Tr^U(f))(w,x) = (z,y)$.

— **Yanking**: $s_{UU} \in \mathbb{T}^U_{U,U}$; indeed $\pi_2 s_{u_1}$ is the constant u_1 function, hence it has a unique fixed point, namely u_1. Moreover, $Tr^U(s)(x) = \pi_1 s_x(u)$ where u is the unique fixed point of $\pi_2 s_x$, thus $u = x$ and $Tr^U(s)(x) = \pi_1(u,x) = u = x$ and hence $Tr^U(s) = 1_U$.

$\square$

Proposition 2.6 remains valid for the category $(\mathbf{Sets}, \times)$ of sets and mappings. The latter then becomes a partially traced category with the same definition for trace class morphisms as in **CMet**. However, this fails for the category $(\mathbf{Rel}, \times)$, of sets and relations, as Lemma 2.7 is no longer valid: consider the sets $A = \{a\}$, $B = \{b, b'\}$, and let $f = \{(a,b), (a,b')\}$ and $g = \{(b,a), (b',a)\}$.

(c) Monoidal Subcategories of Traced Categories

The following results have been obtained recently by O. Malherbe (Mal10) and yield many interesting examples of partial traces, especially related to categories arising in the semantics of quantum programming languages and higher-order quantum computation (Sel04; Sel04a).

Example 2.8 (O. Malherbe). Let $(\mathbb{C}, \otimes, I, s)$ be a symmetric monoidal category, with monoidal subcategory $\mathbb{D}$. Suppose $\mathbb{C}$ is partially traced, with trace classes $\mathbb{T}^U_{X,Y} \subseteq \mathbb{C}(X \otimes U, Y \otimes U)$ for all objects X, Y of $\mathbb{C}$ and with trace $Tr^U_{X,Y} : \mathbb{T}^U_{X,Y} \longrightarrow \mathbb{C}(X,Y)$.

Then $\mathbb{D}$ is partially traced by defining $f \in \mathbb{D}(X \otimes U, Y \otimes U)$ to be *trace class (in $\mathbb{D}$)* if $f \in \mathbb{T}^U_{X,Y}$ in $\mathbb{C}$ and in addition $Tr^U_{X,Y}(f) \in \mathbb{D}(X,Y)$. Moreover, the trace of $f \in \mathbb{D}$ (when defined), is its value when calculated in $\mathbb{C}$.

In particular, if $\mathbb{D}$ is a monoidal subcategory of $\mathbb{C}$ and $\mathbb{C}$ is a *totally traced* symmetric monoidal category, then $\mathbb{D}$ is partially traced. In this case, a map $f \in \mathbb{D}$ is trace class if, when considered as a $\mathbb{C}$ map, its trace lands in $\mathbb{D}$, and in this case, the value of the trace is its value in $\mathbb{C}$.

(d) Total Traces

Of course, all (totally-defined) traces in the usual definition of a traced monoidal category yield a trace class, namely the entire homset is the domain of Tr. In particular, all the examples in our previous work on uni-object GoI based on unique decomposition categories, (HS04a; HS04b), still apply here.

Remark 2.9 (A Non-Example). Consider the structure (**CMet** , $\times$). Defining the trace class morphisms as those f such that $\pi_2 \lambda u.f(x,u) : U \longrightarrow U$ is a contraction, for every $x \in X$, does not yield a partially traced category: all axioms are true except for dinaturality and Vanishing II.

3. Orthogonality Relations

Girard originally introduced orthogonality relations into linear logic to model formulas (or types) as sets equal to their biorthogonal (e.g. in the phase semantics of the original paper (Gir87) and in GoI 1 (Gir88)). Recently M. Hyland and A. Schalk gave an abstract approach to orthogonality relations in symmetric monoidal closed categories (HylSch03). They also point out that an orthogonality on a traced symmetric monoidal category $\mathbb{C}$ can be obtained by first considering their axioms applied to $Int(\mathbb{C})$, the compact closure of $\mathbb{C}$, and then translating them down to $\mathbb{C}$. Below we give this translation (not explicitly calculated in (HylSch03)), using the so-called "GoI construction" $\mathcal{G}(\mathbb{C})$ (Abr96; Hagh00) instead of $Int(\mathbb{C})$. The categories $\mathcal{G}(\mathbb{C})$ and $Int(\mathbb{C})$ are both compact closures of $\mathbb{C}$, and are shown to be isomorphic in (Hagh00). For more on compact closure constructions the interested reader is referred to the above references.

As we are dealing with partial traces we need to take extra care in stating the axioms below; namely, an axiom involving a trace should be read with the proviso: "whenever all traces exist". Finally hereafter, without loss of generality and for readability we consider strict monoidal categories. It is well known that every monoidal category is equivalent to a strict one.

Definition 3.1. Let $\mathbb{C}$ be a traced symmetric monoidal category. An *orthogonality relation* on $\mathbb{C}$ is a family of relations $\perp_{UV}$ between maps $u : V \longrightarrow U$ and $x : U \longrightarrow V$

$$V \xrightarrow{u} U \perp_{UV} U \xrightarrow{x} V$$

subject to the following axioms:

(i) *Isomorphism* : Let $f : U \otimes V' \longrightarrow V \otimes U'$ and $\hat{f} : U' \otimes V \longrightarrow V' \otimes U$ be such that $Tr^{V'}(Tr^{U'}((1 \otimes 1 \otimes s_{U',V'})\alpha^{-1}(f \otimes \hat{f})\alpha)) = s_{U,V}$ and $Tr^{V}(Tr^{U}((1 \otimes 1 \otimes s_{U,V})\alpha^{-1}(\hat{f} \otimes f)\alpha)) = s_{U',V'}$. Here $\alpha = (1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)$ with s at appropriate types. Note that this simply means that $f : (U,V) \longrightarrow (U',V')$ and $\hat{f} : (U',V') \longrightarrow (U,V)$ are inverses of each other in $\mathcal{G}(\mathbb{C})$.
Then for all $u : V \longrightarrow U$ and $x : U \longrightarrow V$,

$$u \perp_{UV} x \ \text{ iff } \ Tr^{U}_{V',U'}(s_{U,U'}(u \otimes 1_{U'})f s_{V',U}) \perp_{U'V'} Tr^{V}_{U',V'}((1_{V'} \otimes x)\hat{f});$$

that is, orthogonality is invariant under isomorphism.

(ii) *Tensor* : For all $u : V \longrightarrow U$, $v : V' \longrightarrow U'$ and $h : U \otimes U' \longrightarrow V \otimes V'$,

$$u \perp_{UV} Tr^{U'}_{U,V}((1_V \otimes v)h) \ \text{ and } \ v \perp_{U'V'} Tr^{U}_{U',V'}(s_{U,V'}(u \otimes 1_{V'})h s_{U',U})$$

$$\text{imply} \qquad (u \otimes v) \perp_{U\otimes U', V\otimes V'} h.$$

(iii) *Identity* : For all $u : V \longrightarrow U$ and $x : U \longrightarrow V$,

$$u \perp_{UV} x \text{ implies } 1_I \perp_{II} Tr^V_{I,I}(xu).$$

(iv) *Symmetry* : For all $u : V \longrightarrow U$ and $x : U \longrightarrow V$,

$$u \perp_{UV} x \text{ iff } x \perp_{VU} u.$$

Remark 3.2.

(i) The above axiomatisation is slightly simplified: there is an additional axiom considered in (HylSch03)
Implication : For all $u : V \longrightarrow U$, $y : U' \longrightarrow V'$ and $f : U \otimes V' \longrightarrow V \otimes U'$

$$u \perp_{UV} Tr^{V'}_{U,V}((1_V \otimes y)f) \text{ and } Tr^{U}_{V',U'}(s_{U,U'}(u \otimes 1_{U'})fs_{V',U}) \perp_{U'V'} y$$

$$\text{imply} \qquad f \perp_{V\otimes U', U\otimes V'} (u \otimes y).$$

For a (partially) traced symmetric monoidal category, the Tensor and Implication axioms are equivalent in the presence of the Symmetry axiom, as can be easily seen (cf. (HylSch03).) Thus we shall not consider the Implication axiom below.

(ii) Our work on GoI reveals that one needs another axiom which we observe as the converse of the Tensor axiom. This is related to abstract computation and the notion of datum in GoI. Hence, we shall replace the Tensor axiom by the following stronger Tensor axiom which turns out to be the same as the Precise Tensor axiom of (HylSch03).

Precise Tensor:
For all $u : V \longrightarrow U$, $v : V' \longrightarrow U'$ and $h : U \otimes U' \longrightarrow V \otimes V'$,

$$u \perp_{UV} Tr^{U'}_{U,V}((1_V \otimes v)h) \text{ and } v \perp_{U'V'} Tr^{U}_{U',V'}(s_{U,V'}(u \otimes 1_{V'})hs_{U',U})$$

$$\text{iff} \qquad (u \otimes v) \perp_{U\otimes U', V\otimes V'} h.$$

whenever all traces exist.

Definition 3.3. Let $\mathbb{C}$ be a traced symmetric monoidal category. A *(strong) orthogonality* relation is defined as in Definition 3.1 but with the Tensor axiom replaced by the Precise Tensor axiom above.

In the context of GoI, we will be working with strong orthogonality relations on endomorphism sets of objects in the underlying categories. Biorthogonally closed (i.e. $X = X^{\perp\perp}$) subsets of certain endomorphism sets are important as they define *types* (GoI interpretation of formulae.) We have observed that all the orthogonality relations that we work with in this paper can be characterized using trace classes. This suggests the following, which seems to cover many known examples.

Example 3.4 (Orthogonality as trace class). Let $(\mathbb{C}, \otimes, I, Tr)$ be a partially traced category where $\otimes$ is the monoidal product with unit I, and Tr is the partial trace operator as in Section 2. Let A and B be objects of $\mathbb{C}$. For $f : A \longrightarrow B$ and $g : B \longrightarrow A$, we can define an orthogonality relation by declaring $f \perp_{BA} g$ iff $gf \in \mathbb{T}^A_{I,I}$. Axioms can be checked easily and we shall not include the verification here. It turns out that this is a variation of the notion of *Focussed orthogonality* of Hyland and Schalk (HylSch03).

Hence, from our previous discussion on traces, we obtain the following examples:

— $\mathbf{Vec}_{fd}$. For $A \in \mathbf{Vec}_{fd}$, $f, g \in End(A)$, define $f \perp g$ iff $I - gf$ is invertible. Here I is the identity matrix of size $dim(A)$.

— **CMet** . Let $M \in \mathbf{CMet}$. For $f, g \in End(M)$, define $f \perp g$ iff gf has a unique fixed point.

4. Multi-object (Typed) GoI: the multiplicative level

As presented in the Introduction, the principal aim of this paper is to give a typed version of GoI, called Multiobject GoI for both multiplicative linear logic (**MLL**) and multiplicative exponential linear logic (**MELL**).

We generalize the original GoI interpretation of formal proofs in linear logic from an untyped to a typed setting. Moreover, we also interpret the units. This involves moving from interpreting proofs in the endomorphism monoid of a reflexive object U in appropriate totally traced categories, as in (AHS02; HS04a)), to a more abstract setting: general endomorphism monoids in appropriate partially-traced monoidal $*$-categories. We begin with the simpler case of **MLL** and in the next section discuss the exponential rules for **MELL**.

4.1. *The MGoI Interpretation of* **MLL**

In this subsection we introduce the Multiobject Geometry of Interaction (MGoI) semantics for multiplicative linear logic in a partially traced symmetric monoidal category $(\mathbb{C}, \otimes, I, Tr, \perp)$ equipped with an orthogonality relation $\perp$ as in the previous sections. Here $\otimes$ is the monoidal product with unit I and Tr is a partial trace operator as in Section 2. We do not require that the category $\mathbb{C}$ have a reflexive object, so uni-object GoI semantics ((Gir89a; HS04a)) may not be possible to carry out in $\mathbb{C}$.

The MGoI semantics, denoted θ, interprets formulas and proofs inductively in a structure $(\mathbb{C}, \otimes, I, Tr, \perp)$. The ideas below were inspired by Girard's original uni-object GoI semantics referred to above.

Interpreting formulas:

Let A be an object of $\mathbb{C}$ and let $f, g \in End(A)$. We say that f is *orthogonal to* g, denoted $f \perp g$, if $(f, g) \in \perp$. Also given $X \subseteq End(A)$ we define

$$X^{\perp} = \{f \in End(A) \,|\, \forall g \in X, f \perp g\}.$$

We now define an operator on the objects of $\mathbb{C}$ as follows: given an object A, $\mathcal{T}(A) = \{X \subseteq End(A) \,|\, X^{\perp\perp} = X\}$. Elements of $\mathcal{T}(A)$ are often called *types.*

We first define a "compact" interpretation map $[\![-]\!]$ on the formulas of **MLL** as follows. Given the value of $[\![-]\!]$ on the atomic propositions as objects of $\mathbb{C}$, we extend it to all formulas by:

— $[\![\mathbf{1}]\!] = [\![\perp]\!] = I$ where I is the unit of $\mathbb{C}$.
— $[\![A^\perp]\!] = [\![A]\!]$
— $[\![A \parr B]\!] = [\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$.

We then define the MGoI-interpretation θ for formulas as follows.

— $\theta(\mathbf{1}) = \{1_I\}^{\perp\perp}$, and $\theta(\perp) = \{1_I\}^\perp$.
— $\theta(\alpha) \in \mathcal{T}([\![\alpha]\!])$, where α is an atomic formula.
— $\theta(\alpha^\perp) = \theta(\alpha)^\perp$, where α is an atomic formula.
— $\theta(A \otimes B) = \{a \otimes b \mid a \in \theta(A), b \in \theta(B)\}^{\perp\perp}$
— $\theta(A \parr B) = \{a \otimes b \mid a \in \theta(A)^\perp, b \in \theta(B)^\perp\}^\perp$

Easy consequences of the definition are: (i) for any formula A, $(\theta A)^\perp = \theta(A^\perp)$, (ii) $\theta(A) \subseteq End([\![A]\!])$, and (iii) $\theta(A)^{\perp\perp} = \theta(A)$. Hence, θ *interprets formulas as types.*

Interpreting proofs:

We define the MGoI interpretation for proofs of **MLL**, similarly to (HS04a). Every **MLL** sequent will be of the form $\vdash [\Delta], \Gamma$ where Γ is a sequence of formulas and Δ is a sequence of cut formulas that have already been made in the proof of $\vdash \Gamma$ (see (Gir89a; HS04a)). This device is used to keep track of the cuts in a proof of $\vdash \Gamma$. A proof Π of $\vdash [\Delta], \Gamma$ is represented by a morphism $\theta(\Pi) \in End(\otimes[\![\Gamma]\!] \otimes [\![\Delta]\!])$. With $\Gamma = A_1, \cdots, A_n$, $\otimes[\![\Gamma]\!]$ stands for $[\![A_1]\!] \otimes \cdots \otimes [\![A_n]\!]$, similarly for Δ. We drop the double brackets wherever there is no danger of confusion. We also define $\sigma = s \otimes \cdots \otimes s$ (m-copies) where s is the symmetry map at different types (omitted for convenience), and $|\Delta| = 2m$. The morphism σ represents the cuts in the proof of $\vdash \Gamma$, i.e. it models Δ. In the case where Δ is empty (that is for a cut-free proof), we define $\sigma : I \longrightarrow I$ to be 1_I where I is the unit of the monoidal product in $\mathbb{C}$.

Let Π be a proof of $\vdash [\Delta], \Gamma$. We define the MGoI interpretation of Π, denoted by $\theta(\Pi)$, by induction on the length of the proof as follows.

1 Π be the axiom $\vdash \mathbf{1}$, then $\theta(\Pi) = 1_I$.
2 Π is obtained using the $\perp$ *rule* applied to the proof Π' of $\vdash [\Delta], \Gamma'$. Then $\theta(\Pi) = \theta(\Pi') \otimes 1_I = \theta(\Pi')$, as we are working with strict monoidal categories.
3 Π is an *axiom* $\vdash A, A^\perp$, $\theta(\Pi) := s_{V,V}$ where $[\![A]\!] = [\![A^\perp]\!] = V$.
4 Π is obtained using the *cut* rule on Π' and Π'' that is,

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta'], \Gamma', A\end{array} \quad \begin{array}{c}\Pi'' \\ \vdots \\ \vdash [\Delta''], A^\perp, \Gamma''\end{array}}{\vdash [\Delta', \Delta'', A, A^\perp], \Gamma', \Gamma''} \; cut$$

Define $\theta(\Pi) = \tau^{-1}(\theta(\Pi') \otimes \theta(\Pi''))\tau$, where τ is the permutation
$\Gamma' \otimes \Gamma'' \otimes \Delta' \otimes \Delta'' \otimes A \otimes A^\perp \xrightarrow{\tau} \Gamma' \otimes A \otimes \Delta' \otimes A^\perp \otimes \Gamma'' \otimes \Delta''$.

5 Π is obtained using the *exchange* rule on the formulas A_i and A_{i+1} in Γ'. That is Π

is of the form

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma'\end{array}}{\vdash [\Delta], \Gamma}\ \textit{exchange}$$

where $\Gamma' = \Gamma'_1, A_i, A_{i+1}, \Gamma'_2$ and $\Gamma = \Gamma'_1, A_{i+1}, A_i, \Gamma'_2$. Then, $\theta(\Pi) = \tau^{-1}\theta(\Pi')\tau$, where $\tau = 1_{\Gamma'_1} \otimes s \otimes 1_{\Gamma'_2 \otimes \Delta}$.

6 Π is obtained using an application of the *par* rule, that is Π is of the form:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', A, B\end{array}}{\vdash [\Delta], \Gamma', A \parr B}\ \parr \quad . \text{ Then } \theta(\Pi) = \theta(\Pi').$$

7 Π is obtained using an application of the *times* rule, that is Π is of the form:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta'], \Gamma', A\end{array} \quad \begin{array}{c}\Pi'' \\ \vdots \\ \vdash [\Delta''], \Gamma'', B\end{array}}{\vdash [\Delta', \Delta''], \Gamma', \Gamma'', A \otimes B}\ \otimes$$

Then $\theta(\Pi) = \tau^{-1}(\theta(\Pi') \otimes \theta(\Pi''))\tau$, where τ is the permutation $\Gamma' \otimes \Gamma'' \otimes A \otimes B \otimes \Delta' \otimes \Delta'' \xrightarrow{\tau} \Gamma' \otimes A \otimes \Delta' \otimes \Gamma'' \otimes B \otimes \Delta''$.

When Δ' and Δ'' are empty sequences, this corresponds to the definition of tensor product in Abramsky's $\mathcal{G}(\mathbb{C})$ (see (Abr96; Hagh00).)

Example 4.1.

(a) Let Π be the following proof:

$$\dfrac{\vdash A, A^\perp \quad \vdash A, A^\perp}{\vdash [A^\perp, A], A, A^\perp}\ \textit{cut}$$

Then the MGoI semantics of this proof is given by

$$\theta(\Pi) = \tau^{-1}(s \otimes s)\tau = s_{V\otimes V, V\otimes V}$$

where $\tau = (1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)$ and $[\![A]\!] = [\![A^\perp]\!] = V$.

(b) Now consider the following proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash B, B^\perp \quad \vdash C, C^\perp}{\vdash B, C, B^\perp \otimes C^\perp}}{\vdash B, B^\perp \otimes C^\perp, C}}{\vdash B^\perp \otimes C^\perp, B, C}}{\vdash B^\perp \otimes C^\perp, B \parr C}\ .$$

Its denotation is $s_{V\otimes W, V\otimes W}$, where $[\![B]\!] = [\![B^\perp]\!] = V$ and $[\![C]\!] = [\![C^\perp]\!] = W$. □

4.2. *Dynamics*

Dynamics is at the heart of the GoI interpretation as compared to denotational semantics and it is hidden in the cut-elimination process. The mathematical model of cut-elimination is given by the so called *execution formula* defined as follows:

$$EX(\theta(\Pi), \sigma) = Tr^{\otimes\Delta}_{\otimes\Gamma,\otimes\Gamma}((1 \otimes \sigma)\theta(\Pi)) \tag{2}$$

where Π is a proof of the sequent $\vdash [\Delta], \Gamma$, $\sigma = s \otimes \cdots \otimes s$ (m times) models Δ, and $2m$ is the number of formulas in Δ. Note that $EX(\theta(\Pi), \sigma)$ is a morphism from $\otimes\Gamma \longrightarrow \otimes\Gamma$, when it exists. We shall prove below (see Theorem 4.6) that the execution formula always exists for any **MLL** proof Π.

Example 4.2.

Consider the proof Π in Example 4.1 above. Recall also that $\sigma = s$ in this case ($m = 1$). Then $EX(\theta(\Pi), \sigma) = Tr((1 \otimes s_{V,V})s_{V\otimes V, V\otimes V}) = s_{V,V}$.

Note that in this case we have obtained the MGoI interpretation of the cut-free proof of $\vdash A, A^{\perp}$, obtained by applying Gentzen's Hauptsatz to the proof Π. □

4.3. *Soundness of the MGoI Interpretation for* **MLL**

In this section we present one of the main results of this paper: the soundness of the MGoI interpretation. We show that if a proof Π is reduced (via cut-elimination) to another proof Π', then $EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), \tau)$; that is, $EX(\theta(\Pi), \sigma)$ is an invariant of reduction. In particular, if Π' is cut-free (i.e. a normal form) we have $EX(\theta(\Pi), \sigma) = \theta(\Pi')$. Intuitively this says that if one thinks of cut-elimination as computation then $\theta(\Pi)$ can be thought of as an algorithm. The computation takes place as follows: if $EX(\theta(\Pi), \sigma)$ exists then it yields a datum (cf. cut-free proof). This intuition will be made precise below (Theorems 4.6 & 4.8).

The next fundamental lemma (which features in several of Girard's papers) follows directly from our trace axioms. It is essentially a version of the Church-Rosser theorem.

Lemma 4.3 (Associativity of cut). *Let Π be a proof of $\vdash [\Gamma, \Delta], \Lambda$ and σ and τ be the morphisms representing the cut-formulas in Γ and Δ respectively. Then*

$$EX(\theta(\Pi), \sigma \otimes \tau) = EX(EX(\theta(\Pi), \tau), \sigma) = EX(EX((1 \otimes s)\theta(\Pi)(1 \otimes s), \sigma), \tau),$$

whenever all traces exist.

Proof.
$EX(EX(\theta(\Pi), \tau), \sigma) =$
$= Tr((1 \otimes \sigma)Tr((1 \otimes \tau)\theta(\Pi)))$ definition of EX formula
$= Tr(Tr((1 \otimes \sigma \otimes 1)(1 \otimes \tau)\theta(\Pi)))$ naturality of trace
$= Tr((1 \otimes \sigma \otimes \tau)\theta(\Pi))$ vanishing II property of trace
$= EX(\theta(\Pi), \sigma \otimes \tau)$.

As for the second equality: (we drop the subscripts for s, as there is no danger of confusion!)

$EX(EX((1\otimes s)\theta(\Pi)(1\otimes s),\sigma),\tau) =$
$= Tr((1\otimes\tau)Tr((1\otimes\sigma)(1\otimes s)\theta(\Pi)(1\otimes s)))$ def. of EX formula
$= Tr(Tr((1\otimes\tau\otimes 1)(1\otimes 1\otimes\sigma)(1\otimes s)\theta(\Pi)(1\otimes s)))$ naturality of trace
$= Tr(Tr((1\otimes\tau\otimes\sigma)(1\otimes s)\theta(\Pi)(1\otimes s)))$ functoriality of tensor
$= Tr(Tr((1\otimes s)(1\otimes\sigma\otimes\tau)\theta(\Pi)(1\otimes s)))$ naturality of symmetry
$= Tr(Tr((1\otimes\sigma\otimes\tau)\theta(\Pi)))$ dinaturality of trace
$= Tr((1\otimes\sigma\otimes\tau)\theta(\Pi))$ vanishing II property of trace
$= EX(\theta(\Pi),\sigma\otimes\tau)$.

□

The next definitions, of fundamental importance to the original GoI framework, are analogous to concepts arising in realizability and Girard's method of *candidats* (cf. (GLT)).

In the sequel we shall be working in a partially traced symmetric monoidal category equipped with an orthogonality relation. We shall suppress mentioning the use of the Symmetry axiom of the orthogonality relation.

Definition 4.4. Let $\Gamma = A_1,\cdots,A_n$ and $V_i = [\![A_i]\!]$.

- A *datum of type* $\theta\Gamma$ is a morphism $M : \otimes_i V_i \longrightarrow \otimes_i V_i$ such that for any $a_i \in \theta(A_i^\perp)$, $\otimes_i a_i \perp M$ and

$$M\bullet a_1 := Tr^{V_1}(s^{-1}_{\otimes_{i\neq 1}V_i,V_1}(a_1\otimes 1_{V_2}\otimes\cdots\otimes 1_{V_n})Ms_{\otimes_{i\neq 1}V_i,V_1})$$

  and

$$M\hat{\bullet}(a_2\otimes\cdots\otimes a_n) := Tr^{V_2\otimes\cdots\otimes V_n}((1\otimes a_2\otimes\cdots\otimes a_n)M)$$

  both exist.
- An *algorithm of type* $\theta\Gamma$ is a morphism $M : \otimes_i V_i\otimes[\![\Delta]\!] \longrightarrow \otimes_i V_i\otimes[\![\Delta]\!]$ for some $\Delta = B_1, B_2,\cdots,B_{2m}$ with m a nonnegative integer and $B_{i+1} = B_i^\perp$ for $i = 1,3,\cdots,2m-1$, such that if $\sigma : \otimes_{i=1}^{2m}[\![B_i]\!] \longrightarrow \otimes_{i=1}^{2m}[\![B_i]\!]$ is $\otimes_{i=1,odd}^{2m-1} s_{[\![B_i]\!],[\![B_{i+1}]\!]}$, $EX(M,\sigma)$ exists and is a datum of type $\theta\Gamma$.

Lemma 4.5. *Let $\widetilde{\Gamma} = A_2,\cdots,A_n$ and $\Gamma = A_1,\widetilde{\Gamma}$. Let $V_i = [\![A_i]\!]$, and $M : \otimes_i V_i \longrightarrow \otimes_i V_i$, for $i = 1,\cdots,n$. Then, M is a datum of type $\theta(\Gamma)$ iff for all $a_i \in \theta(A_i^\perp)$, $M\bullet a_1$ and $M\hat{\bullet}(a_2\otimes\cdots\otimes a_n)$ (defined as above) exist and are in $\theta(\widetilde{\Gamma})$, and $\theta(A_1)$, respectively.*

Proof. First note that we interpret $\theta(\widetilde{\Gamma})$ as $\theta(A_2 \parr \cdots \parr A_n)$. Let $a_i \in \theta(A_i^\perp)$ for $i = 1,\cdots,n$, suppose M is a datum of type $\theta(\Gamma)$. Then $(a_1\otimes(a_2\otimes\cdots\otimes a_n)) \perp M$, and $M\bullet a_1 = Tr(s(a_1\otimes 1_{V_2}\otimes\cdots\otimes 1_{V_n})Ms)$ and $M\hat{\bullet}(a_2\otimes\cdots\otimes a_n) = Tr((1\otimes a_2\otimes\cdots\otimes a_n)M)$ both exist by definition. By the Precise Tensor axiom of the orthogonality relation, $M\bullet a_1 \perp (a_2\otimes\cdots\otimes a_n)$ and $M\hat{\bullet}(a_2\otimes\cdots\otimes a_n) \perp a_1$, so $M\bullet a_1 \in \theta(\widetilde{\Gamma})$ and $M\hat{\bullet}(a_2\otimes\cdots\otimes a_n) \in \theta(A_1)$. Conversely, suppose that for all $a_i \in \theta(A_i^\perp)$ $(i = 1,\cdots,n)$, $M\bullet a_1$ and $M\hat{\bullet}(a_2\otimes\cdots\otimes a_n)$ exist and are in $\theta(\widetilde{\Gamma})$ and $\theta(A_1)$ respectively. Then for all $a_i \in \theta(A_i^\perp)$, $M\bullet a_1 \perp (a_2\otimes\cdots\otimes a_n)$, and $M\hat{\bullet}(a_2\otimes\cdots\otimes a_n) \perp a_1$, so again by the Precise Tensor axiom, $M \perp (a_1\otimes\cdots\otimes a_n)$. Hence M is datum of type $\theta(\Gamma)$. □

Theorem 4.6 (Proofs as algorithms). *Let Π be an* **MLL** *proof of a sequent $\vdash [\Delta],\Gamma$. Then $\theta(\Pi)$ is an algorithm of type $\theta\Gamma$.*

Proof.

— Let Π be the axiom $\vdash \mathbf{1}$, clearly $\theta(\Pi) = 1_I \perp a$ for every $a \in \theta(\mathbf{1})^\perp$ and so $\theta(\Pi)$ is an algorithm of type $\theta(\mathbf{1})$.

— Let Π be the proof obtained using the $\perp$ *rule* applied to the proof Π' of $\vdash [\Delta], \Gamma'$, also suppose that $\theta(\Pi')$ is an algorithm of type $\theta(\Gamma')$. Recall that $\theta(\Pi) = \theta(\Pi') \otimes 1_I = \theta(\Pi')$, and thus $\theta(\Pi)$ is an algorithm of type $\theta(\Gamma') = \theta(\Gamma' \parr \perp) = \theta(\Gamma', \perp)$.

— Let Π be an axiom, where $\Gamma = A, A^\perp$ and Δ is empty. Let $a \in \theta A^\perp$ and $b \in \theta A$, and $[\![A]\!] = V$, so $a, b : V \longrightarrow V$ and $a \perp b$. Note that $s_{V,V} \bullet a = Tr(s(a \otimes 1)ss) = a$ by generalized yanking. By the Precise Tensor axiom, $a \otimes b \perp s_{V,V}$. Thus $EX(\theta(\Pi), 1_I) = \theta(\Pi)$ is a datum of type $\theta\Gamma$.

— Suppose Π is obtained by applying the cut rule to the proofs Π' and Π'' of $\vdash [\Delta'], \Gamma', A$ and $\vdash [\Delta''], A^\perp, \Gamma''$ respectively. We assume first that Δ' and Δ'' are empty and $\Gamma' = B'$ and $\Gamma'' = B''$ are single formulas. Recall that $\theta(\Pi) = \tau^{-1}(\theta(\Pi') \otimes \theta(\Pi''))\tau$. We need to show that (with $\sigma = s$):

  (i) $EX(\theta(\Pi), \sigma)$ exists.

  (ii)$EX(\theta(\Pi), \sigma)$ is a datum of type $\theta(B', B'')$.

  Let $b' \in \theta({B'}^\perp)$, and a denote $\theta(\Pi') \bullet b'$. Then $a \in \theta(A)$, and $\theta(\Pi'') \bullet a \in \theta(B'')$, by inductive hypothesis. We shall show that $EX(\theta(\Pi), \sigma) \bullet b' \in \theta(B'')$. Many steps in the following equations have been compressed: they all follow from trace properties and naturality of symmetry morphisms. Let $[\![A]\!] = V$, $[\![B']\!] = U$ and $[\![B'']\!] = W$

  $\theta(\Pi'') \bullet a$
  $= Tr^V(s_{V,W}(a \otimes 1)\theta(\Pi'')s_{W,V})$
  $= Tr^V(s_{V,W}(Tr^U(s_{V,U}(b' \otimes 1)\theta(\Pi')s_{U,V}) \otimes 1)\theta(\Pi)s_{W,V})$
  $= Tr^U(s_{U,W}(b' \otimes 1)(Tr^V((1 \otimes s_{V,W})(\theta(\Pi') \otimes 1)(s_{V,U} \otimes 1)(1 \otimes s_{W,U})$
  $(\theta(\Pi) \otimes 1)(s_{W,V} \otimes 1)(1 \otimes s_{U,V}))))$
  $= Tr^U(s_{U,W}(b' \otimes 1)Tr^{V \otimes V}((1 \otimes 1 \otimes s_{V,V})(1 \otimes s_{V,W} \otimes 1)(1 \otimes 1 \otimes s_{V,W})$
  $(\theta(\Pi') \otimes \theta(\Pi))(1 \otimes 1 \otimes s_{W,V})(1 \otimes s_{W,V} \otimes 1))s_{W,U})$
  $= EX(\theta(\Pi), \sigma) \bullet b'$

  However, by inductive hypothesis $\theta(\Pi'') \bullet a \in \theta(B'')$. Now, let $b'' \in \theta(B'')^\perp$ and let a denote $\theta(\Pi'') \hat{\bullet} b''$. By inductive hypothesis $a \in \theta A^\perp$ and $\theta(\Pi') \hat{\bullet} a \in \theta B'$. One can, similarly to above, show that $EX(\theta(\Pi), \sigma) \hat{\bullet} b'' = \theta(\Pi') \hat{\bullet} a$. The case of nonsingleton Γ' and Γ'' is similar. This proves (ii) above. As for (i), consider $Tr^U(s_{U,W}(b' \otimes 1)Tr^{V \otimes V}((1 \otimes 1 \otimes s_{V,V})(1 \otimes s_{V,W} \otimes 1)(1 \otimes 1 \otimes s_{V,W})(\theta(\Pi') \otimes \theta(\Pi''))(1 \otimes 1 \otimes s_{W,V})(1 \otimes s_{W,V} \otimes 1))s_{W,U})$ and the fact that $Tr^{V \otimes V}((1 \otimes 1 \otimes s_{V,V})(1 \otimes s_{V,W} \otimes 1)(1 \otimes 1 \otimes s_{V,W})(\theta(\Pi') \otimes \theta(\Pi''))(1 \otimes 1 \otimes s_{W,V})(1 \otimes s_{W,V} \otimes 1)) = EX(\theta(\Pi), \sigma)$

  In this and all the following cases we assume that Δ is empty. The nonempty case can be reduced to the empty case using the associativity of cut. More explicitly, we would like to prove the result for $EX(\theta(\Pi), \sigma)$, where σ represents the cut formulas in Δ. We remove all the cuts in Π except the one occurring as the last rule by first pre- and post-composing $\theta(\Pi)$ with appropriate permutations (see the rightmost formula in Lemma 4.3) and then applying the execution formula. Then we apply this theorem and get back to $EX(\theta(\Pi), \sigma)$ using the associativity of cut.

— Suppose that Π is obtained from a proof Π' of Γ' by an application of an exchange rule. Let $\Gamma' = A_1, \cdots, A_i, A_{i+1}, \cdots, A_n$ and $\Gamma = A_1, \cdots, A_{i+1}, A_i, \cdots, A_n$. By inductive hypothesis $\theta(\Pi')$ is a datum of type $\theta(\Gamma')$, so for $a_i \in \theta(A_i^{\perp})$, we have that $(a_1 \otimes \cdots \otimes a_n) \perp \theta(\Pi')$ and so

$$\begin{array}{rcl} \tau^{-1}(a_1 \otimes \cdots \otimes a_n) & \perp & \theta(\Pi')\tau \text{ Isomorphism axiom} \\ (a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_n)\tau^{-1} & \perp & \theta(\Pi')\tau \\ (a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_n) & \perp & \tau^{-1}\theta(\Pi')\tau = \theta(\Pi). \end{array}$$

— Suppose that Π is obtained from the proofs Π' and Π'' of $\vdash \Gamma', A$ and $\vdash \Gamma'', B$ respectively by an application of a $\otimes$-rule. We let $\Gamma' = C'$ and $\Gamma'' = C''$ be single formulas, the general case is similar. Let $[\![A]\!] = V, [\![B]\!] = W, [\![C']\!] = U$, and $[\![C'']\!] = Y$. Recall that $\theta(\Pi) = \tau^{-1}(\theta(\Pi') \otimes \theta(\Pi''))\tau$.
We need to show that $\theta(\Pi)$ is a datum of type $\theta(C', C'', A \otimes B)$ which by Precise Tensor axiom and Lemma 4.5 is equivalent to showing that 1. for all $\gamma' \in \theta(C'^{\perp})$, $\theta(\Pi)\cdot\gamma' \in \theta(C'', A \otimes B)$ and 2. for all $\gamma'' \in \theta(C'')^{\perp}, \alpha \in \theta(A \otimes B)^{\perp}, \theta(\Pi)\hat{\cdot}(\gamma'' \otimes \alpha) \in \theta(C')$.
To show (1), we need to prove that for γ', γ'', and α as above, $(\theta(\Pi)\cdot\gamma')\cdot\gamma'' \perp \alpha$, and $(\theta(\Pi)\cdot\gamma')\hat{\cdot}\alpha \perp \gamma''$. We show these in order below:
By inductive hypotheses, $a = \theta(\Pi')\cdot\gamma'$ and $b = \theta(\Pi'')\cdot\gamma''$ exist, and $a \in \theta A$ and $b \in \theta B$.

$$\begin{array}{rcl} a \otimes b & = & a \otimes Tr^Y(s_{Y,W}(\gamma'' \otimes 1)\theta(\Pi'')s_{W,Y}) \\ & = & Tr^Y(a \otimes s_{Y,W}(\gamma'' \otimes 1)\theta(\Pi'')s_{W,Y}) \\ & = & Tr^Y(Tr^U(s_{U,V}(\gamma' \otimes 1)\theta(\Pi')s_{V,U}) \otimes s_{Y,W}(\gamma'' \otimes 1)\theta(\Pi'')s_{W,Y}) \\ & = & Tr^U(Tr^Y(1 \otimes s_{U,W} \otimes 1)(s_{U,V}(\gamma' \otimes 1)\theta(\Pi')s_{V,U} \otimes \\ & & s_{Y,W}(\gamma'' \otimes 1)\theta(\Pi'')s_{W,Y})(1 \otimes s_{W,U} \otimes 1)) \\ & = & Tr^{U \otimes Y}(\alpha^{-1}(\gamma' \otimes \gamma'' \otimes 1 \otimes 1)(1 \otimes s_{V,Y} \otimes 1)(\theta(\Pi') \otimes \theta(\Pi'')) \\ & & (1 \otimes s_{Y,V} \otimes 1)\alpha) \\ & & \text{where } \alpha = (1 \otimes s_{V,Y} \otimes 1)(s_{V,U} \otimes s_{W,Y})(1 \otimes s_{W,U} \otimes 1) \\ & = & (\theta(\Pi)\cdot\gamma')\cdot\gamma'' \end{array}$$

by definition of type for tensor, $(\theta(\Pi)\cdot\gamma')\cdot\gamma'' \in \theta(A \otimes B)$.
Similarly using naturality properties and trace axioms one can show that $(\theta(\Pi)\cdot\gamma')\hat{\cdot}\alpha = \theta(\Pi'')\hat{\cdot}(\alpha\cdot(\theta(\Pi')\cdot\gamma'))$ which is in $\theta(C'')$ using inductive hypotheses.
To show part (2), note that using trace axioms and naturality properties one can show that $\theta(\Pi)\hat{\cdot}(\gamma'' \otimes \alpha) = \theta(\Pi')\hat{\cdot}(\alpha\hat{\cdot}(\theta(\Pi'')\cdot\gamma''))$ and the latter is in $\theta(C')$ by inductive hypotheses.

— Suppose Π is obtained from a proof Π' of $\vdash \Gamma', A, B$ by an application of a par rule. Then $\theta(\Pi) = \theta(\Pi')$ and there is nothing to prove.

$\square$

Corollary 4.7 (Existence of Dynamics). Let Π be an **MLL** proof of a sequent $\vdash [\Delta], \Gamma$. Then $EX(\theta(\Pi), \sigma)$ exists.

Theorem 4.8 (EX is an invariant). *Let Π be an* **MLL** *proof of a sequent $\vdash [\Delta], \Gamma$. Then,*

(i) If Π reduces to Π' by any sequence of cut-eliminations, then $EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), \tau)$. So $EX(\theta(\Pi), \sigma)$ is an invariant of reduction.

(ii) In particular, if Π' is any cut-free proof obtained from Π by cut-elimination, then $EX(\theta(\Pi), \sigma) = \theta(\Pi')$.

Proof. It suffices to check the following key cases:

1 Suppose Π is of the form

$$\cfrac{\cfrac{\cfrac{\Pi'}{\vdash \Gamma}}{\vdash \Gamma, \bot} \quad \vdash \mathbf{1}}{\vdash [\bot, \mathbf{1}], \Gamma}\ cut$$

$$\begin{array}{rcl}
EX(\theta(\Pi), \sigma) & = & Tr^{I \otimes I}\left((1 \otimes \sigma)\theta(\Pi)\right) \\
& = & Tr^{I \otimes I}\left((1 \otimes s_{I,I})(\theta(\Pi') \otimes 1_I \otimes 1_I)\right) \\
& = & Tr^{I \otimes I}\left(\theta(\Pi') \otimes 1_I \otimes 1_I\right) \\
& = & Tr^{I}\left(\theta(\Pi') \otimes 1_I\right),\ \mathbb{C} \text{ is strict} \\
& = & \theta(\Pi'),\ \text{Vanishing I and strictness.}
\end{array}$$

2 Suppose Π' is a cut-free proof of $\vdash \Gamma, A$ and Π is obtained by applying the cut rule to Π' and the axiom $\vdash A^\perp, A$. Then

$EX(\theta(\Pi), \sigma)$

$= Tr\left((1 \otimes 1 \otimes s)\tau^{-1}(\theta(\Pi') \otimes s)\tau\right)$

$= Tr\left((1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)(\theta(\Pi') \otimes 1 \otimes 1)(1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)\right)$

$= (1 \otimes Tr(s))\theta(\Pi')(1 \otimes Tr(s)) = \theta(\Pi')$.

3 Suppose Π is of the form

$$\cfrac{\cfrac{\Pi'}{\vdash \Gamma', A} \quad \cfrac{\Pi''}{\vdash A^\perp, \Gamma''}}{\vdash [A, A^\perp], \Gamma', \Gamma''}\ cut$$

We assume that the last rules in Π' and Π'' are logical rules applied to A or $A^\perp$. Hence in the syntax the cut rule for A will be replaced by other cuts. We use σ to represent the cuts of Π and τ for those of Ξ, which is obtained from Π by one step reduction (cut-elimination). We shall ignore the exchange rule.

There is only one case: $A \equiv B \otimes C$ and hence $A^\perp \equiv B^\perp \parr C^\perp$. Hence Π' is obtained from Π'_1 of $\vdash \Gamma'_1, B$ and Π'_2 of $\vdash \Gamma'_2, C$ using the times rule. Also Π'' is obtained from Π''_1 of $\vdash B^\perp, C^\perp, \Gamma''$ using the par rule. Ξ is obtained by first applying the cut rule to Π'_1 and Π''_1 to get Π_0 and then by applying the cut rule again to Π_0 and Π'_2. We shall, without loss of generality, assume that Γ'_1, Γ'_2 and Γ'' consist of single formulas. The following derivation contains many compressed steps, all follow from trace axioms. Below, $\alpha, \alpha', \beta, \eta, \gamma$, and ρ are appropriate permutations.

$EX(\theta(\Pi), \sigma)$
$= Tr(\alpha'\alpha^{-1}(\theta(\Pi_1') \otimes \theta(\Pi_2') \otimes \theta(\Pi_1''))\alpha)$
$= Tr(\beta^{-1}(1 \otimes 1 \otimes \theta(\Pi_1''))(1 \otimes s \otimes 1 \otimes 1)(\theta(\Pi_1') \otimes \theta(\Pi_2') \otimes 1)(1 \otimes s \otimes 1 \otimes 1)\beta)$
$= Tr((1 \otimes 1 \otimes 1 \otimes s)\eta Tr((1 \otimes 1 \otimes 1 \otimes 1 \otimes s)(1 \otimes 1 \otimes 1 \otimes \theta(\Pi_2') \otimes 1)$
$(1 \otimes 1 \otimes 1 \otimes s \otimes 1)(1 \otimes 1 \otimes 1 \otimes 1 \otimes s)(\theta(\Pi_1') \otimes \theta(\Pi_1'') \otimes 1)$
$(1 \otimes 1 \otimes 1 \otimes s \otimes 1)(1 \otimes 1 \otimes 1 \otimes 1 \otimes s))\gamma)$
$= Tr(\rho^{-1}(\theta(\Pi_1') \otimes \theta(\Pi_1'') \otimes \theta(\Pi_2'))\rho)$
$= EX(\theta(\Xi), \tau)$

$\square$

5. MGoI for MELL in *-Categories

We now wish to extend the above multi-object GoI interpretation for multiplicative linear logic to the exponential structure of **MELL**. To this end, we add additional structure to a monoidal category, namely a contravariant involutive endofunctor $(-)^*$, to obtain a notion of a monoidal $*$-category. We then introduce GoI categories as $*$-categories with an orthogonality, an additional endofunctor T, and certain monoidal retractions suitable for treating the exponential structure, as in (AHS02).

In the following we are motivated by the definition of monoidal $*$-categories from (ABP99). Nevertheless, our definition is different from theirs, as we do not require a conjugation functor, and we demand stronger conditions on $(-)^*$. Categories such as these with further structure on the homsets (W^*-categories) were first introduced in (GLR85). The idea there was to generalize the notions and machinery of von Neumann algebras to a categorical setting. Later, similar categories (C^*-categories) were defined in (DopR89) and studied in depth. The motivation in this work was to present a new duality theory for compact groups, itself motivated by the work in the early seventies on superselection structure in quantum field theory. Both (GLR85) and (DopR89) are excellent sources for examples of the kinds of $*$-categories we define here.

Definition 5.1. A *symmetric monoidal $*$-category* $\mathbb{C}$ is a symmetric monoidal category with a strict symmetric monoidal functor $(\ )^* : \mathbb{C}^{op} \longrightarrow \mathbb{C}$ which is strictly involutive and the identity on objects. Note that this in particular implies that $(f \otimes g)^* = f^* \otimes g^*$, and $s^*_{A,B} = s_{B,A}$ where $s_{A,B}$ is the symmetry morphism.

We say that a morphism $f : A \longrightarrow A$ is *Hermitian* if $f^* = f$. A morphism $f : A \longrightarrow B$ is called a *partial isometry* if $f^*ff^* = f^*$ or equivalently, if $ff^*f = f$. A morphism $f : A \longrightarrow A$ is called a *partial symmetry* if it is Hermitian and a partial isometry. That is, if $f^* = f$ and $f^3 = f$. Note that there is no underlying Hilbert space structure on the homsets of $\mathbb{C}$; the terminology here is borrowed from operator algebras to account for the similar properties of such morphisms, which can be expressed in the more general setting of $*$-categories.

An obvious example is the category $\mathbf{Hilb}_\otimes$ of Hilbert spaces and bounded linear maps with tensor product of Hilbert spaces as the monoidal product. Given $f : H \longrightarrow K$,

$f^* : K \longrightarrow H$ is given by the adjoint of f, defined uniquely by $\langle f(x), y\rangle = \langle x, f^*(y)\rangle$. It is not hard to see that all the required properties are satisfied. Note that the category $\mathbf{Hilb}_{\oplus}$ of Hilbert spaces and bounded linear maps but with direct sum as the monoidal product is a $*$-category too, with the same definition for the $(\ )^*$ functor.

Another example is the category $\mathbf{Rel}_{\times}$ of sets and relations with the cartesian product of sets as the monoidal product. Given $f : X \longrightarrow Y$, $f^* = \overline{f}$ where $\overline{f}$ is the converse relation. Again, note that the category $\mathbf{Rel}_{\oplus}$ of sets and relations with monoidal product, the disjoint union (categorical biproduct) is a monoidal $*$-category too, with the same definition for the $(\,)^*$ functor.

Yet another example that shows up frequently in the context of GoI is the category $\mathbf{PInj}_{\uplus}$ of sets and partial injective maps, with disjoint union as the monoidal product. Given $f : X \longrightarrow Y$, $f^* = f^{-1}$.

Other examples include $\mathbf{Hilb}_{fd}$ of finite dimensional Hilbert spaces and bounded linear maps, $\mathbf{URep}(G)$, finite representations of a compact group G, etc. For more details, examples and the ways that such categories show up in logic and computer science, see (ABP99).

Definition 5.2. A *GoI category* is a triple $(\mathbb{C}, T, \perp)$ where $\mathbb{C}$ is a partially traced $*$-category, $T = (T, \psi, \psi_I) : \mathbb{C} \longrightarrow \mathbb{C}$ is a traced symmetric monoidal functor, that is if $f \in \mathbb{T}^U_{X,Y}$, then $\psi^{-1}_{Y,U} T(f)\psi_{X,U} \in \mathbb{T}^{TU}_{TX,TY}$ and $Tr^{TU}_{TX,TY}(\psi^{-1}_{Y,U} T(f)\psi_{X,U}) = T(Tr^U{}_{X,Y}(f))$. Here $\perp$ is an orthogonality relation on $\mathbb{C}$ as in the above. Furthermore, we require that

• The following monoidal natural retractions exist (here $\mathcal{K}_I$ denotes the constant I functor).

(i) $\mathcal{K}_I \lhd T\ (w, w^*)$, i.e. retractions, natural in X, $I \lhd TX\ (w_X, w^*_X)$.
(ii) $Id \lhd T\ (d, d^*)$ i.e. retractions, natural in X, $X \lhd TX\ (d_X, d^*_X)$.
(iii) $T^2 \lhd T\ (e, e^*)$ i.e. retractions, natural in X, $TTX \lhd TX\ (e_X, e^*_X)$.
(iv) $T \otimes T \lhd T\ (c, c^*)$ i.e. retractions, natural in X, $TX \otimes TX \lhd TX\ (c_X, c^*_X)$.

• The orthogonality relation must be *GoI compatible*, that is, it must satisfy the following additional axioms:

(c0) For all $f : V \longrightarrow U$ and $g : U \longrightarrow V$,

$$f \perp_{U,V} g \text{ implies } Tf \perp_{TU,TV} Tg.$$

(c1) For all $f : TV \longrightarrow TU$ and $g : U \longrightarrow V$,

$$f \perp_{TU,TV} Tg \text{ implies } e_U(Tf)e^*_V \perp_{TU,TV} Tg.$$

(c2) For all $f : V \longrightarrow U$, $g : U \longrightarrow V$,

$$f \perp_{U,V} g \text{ implies } d_U f d^*_V \perp_{TU,TV} Tg.$$

(c3) For all $f : U \longrightarrow U$ and $g : I \longrightarrow I$,

$$w_U g w^*_U \perp_{TU,TU} Tf.$$

(c4) For all $f : TV \otimes TV \longrightarrow TU \otimes TU$ and $g : U \longrightarrow V$,

$$f \perp_{TU\otimes TU, TV\otimes TV} Tg \otimes Tg \text{ implies } c_U f c^*_V \perp_{TU,TV} Tg.$$

• The functor T commutes with $(\ )^*$, that is $(T(f))^* = T(f^*)$. Moreover, $\psi^* = \psi^{-1}$ and $\psi_I^* = \psi_I^{-1}$.

Proposition 5.3. *Suppose $\mathbb{C}$ is a partially traced $*$-category that is in addition equipped with an endofunctor T and monoidal retractions as in Definition 5.2. Then, the orthogonality relation $\perp$ defined as in Example 3.4 is GoI compatible.*

Proof. We shall verify the compatibility axioms of Definition 5.2.

(c0) $Tr^{TV}(T(g)T(f)) = T(Tr^V(gf))$.

(c1) $Tr^{TV}(T(g)e_U T(f)e_V^*) = Tr^{TV}(e_V T^2(g)T(f)e_V^*) = Tr^{T^2V}(T^2(g)T(f)) = T(Tr^{TV}(T(g)f))$.

(c2) $Tr^{TV}(T(g)d_U f d_V^*) = Tr^{TV}(d_V g f d_V^*) = Tr^V(gf)$.

(c3) $Tr^{TU}(T(f)w_U g w_U^*) = Tr^{TU}(w_U g w_U^*) = Tr^I(g)$.

Recall that $\mathbb{T}_{I,I}^I = \mathbb{C}(I, I)$.

(c4) $Tr^{TV}(T(g)c_U f c_V^*) = Tr^{TV}(c_V(Tg \otimes Tg)f c_V^*) = Tr^{TV\otimes TV}((Tg \otimes Tg)f)$. □

GoI categories are the main mathematical structures in our semantic interpretation in the following section. Here are a few examples of GoI categories.

Examples 5.4.

(a) $(\mathbf{PInj}_{\uplus}, T, \perp)$

We define $f \perp g$ iff gf is nilpotent. It can be easily checked that this definition satisfies the axioms for an orthogonality relation. We also define $TA = \mathbb{N} \times A$ for any set A, with monoidal retractions as defined in (HS04a).

Let us verify the compatibility axioms:

— For $f : V \longrightarrow U$ and $g : U \longrightarrow V$, suppose gf is nilpotent, say $(gf)^n = 0$, then $(T(g)T(f))^n = T((gf)^n) = 0$ as T is a traced and thus an additive functor.

— For $f : TV \longrightarrow TU$ and $g : U \longrightarrow V$, suppose $T(g)f$ is nilpotent, say $(T(g)f)^n = 0$, then $(T(g)e_U T(f)e_V^*)^n = (e_V T^2(g)T(f)e_V^*)^n = (T^2(g)T(f))^n = T((T(g)f)^n) = 0$.

— For $f : V \longrightarrow U$ and $g : U \longrightarrow V$, suppose gf is nilpotent, say $(gf)^n = 0$, then $(T(g)d_U f d_V^*)^n = (d_V g f d_V^*)^n$ by naturality of d_U, but as $d_V^* d_V = 1_V$ we have $(d_V g f d_V^*)^n = d_V (gf)^n d_V^* = 0$.

— As $I = \emptyset$ and $w_I = 0$, we have that $T(f)w_U g w_U^*$ is nilpotent.

— For $f : TV \otimes TV \longrightarrow TU \otimes TU$ and $g : U \longrightarrow V$, suppose $(Tg \otimes Tg)f$ is nilpotent, say $((Tg \otimes Tg)f)^n = 0$, Then $(T(g)c_U f c_V^*)^n = (c_V(Tg \otimes Tg)f c_V^*)^n$, by naturality of c_V, but as $c_V^* c_V = 1_{TV\otimes TV}$ we have $(c_V(Tg \otimes Tg)f c_V^*)^n = c_V((Tg \otimes Tg)f)^n c_V^* = 0$.

Finally, for any $f : X \longrightarrow Y$, $(Tf)^* = T(f^*)$.

(b) $(\mathbf{Hilb}_{\oplus}, T, \perp)$, where **Hilb** is the category of Hilbert spaces and bounded linear maps. The monoidal product is the direct sum of Hilbert spaces. It turns out that $\mathbf{Hilb}_{\oplus}$ is a partially traced $*$-category: the partial trace is defined as in the case of finite dimensional vector spaces and the proof uses Lemmas 2.4 and 2.5 that remain valid for operator matrices. $T(H) = \ell^2 \otimes H$ where ℓ^2 is the space of square summable sequences. The monoidal retractions are as defined in (HS04a).

We define $f \perp g$ iff $(1-gf)$ is an invertible linear transformation. Compatibility follows from Proposition 5.3, because for $f : H \longrightarrow K$, $g : K \longrightarrow H$, $f \perp g$ iff $gf \in \mathbb{T}^H$.

Finally, as $\mathbf{Hilb}_\oplus$ is also a $*$-category with f^* the adjoint of f, we have that for any $f : H \longrightarrow K$, $(Tf)^* = T(f^*)$.

(c) $(\mathbf{Rel}_\oplus, T, \perp)$ is a GoI-category with the same definitions for T and $\perp$ as in the case of **PInj**. Note that disjoint union, denoted $\oplus$, is in fact the categorical biproduct in **Rel**.

Multiobject Geometry of Interaction (MGoI) was introduced in (HS05a) and was used to interpret **MLL** without units. It was later extended to exponentials in (Hagh06). The main idea in (HS05a) was to keep the types of the formulas that were defined by a denotational semantics map during the GoI interpretation. For the multiplicative case this also implied that, in contrast to the usual GoI, there was no need for a reflexive object U and this made the interpretation possible in categories like finite dimensional vector spaces. On the other hand, for **MELL** it is no longer possible to carry out the MGoI interpretation in finite dimensions, as for example we are forced to admit a retraction $TT \lhd T$ in the model category. Note that, although in this way reflexive objects reappear, they are not used to collapse types as in the untyped GoI interpretation using a single object U (HS04a; HS10).

5.1. *MGoI Interpretation of formulas*

Given a GoI category $(\mathbb{C}, T, \perp)$, with the definition of orthogonality as in Section 4 we extend the interpretation map $[\![-]\!]$ on the formulas of **MELL** as follows.

— $[\![!A]\!] = [\![?A]\!] = T[\![A]\!]$.

The MGoI-interpretation for formulas is extended as follows.

— $\theta(!A) = \{Ta \,|\, a \in \theta(A)\}^{\perp\perp}$

— $\theta(?A) = \{Ta \,|\, a \in \theta(A^\perp)\}^{\perp}$

The following are still valid facts: (i) for any formula A, $(\theta A)^\perp = \theta A^\perp$, (ii) $\theta(A) \subseteq End([\![A]\!])$, and (iii) $\theta(A)^{\perp\perp} = \theta(A)$.

5.2. *MGoI Interpretation of proofs*

In this section we define the MGoI interpretation for proofs of **MELL**. All references from now on refer to this MGoI interpretation unless stated otherwise.

As before, every **MELL** sequent will be of the form $\vdash [\Delta], \Gamma$ where Γ is a sequence of formulas and Δ is a sequence of cut formulas that have already been made in the proof of $\vdash \Gamma$. A proof Π of $\vdash [\Delta], \Gamma$ is represented by a morphism $\theta(\Pi) \in End(\otimes[\![\Gamma]\!] \otimes [\![\overline{\Delta}]\!])$. With $\Gamma = A_1, \cdots, A_n$, $\otimes[\![\Gamma]\!]$ stands for $[\![A_1]\!] \otimes \cdots \otimes [\![A_n]\!]$, and with $\Delta = B_1, B_1^\perp, \cdots B_m, B_m^\perp$, $[\![\overline{\Delta}]\!] = T^k([\![B_1]\!] \otimes \cdots \otimes [\![B_m^\perp]\!])$, for some non-negative integer k, with T^0 being the identity functor. Note that this is slightly different from the interpretation in Section 4 due to the presence of the functor T and its powers. However, the **MLL** case can be recovered easily by letting $k = 0$ and recalling that $T^0 = id$, the identity functor. As before, we drop the double brackets wherever there is no danger of confusion.

Definition 5.5 (The MGoI Interpretation). Let Π be a proof of $\vdash [\Delta], \Gamma$. We define *the MGoI interpretation of* Π, denoted by $\theta(\Pi)$, by induction on the length of the proof as follows. Note that some cases are identical to those in Section 4. However, we include all **MLL** cases here due to appearance of T^k in their interpretation in the presence of exponential connectives.

1 Π be the axiom $\vdash \mathbf{1}$, then $\theta(\Pi) = 1_I$.

2 Π is obtained using the $\perp$ *rule* applied to the proof Π' of $\vdash [\Delta], \Gamma'$. Then $\theta(\Pi) = \theta(\Pi') \otimes 1_I = \theta(\Pi')$.

3 Π is an *axiom* $\vdash A, A^\perp$, $\theta(\Pi) := s_{V,V}$ where $[\![A]\!] = [\![A^\perp]\!] = V$.

4 Π is obtained using the *cut* rule on Π' and Π'' that is,

$$\dfrac{\begin{array}{cc} \Pi' & \Pi'' \\ \vdots & \vdots \\ \vdash [\Delta'], \Gamma', A & \vdash [\Delta''], A^\perp, \Gamma'' \end{array}}{\vdash [\Delta', \Delta'', A, A^\perp], \Gamma', \Gamma''}\ \mathit{cut}$$

Define $\theta(\Pi) = \tau^{-1}(\theta(\Pi') \otimes \theta(\Pi''))\tau$, where τ is the permutation
$\Gamma' \otimes \Gamma'' \otimes \overline{\Delta'} \otimes \overline{\Delta''} \otimes A \otimes A^\perp \xrightarrow{\tau} \Gamma' \otimes A \otimes \overline{\Delta'} \otimes A^\perp \otimes \Gamma'' \otimes \overline{\Delta''}$.
(double brackets and $\otimes$ are dropped for the sake of readability).

5 Π is obtained using the *exchange* rule on the formulas A_i and A_{i+1} in Γ'. That is Π is of the form

$$\dfrac{\begin{array}{c} \Pi' \\ \vdots \\ \vdash [\Delta], \Gamma' \end{array}}{\vdash [\Delta], \Gamma}\ \mathit{exchange}$$

where $\Gamma' = \Gamma'_1, A_i, A_{i+1}, \Gamma'_2$ and $\Gamma = \Gamma'_1, A_{i+1}, A_i, \Gamma'_2$. Then, $\theta(\Pi)$ is obtained from $\theta(\Pi')$ by interchanging the rows i and $i + 1$. So, $\theta(\Pi) = \tau^{-1}\theta(\Pi')\tau$, where $\tau = 1_{\Gamma'_1} \otimes s \otimes 1_{\Gamma'_2 \otimes \overline{\Delta}}$.

6 Π is obtained using an application of the *par* rule, that is Π is of the form:

$$\dfrac{\begin{array}{c} \Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', A, B \end{array}}{\vdash [\Delta], \Gamma', A \mathbin{⅋} B}\ ⅋ \qquad . \text{ Then } \theta(\Pi) = \theta(\Pi').$$

7 Π is obtained using an application of the *times* rule, that is Π is of the form:

$$\dfrac{\begin{array}{cc} \Pi' & \Pi'' \\ \vdots & \vdots \\ \vdash [\Delta'], \Gamma', A & \vdash [\Delta''], \Gamma'', B \end{array}}{\vdash [\Delta', \Delta''], \Gamma', \Gamma'', A \otimes B}\ \otimes$$

Then $\theta(\Pi) = \tau^{-1}(\theta(\Pi') \otimes \theta(\Pi''))\tau$, where τ is the permutation
$\Gamma' \otimes \Gamma'' \otimes A \otimes B \otimes \overline{\Delta'} \otimes \overline{\Delta''} \xrightarrow{\tau} \Gamma' \otimes A \otimes \overline{\Delta'} \otimes \Gamma'' \otimes B \otimes \overline{\Delta''}$.

8 Π is obtained from Π' by an *of course* rule, that is Π has the form :

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], ?\Gamma', A\end{array}}{\vdash [\Delta], ?\Gamma', !A}\ \text{of course}$$

Then $\theta(\Pi) = (e_{\Gamma'} \otimes 1_{TA} \otimes 1_{\overline{\Delta}})\varphi^{-1}T(\theta(\Pi'))\varphi(e^*_{\Gamma'} \otimes 1_{TA} \otimes 1_{\overline{\Delta}})$, where $TT \lhd T\,(e, e^*)$, with $\Gamma' = A_1, \cdots, A_n$, $e_{\Gamma'} = e_{A_1} \otimes \cdots \otimes e_{A_n}$, similarly for e^*, and φ is the canonical isomorphism: The isomorphism $\varphi : T^2(\Gamma') \otimes TA \otimes T(\overline{\Delta}) \longrightarrow T(T(\Gamma') \otimes A \otimes \overline{\Delta})$ is defined using the isomorphism $\psi_{X,Y} : TX \otimes TY \longrightarrow T(X \otimes Y)$. With $\Gamma' = A_1, \cdots, A_n$, $T(\Gamma')$ is a shorthand for $TA_1 \otimes \cdots \otimes TA_n$, and $\overline{\Delta}$ is as before.

9 Π is obtained from Π' by the *dereliction* rule, that is, Π is of the form :

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', A\end{array}}{\vdash [\Delta], \Gamma', ?A}\ \text{dereliction}$$

Then $\theta(\Pi) = (1_{\Gamma'} \otimes d_A \otimes 1_{\overline{\Delta}})\theta(\Pi')(1_{\Gamma'} \otimes d^*_A \otimes 1_{\overline{\Delta}})$ where $Id \lhd T\,(d, d^*)$.

10 Π is obtained from Π' by the *weakening* rule, that is, Π is of the form:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma'\end{array}}{\vdash [\Delta], \Gamma', ?A}\ \text{weakening}$$

Then $\theta(\Pi) = (1_{\Gamma'} \otimes w_A \otimes 1_{\overline{\Delta}})\theta(\Pi')(1_{\Gamma'} \otimes w^*_A \otimes 1_{\overline{\Delta}})$, where $\mathcal{K}_I \lhd T\,(w, w^*)$.

11 Π is obtained from Π' by the *contraction* rule, that is, Π is of the form :

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', ?A, ?A\end{array}}{\vdash [\Delta], \Gamma', ?A}\ \text{contraction}$$

Then $\theta(\Pi) = (1_{\Gamma'} \otimes c_A \otimes 1_{\overline{\Delta}})\theta(\Pi')(1_{\Gamma'} \otimes c^*_A \otimes 1_{\overline{\Delta}})$, where $T \otimes T \lhd T\,(c, c^*)$.

Examples 5.6. *(a) Let* Π *be the following proof:*

$$\dfrac{\dfrac{\dfrac{\vdash A, A^\perp}{\vdash A \parr A^\perp}}{\vdash !(A \parr A^\perp)} \qquad \dfrac{\vdash B, B^\perp}{\vdash ?(A^\perp \otimes A), B, B^\perp}}{\vdash [!(A \parr A^\perp), ?(A^\perp \otimes A)], B, B^\perp}\ \mathit{cut}$$

Given $[\![A]\!] = V$ *and* $[\![B]\!] = W$*, we have*
$\theta(\Pi) = \tau^{-1}(T(s_{V,V}) \otimes ((w_{T(V\otimes V)} \otimes 1_{W\otimes W})s_{W,W}(w^*_{T(V\otimes V)} \otimes 1_{W\otimes W})))\tau$ *where* τ *is the permutation:* $[\![B]\!] \otimes [\![B^\perp]\!] \otimes [\![!(A \parr A^\perp)]\!] \otimes [\![?(A^\perp \otimes A)]\!] \xrightarrow{\tau} [\![!(A \parr A^\perp)]\!] \otimes [\![?(A^\perp \otimes A)]\!] \otimes [\![B]\!] \otimes [\![B^\perp]\!]$.

(b) Now consider the following proof

$$\dfrac{\dfrac{\dfrac{\vdash A, A^\perp}{\vdash A, ?A^\perp}}{\vdash !A, ?A^\perp} \quad \vdash B, B^\perp}{\vdash !A \otimes B, ?A^\perp \parr B^\perp}$$

Given $[\![A]\!] = V$ *and* $[\![B]\!] = W$*, we have* $\theta(\Pi) = (1 \otimes s \otimes 1)(1 \otimes e \otimes 1 \otimes 1)(\psi^{-1}T(h)\psi \otimes s)(1 \otimes e^* \otimes 1 \otimes 1)(1 \otimes s \otimes 1)$ *where* $h = (1 \otimes d_V)s(1 \otimes d^*_V)$*.*

Proposition 5.7. *Let* Π *be an* **MELL** *proof of* $\vdash [\Delta], \Gamma$*. Then* $\theta(\Pi)$ *is a partial symmetry.*

Proof. Proof follows by induction on the length of the proofs, noting that the functor $(\ )^*$ is a strict symmetric monoidal functor, $T(f)^* = T(f^*)$, $\psi^* = \psi^{-1}$, and $\psi^*_I = \psi^{-1}_I$. □

5.3. *Interpretation of cut-elimination*

As we saw previously, the mathematical model of cut-elimination is given by the *execution formula* as in (2), defined as follows:

$$EX(\theta(\Pi), \sigma) = Tr^{\otimes\Delta}_{\otimes\Gamma,\otimes\Gamma}((1 \otimes \sigma)\theta(\Pi))$$

where Π is a proof of the sequent $\vdash [\Delta], \Gamma$, and $\sigma = s^{\otimes m}$ models Δ, where $|\Delta| = 2m$. Note that $EX(\theta(\Pi), \sigma)$ is a morphism from $\otimes\Gamma \longrightarrow \otimes\Gamma$, when it exists. We shall prove below (see Theorem 5.12) that the execution formula always exists for any **MELL** proof Π.

Example 5.8. Let Π be the following proof from Example 5.6:

$$\dfrac{\dfrac{\dfrac{\vdash A, A^\perp}{\vdash A \parr A^\perp}}{\vdash !(A \parr A^\perp)} \quad \dfrac{\vdash B, B^\perp}{\vdash ?(A^\perp \otimes A), B, B^\perp}}{\vdash [!(A \parr A^\perp), ?(A^\perp \otimes A)], B, B^\perp}\ cut$$

Recall that given $[\![A]\!] = V$ and $[\![B]\!] = W$, we have
$\theta(\Pi) = \tau^{-1}(T(s_{V,V}) \otimes ((w_{T(V\otimes V)} \otimes 1_{W\otimes W})s_{W,W}(w^*_{T(V\otimes V)} \otimes 1_{W\otimes W}))\tau$ where τ is the appropriate permutation. In this case, $\sigma = s$, $(m = 1)$.

Then
$EX(\theta(\Pi), \sigma) = Tr((1 \otimes s_{T(V\otimes V),T(V\otimes V)})\theta(\Pi)) = s_{W,W}$, the GoI interpretation of $\vdash B, B^\perp$ to which Π reduces after cut-elimination.

5.4. *Soundness of the Interpretation*

In this section we discuss the soundness of the MGoI interpretation. We show that if a proof Π is reduced (via cut-elimination) to another proof Π', then $EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), \tau)$; that is, $EX(\theta(\Pi), \sigma)$ is an invariant of reduction. In particular, if Π' is cut-free (i.e. a normal form) we have $EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), 1_I) = \theta(\Pi')$.

The following lemma is valid for **MELL** too with the exact same proof as in Section 4.

Lemma 5.9 (Associativity of cut). *Let* Π *be a proof of* $\vdash [\Gamma, \Delta], \Lambda$ *and* σ *and* τ *be the morphisms representing the cut-formulas in* Γ *and* Δ *respectively. Then*

$$EX(\theta(\Pi), \sigma \otimes \tau) = EX(EX(\theta(\Pi), \tau), \sigma) = EX(EX((1 \otimes s)\theta(\Pi)(1 \otimes s), \sigma), \tau),$$

whenever all traces exist.

The definition of a datum is the same as in the case of **MLL** and is repeated for ease of reference. However, the definition of an algorithm needs to be slightly changed to take into account the presence of the functor T and its powers.

Definition 5.10. Let $\Gamma = A_1, \cdots, A_n$ and $V_i = [\![A_i]\!]$.

- A *datum of type* $\theta\Gamma$ is a morphism $M : \otimes_i V_i \longrightarrow \otimes_i V_i$ such that for any $a_i \in \theta(A_i^\perp)$, $\otimes_i a_i \perp M$ and

$$M \bullet a_1 := Tr^{V_1}(s^{-1}_{\otimes_{i \neq 1} V_i, V_1}(a_1 \otimes 1_{V_2} \otimes \cdots \otimes 1_{V_n}) M s_{\otimes_{i \neq 1} V_i, V_1})$$

  and

$$M \hat{\bullet} (a_2 \otimes \cdots \otimes a_n) := Tr^{V_2 \otimes \cdots \otimes V_n}((1 \otimes a_2 \otimes \cdots \otimes a_n) M)$$

  both exist.
- An *algorithm of type* $\theta\Gamma$ is a morphism $M : \otimes_i V_i \otimes [\![\overline{\Delta}]\!] \longrightarrow \otimes_i V_i \otimes [\![\overline{\Delta}]\!]$ for some $\Delta = B_1, B_2, \cdots, B_{2m}$ with m a nonnegative integer and $B_{i+1} = B_i^\perp$ for $i = 1, 3, \cdots, 2m-1$, and $\overline{\Delta}$ as before, such that with $\sigma : T^k(\otimes_{i=1}^{2m} [\![B_i]\!]) \longrightarrow T^k(\otimes_{i=1}^{2m} [\![B_i]\!])$ defined as $T^k(\otimes_{i=1,odd}^{2m-1} s_{[\![B_i]\!], [\![B_{i+1}]\!]})$, for some non-negative integer k, $EX(M, \sigma)$ exists and is a datum of type $\theta\Gamma$.

The following lemma is valid for **MELL** with the exact same proof as in Section 4.

Lemma 5.11. *Let* $\widetilde{\Gamma} = A_2, \cdots, A_n$ *and* $\Gamma = A_1, \widetilde{\Gamma}$. *Let* $V_i = [\![A_i]\!]$, *and* $M : \otimes_i V_i \longrightarrow \otimes_i V_i$, *for* $i = 1, \cdots, n$. *Then,* M *is a datum of type* $\theta(\Gamma)$ *iff for all* $a_i \in \theta(A_i^\perp)$, $M \bullet a_1$ *and* $M \hat{\bullet} (a_2 \otimes \cdots \otimes a_n)$ *(defined as above) exist and are in* $\theta(\widetilde{\Gamma})$, *and* $\theta(A_1)$, *respectively.*

Theorem 5.12 (Proofs as algorithms). *Let* Π *be an* **MELL** *proof of a sequent* $\vdash [\Delta], \Gamma$. *Then* $\theta(\Pi)$ *is an algorithm of type* $\theta\Gamma$.

Proof. The proof of the cases that do not involve exponentials was already given in Section 4 and we shall not repeat them here. The modification due to T and its powers does not change the proof for **MLL** proofs.

In all the cases below we shall assume that Δ is an empty sequence and that the context $\Gamma' = B$ is a single formula. The general case of $|\Gamma'| > 1$ follows similarly and the case of non-empty Δ follows from the associativity of cut. For a discussion regarding these assumptions in the case of uni-object GoI, see (HS04a). We shall use A for $[\![A]\!]$, for any formula A.

— Suppose Π is obtained from Π' using the *of course* rule. Recall that $\theta(\Pi) = (e_B \otimes 1)\varphi^{-1}(T\theta(\Pi'))\varphi(e_B^* \otimes 1)$. Let $b = T(\alpha)$ with $\alpha \in \theta(B^\perp)$, we then have,

$$\begin{aligned}
\theta(\Pi)\centerdot b &= [(e_B \otimes 1)\varphi^{-1}(T\theta(\Pi'))\varphi(e_B^* \otimes 1)]\centerdot b \\
&= Tr(s(b \otimes 1)(e_B \otimes 1)\varphi^{-1}(T\theta(\Pi'))\varphi(e_B^* \otimes 1)s) \\
&= Tr(s(e_B \otimes 1)(Tb \otimes 1)\varphi^{-1}(T\theta(\Pi'))\varphi(e_B^* \otimes 1)s), \text{naturality of } e \\
&= Tr(s(Tb \otimes 1)\varphi^{-1}(T\theta(\Pi'))\varphi)s), \text{dinaturality of trace and } e_B^* e_B = 1 \\
&= T(Tr(s(b \otimes 1)\theta(\Pi')s)) \\
&= T(\theta(\Pi')\centerdot b)
\end{aligned}$$

By inductive hypothesis $\theta(\Pi')\centerdot b \in \theta(A)$, and then we have that $\theta(\Pi)\centerdot b = T(\beta)$ for some $\beta \in \theta(A)$, thus $\theta(\Pi)\centerdot b \in \theta(!A)$. Note that we have proven the result for every $b = T(\alpha)$ with $\alpha \in \theta(B^\perp)$, and that $\{T\alpha \,|\, \alpha \in \theta(B^\perp)\}$ is a dense subset of $\theta(!B^\perp)$ with respect to biorthogonality, and hence we conclude the result for all $b \in \theta(!B^\perp)$, (see also (Gir89a), page 247.)

Now suppose $\alpha \in \theta(?A^\perp)$, we need to show that $\theta(\Pi)\hat{\centerdot}\alpha \in \theta(?B)$. Using (c0) and the properties of orthogonality relation on types, it suffices to show that $\theta(\Pi)\hat{\centerdot}Ta \in \theta(?B)$ for all $a \in \theta A^\perp$. By inductive hypothesis, $\theta(\Pi')\hat{\centerdot}a \in \theta(?B)$ and thus by (c1) $e_B(T(\theta(\Pi'))\hat{\centerdot}Ta)e_B^* \in \theta(?B)$. The latter is nothing but $\theta(\Pi)\hat{\centerdot}Ta$ using the trace axioms.

Note that this is where infinity sneaks in: the equations and inductive hypothesis force us to have $e_B^* e_B = 1$ and this cannot be realized in finite dimensions.

— Suppose Π is obtained from Π' using the *dereliction* rule. Recall that $\theta(\Pi) = (1 \otimes d_A)\theta(\Pi')(1 \otimes d_A^*)$. Let $b \in \theta(B^\perp)$ and $a \in \theta(A^\perp)$, by induction hypothesis $\theta(\Pi')\centerdot b \in \theta(A)$ and thus $\theta(\Pi')\centerdot b \perp a$. On the other hand, $\theta(\Pi)\centerdot b = [(1 \otimes d_A)\theta(\Pi')(1 \otimes d_A^*)]\centerdot b = d_A(\theta(\Pi')\centerdot b)d_A^*$ using trace axioms. Now, by axiom (c2) of compatibility we have $d_A(\theta(\Pi')\centerdot b)d_A^* \perp Ta$, which shows that $\theta(\Pi)\centerdot b \in \theta(?A)$.

For $a \in \theta A^\perp$, $Ta \in \theta(!A^\perp) = \theta(?A)^\perp$. We have $\theta(\Pi)\hat{\centerdot}Ta = \theta(\Pi')\hat{\centerdot}a$ using trace axioms and $d_A^* d_A = 1_A$. Also by inductive hypothesis $\theta(\Pi')\hat{\centerdot}a \in \theta(B)$. We conclude by noting that $\{Ta \,|\, a \in \theta A^\perp\}$ is dense in $\theta(!A^\perp)$.

— Suppose Π is obtained from Π' using the *weakening* rule. Recall that $\theta(\Pi) = (1 \otimes w_A)\theta(\Pi')(1 \otimes w_A^*)$. Let $b \in \theta(B^\perp)$ and note that $\theta(\Pi)\centerdot b = [(1 \otimes w_A)\theta(\Pi')(1 \otimes w_A^*)]\centerdot b = w_A(\theta(\Pi')\centerdot b)w_A^*$ using trace axioms. Note, further that $\theta(\Pi')\centerdot b : I \longrightarrow I$. Now, by axiom (c3) of compatibility, $w_A(\theta(\Pi')\centerdot b)w_A^* \perp Ta$ for all $a : A \longrightarrow A$, thus in particular for those $a \in \theta(A^\perp)$. Thus, $\theta(\Pi)\centerdot b \in \theta(?A)$.

For $a \in \theta A^\perp$, $Ta \in \theta(!A^\perp) = \theta(?A)^\perp$. We have $\theta(\Pi)\hat{\centerdot}Ta = \theta(\Pi')$ using trace axioms and $w_A^* w_A = 1_I$. Also by inductive hypothesis $\theta(\Pi') \in \theta(B)$. We conclude by noting that $\{Ta \,|\, a \in \theta A^\perp\}$ is dense in $\theta(!A^\perp)$.

— Suppose Π is obtained from Π' using the *contraction* rule. Recall that $\theta(\Pi) =$

$(1 \otimes c_A)\theta(\Pi')(1 \otimes c_A^*)$. Let $b \in \theta(B^\perp)$ and note that by inductive hypothesis $\theta(\Pi')\hat{\cdot} b \in \theta(?A, ?A)$. Let $a \in \theta(A^\perp)$. Then, $Ta \in \theta(!A^\perp)$ and $\theta(\Pi')\hat{\cdot} b \perp Ta \otimes Ta$. On the other hand, $\theta(\Pi)\hat{\cdot} b = c_A(\theta(\Pi')\hat{\cdot} b)c_A^*$ using trace axioms. Now, by axiom (c4) of compatibility, $c_A(\theta(\Pi')\hat{\cdot} b)c_A^* \perp Ta$ and so $\theta(\Pi)\hat{\cdot} b \in \theta(?A)$.
For $a \in \theta A^\perp$, $Ta \in \theta(!A^\perp) = \theta(?A)^\perp$. We have $\theta(\Pi)\hat{\cdot} Ta = \theta(\Pi')\hat{\cdot}(Ta \otimes Ta)$ using trace axioms and $c_A^* c_A = 1_{TA \otimes TA}$. Also by inductive hypothesis $\theta(\Pi')\hat{\cdot}(Ta \otimes Ta) \in \theta(B)$. We conclude by noting that $\{Ta \mid a \in \theta A^\perp\}$ is dense in $\theta(!A^\perp)$.

□

Corollary 5.13 (Existence of Dynamics). Let Π be an **MELL** proof of a sequent $\vdash [\Delta], \Gamma$. Then $EX(\theta(\Pi), \sigma)$ exists.

Theorem 5.14 (EX is an invariant). *Let Π be an* **MELL** *proof of a sequent $\vdash [\Delta], \Gamma$ such that $?A$ does not occur in Γ for any formula A. Then,*

— *If Π reduces to Π' by any sequence of cut-elimination steps, then $EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), \tau)$. So $EX(\theta(\Pi), \sigma)$ is an invariant of reduction.*
— *In particular, if Π' is any cut-free proof obtained from Π by cut-elimination, then $EX(\theta(\Pi), \sigma) = EX(\theta(\Pi'), 1_I) = \theta(\Pi')$.*

Proof. We suppose Π is of the form

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash \Gamma', A\end{array} \quad \begin{array}{c}\Pi'' \\ \vdots \\ \vdash A^\perp, \Gamma''\end{array}}{\vdash [A, A^\perp], \Gamma', \Gamma''}\ cut$$

We further assume that the last rules in Π' and Π'' are logical rules applied to A or $A^\perp$. Hence in the syntax the cut rule for A will be replaced by other cuts. We use σ to represent the cuts of Π and τ for those of Ξ, which is obtained from Π by one step reduction (cut-elimination). We shall ignore the exchange rule.

It suffices to check the following key cases: note that the key cases of cut versus an axiom and cut between $A \otimes B$ and its dual were covered in the proof of Theorem 4.8 and will not be repeated here.

1 Suppose that $A \equiv\ !B$ and so $A^\perp \equiv\ ?B^\perp$, and Π is given by the following proof.

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1' \\ \vdots \\ \vdash ?\Gamma_1', B\end{array}}{\vdash ?\Gamma_1', !B}\ (!) \qquad \dfrac{\begin{array}{c}\Pi_1'' \\ \vdots \\ \vdash ?B^\perp, ?B^\perp, \Gamma''\end{array}}{\vdash ?B^\perp, \Gamma''}\ (contraction)}{\vdash [!B, ?B^\perp], ?\Gamma_1', \Gamma''}\ (cut)$$

The proof Ξ is obtained by first cutting Π' against Π_1'' to get a proof Π_0 of $\vdash [!B, ?B^\perp], ?B^\perp, ?\Gamma_1', \Gamma''$, next cutting Π' against Π_0 to get Ξ_0 ending with $\vdash [!B, ?B^\perp, !B, ?B^\perp], ?\Gamma_1', ?\Gamma_1', \Gamma''$ and finally doing a sequence of contractions on formulas in $?\Gamma_1'$.
Recall that by assumption, Γ_1' has to be empty. In fact the following equations (in all cases) are not valid otherwise. The problem is that in the presence of a non-empty context, we cannot use the naturality of c, w, e or d which is necessary for the proof. Without loss of generality, in all cases below, we let $\Gamma'' = C$ be a single formula with $[\![C]\!] = C$. Also, let $[\![!B]\!] = [\![?B^\perp]\!] = T(U)$. We shall assume we are working with strict monoidal categories, therefore $A \otimes I = A$ and $f \otimes 1_I = f$ for any object A, and any morphism f.

$EX(\theta(\Pi), \sigma) =$
$= Tr^{TU \otimes TU} [(1 \otimes s)\rho^*(T(\theta\Pi_1') \otimes (c_U \otimes 1)\theta\Pi_1''(c_U^* \otimes 1))\rho]$
$= Tr^{TU \otimes TU \otimes TU}[(1 \otimes \beta)\alpha^*(c_U(T(\theta\Pi_1') \otimes T(\theta\Pi_1')) \otimes \theta\Pi_1''(c_U^* \otimes 1))\alpha]$
dinaturality of trace and naturality of c
$= Tr^{TU^{\otimes 4}} [(1 \otimes \beta')\delta^*(T(\theta\Pi_1') \otimes T(\theta\Pi_1') \otimes \theta\Pi_1'')\delta]$
dinaturality of trace, $c^*c = 1$
$= Tr^{TU^{\otimes 4}} [(1 \otimes s \otimes s)\gamma^*(s(T(\theta(\Pi')) \otimes T(\theta(\Pi')))s \otimes \theta\Pi_1'')\gamma]$
$= EX(\theta(\Xi), s \otimes s)$

Here all lower case Greek letters stand for appropriate permutations. E.g., $\beta = (1 \otimes s)(s \otimes 1)$, etc.

2 Suppose Π is given as:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1' \\ \vdots \\ \vdash B\end{matrix}}{\vdash !B}\,(!) \qquad \dfrac{\begin{matrix}\Pi_1'' \\ \vdots \\ \vdash B^\perp, \Gamma''\end{matrix}}{\vdash ?B^\perp, \Gamma''}\,(dereliction)}{\vdash [!B, ?B^\perp], \Gamma''}\,(cut)$$

and Ξ is obtained as

$$\dfrac{\begin{matrix}\Pi_1' \\ \vdots \\ \vdash B\end{matrix} \qquad \begin{matrix}\Pi_1'' \\ \vdots \\ \vdash B^\perp, \Gamma''\end{matrix}}{\vdash [B, B^\perp], \Gamma''}\,(cut)$$

We have

$EX(\theta(\Pi),\sigma) =$

$= Tr^{TU\otimes TU}\,[(1\otimes s)\rho^*(T(\theta\Pi_1')\otimes(d_U\otimes 1)\theta\Pi_1''(d_U^*\otimes 1))\rho]$

$= Tr^{TU\otimes TU}\,[(1\otimes s)\rho^*(T(\theta\Pi_1')d_U\otimes\theta\Pi_1''(d_U^*\otimes 1))\rho]$

dinaturality of trace

$= Tr^{TU\otimes TU}\,[(1\otimes s)\rho^*(d_U\theta\Pi_1'\otimes\theta\Pi_1''(d_U^*\otimes 1))\rho]$ naturality of d

$= Tr^{U\otimes U}\,[(1\otimes s)\rho^*(\theta\Pi_1'\otimes\theta\Pi_1'')\rho]$ dinaturality of trace and $d'd=1$

$= EX(\theta(\Xi),s)$

Here $\rho=(1\otimes s)(s\otimes 1)$.

3 Suppose Π is given as:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1'\\ \vdots\\ \vdash B\end{matrix}}{\vdash !B}\,(!) \qquad \dfrac{\begin{matrix}\Pi_1''\\ \vdots\\ \vdash \Gamma''\end{matrix}}{\vdash ?B^\perp,\Gamma''}\,(weakening)}{\vdash [!B,?B^\perp],\Gamma''}\,(cut)$$

and $\Xi=\Pi_1''$.

We have

$EX(\theta(\Pi),\sigma) =$

$= Tr^{TU\otimes TU}\,[(1\otimes s)\rho^*(T(\theta\Pi_1')\otimes(w_U\otimes 1)(1_I\otimes\theta\Pi_1'')(w_U^*\otimes 1))\rho]$

$= Tr^{TU\otimes TU}\,[(1\otimes s)\rho^*(T(\theta\Pi_1')w_U\otimes(1_I\otimes\theta\Pi_1'')(w_U^*\otimes 1))\rho]$

dinaturality of trace

$= Tr^{TU\otimes TU}\,[(1\otimes s)\rho^*(w_U\otimes(1_I\otimes\theta\Pi_1'')(w_U^*\otimes 1))\rho]$ naturality of w

$= Tr^{I}\,[s_{I,C}(1_I\otimes\theta\Pi_1'')s_{C,I}]$ dinaturality of trace and $w^*w=1$

$= \theta\Pi_1''$ vanishing I and coherence theorem.

4 The last case is where Π is given as in below. For simplicity we shall ignore the monoidal functor isomorphism φ, its inclusion does not make any changes to the correctness of the proof below.

$$\dfrac{\dfrac{\begin{matrix}\Pi_1'\\ \vdots\\ \vdash B\end{matrix}}{\vdash !B}\,(!) \qquad \dfrac{\begin{matrix}\Pi_1''\\ \vdots\\ \vdash ?B^\perp, C\end{matrix}}{\vdash ?B^\perp,!C}\,(!)}{\vdash [!B,?B^\perp],!C}\,(cut)$$

and Ξ is

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1' \\ \vdots \\ \vdash B\end{array}}{\vdash !B}(!) \qquad \begin{array}{c}\Pi_1'' \\ \vdots \\ \vdash ?B^\perp, C\end{array}}{\vdash [!B, ?B^\perp], C}}{\vdash [!B, ?B^\perp], !C}(!)$$

We have

$$\begin{aligned}
&EX(\llbracket \Pi \rrbracket, \sigma) = \\
&= Tr^{TU\otimes TU}[(1\otimes s)\rho^*(T(\llbracket \Pi_1' \rrbracket) \otimes (e_U \otimes 1)T(\llbracket \Pi_1'' \rrbracket)(e_U^* \otimes 1))\rho] \\
&= Tr^{TU\otimes TU}[(1\otimes s)\rho^*(T\llbracket \Pi_1' \rrbracket e_U \otimes T(\llbracket \Pi_1'' \rrbracket)(e_U^* \otimes 1))\rho] \text{ dinaturality of trace} \\
&= Tr^{T^2U\otimes TU}[(1\otimes s)\rho^*(e_U T^2(\llbracket \Pi_1' \rrbracket) \otimes T(\llbracket \Pi_1'' \rrbracket)(e_U^* \otimes 1))\rho] \text{ naturality of } e \\
&= Tr^{T^2U\otimes TU}[(1\otimes s)\rho^*(T^2(\llbracket \Pi_1' \rrbracket) \otimes T(\llbracket \Pi_1'' \rrbracket))\rho] \text{ dinaturality of trace and } e^*e = 1 \\
&= EX(\Xi, s).
\end{aligned}$$

Here, $\rho = (1\otimes s)(s\otimes 1)$; of course the type of s depends on the particular permutation.

□

6. MGoI and Denotational Semantics

6.1. *Comparison with Int categories*

In the original paper on traced monoidal categories (JSV96), Joyal, Street, and Verity say

This notion of trace also appears in the geometry of interaction as the 'execution formula'.

The authors go on to construct a notion of free compact closure of a traced monoidal category, $Int(\mathbb{C})$, in which composition is given by the trace. An equivalent notion (in the symmetric case) was introduced independently by Abramsky, in his construction $\mathcal{G}(\mathbb{C})$.

There have been several works interpreting various logics in Int-like "GoI" categories, beginning with the Abramsky-Jagadeesan paper (AJ94) as well as unpublished lectures of Hyland, to more recent (and interesting) interpretations of classical logic by Fuhrman and Pym (FP04). In all cases, cut is interpreted as composition in the associated Int category.

We are often asked to compare our view of Girard's GoI (in this article, MGoI) with the above-style denotational interpretations into some $\mathcal{G}(\mathbb{C})$. We show below that: (i) when it makes sense to ask the question, the two semantics are different on formulas and on proofs with cuts, but (ii) they do agree on cut-free proofs (even though the interpretations of the associated formulas are different). Thus, in a rough sense, the denotational (categorical) interpretation of closed (cut-free) proof-terms is the same as the GoI interpretation.

There are, however, some caveats:

(i) In MGoI we work with partial traces. At the moment, there is no suitable published account of compact categories analogous to forming $\mathcal{G}(\mathbb{C})$ or $Int(\mathbb{C})$ for our notion of partial trace. Indeed, this is connected with the notion of partial category. Work in this direction, based on Freyd's paracategories (see Section 6.2 below) will appear in (Mal10). So to make the comparison meaningful, let us suppose we are working in a *totally* traced category $\mathbb{C}$.

(ii) MGoI (and untyped GoI) interpret formulas as types, which of course is not the same as a direct denotational interpretation of formulas as objects in some $\mathcal{G}(\mathbb{C})$.

Nevertheless, let us see to what extent we can compare the two kinds of GoI interpretations in a category $\mathcal{G}(\mathbb{C})$. We shall roughly follow Abramsky and Jagadeesan (AJ94).

Interpreting formulas:

Formulas will be interpreted as *diagonal* objects in $\mathcal{G}(\mathbb{C})$, that is as pairs of objects in $\mathbb{C}$ of the form (V, V). We shall indicate the denotational semantics in $\mathcal{G}(\mathbb{C})$ as $[\![-]\!]_D$ to distinguish it from $[\![-]\!]$, which is used in the MGoI interpretation of formulas given earlier. We assign arbitrary objects to atomic formulas: $[\![p]\!]_D = (V, V)$, where V is an object in $\mathbb{C}$. Relative to such an assignment, we define $[\![A^\perp]\!]_D = [\![A]\!]_D^\perp$. Also, we define $[\![A \otimes B]\!]_D = [\![A \parr B]\!]_D = [\![A]\!]_D \otimes [\![B]\!]_D$. Note that $[\![A^\perp]\!]_D = [\![A]\!]_D$ for every formula A; indeed suppose $[\![A]\!]_D = (V, V)$, then $[\![A^\perp]\!]_D = (V, V)^\perp = (V, V)$.

Remark 6.1. Notice that if we choose $[\![p]\!]_D = ([\![p]\!], [\![p]\!])$ for atomic formulas p, then $[\![A]\!]_D = ([\![A]\!], [\![A]\!])$ for any formula A, (see Proposition 6.2 below.) This is as far as the resemblance goes, however, since the MGoI interpretation of formulas is $\theta(A)$, not $[\![A]\!]$.

Interpreting proofs: In the denotational semantics into the category $\mathcal{G}(\mathbb{C})$ we shall ignore the information about cuts collected in Δ; thus let Π be a proof of $\vdash [\Delta], \Gamma$, with $\Gamma = A_1, \cdots, A_n$. Then $[\![\Pi]\!]_D : I \longrightarrow \otimes[\![A_i]\!]_D$ in $\mathcal{G}(\mathbb{C})$; in other words, $[\![\Pi]\!]_D : V_1 \otimes V_2 \otimes \cdots \otimes V_n \longrightarrow V_1 \otimes V_2 \otimes \cdots \otimes V_n$ in $\mathbb{C}$, where $[\![A_i]\!]_D = (V_i, V_i)$. For a morphism $h : A \otimes B \longrightarrow C$ in $\mathcal{G}(\mathbb{C})$, let $\Lambda(f) : A \longrightarrow B^* \otimes C$ denote its transpose.

— *Axiom* $\vdash A, A^\perp$. Then $[\![\Pi]\!]_D = \Lambda(1_{[\![A]\!]_D})$. Let $[\![A]\!]_D = (V, V)$. Then, when translated into $\mathbb{C}$, we have $[\![\Pi]\!]_D = s_{V,V}$.

— *Cut*: Suppose Π is obtained by applying the cut rule on $A, A^\perp$ to proofs Π' and Π''. Suppose also that $[\![\Pi']\!]_D = f : I \longrightarrow \otimes[\![\Gamma']\!]_D \otimes [\![A]\!]_D$ and $[\![\Pi'']\!]_D = g : I \longrightarrow [\![A^\perp]\!]_D \otimes [\![\Gamma'']\!]_D$. Then

$$[\![\Pi]\!]_D : I \longrightarrow [\![\Gamma']\!]_D \otimes [\![\Gamma'']\!]_D = \Lambda(\Lambda^{-1}(g)\Lambda^{-1}(f)) \quad .$$

Let $\Gamma' = A'_1, \cdots, A'_n$ and $\Gamma'' = A''_1, \cdots, A''_m$, $[\![A'_i]\!]_D = (V'_i, V'_i)$ and $[\![A''_i]\!]_D = (V''_i, V''_i)$, and $[\![A]\!]_D = (V, V)$. When translated into $\mathbb{C}$ we get:
$\Lambda^{-1}(f) : V'_1 \otimes \cdots \otimes V'_n \otimes V \longrightarrow V'_1 \otimes \cdots \otimes V'_n \otimes V$ and
$\Lambda^{-1}(g) : V \otimes V''_1 \otimes \cdots \otimes V''_m \longrightarrow V \otimes V''_1 \otimes \cdots \otimes V''_m$. So

$$[\![\Pi]\!]_D = Tr^{V \otimes V}((1_{\Gamma'} \otimes 1_{\Gamma''} \otimes s_{V,V})\rho^{-1}(\Lambda^{-1}(f) \otimes \Lambda^{-1}(g))\rho)$$

where ρ is the permutation, $\rho : [\![\Gamma']\!]_D \otimes [\![\Gamma'']\!]_D \otimes V \otimes V \longrightarrow [\![\Gamma']\!]_D \otimes V \otimes V \otimes [\![\Gamma'']\!]_D$

— *Exchange*: Given $[\![\Pi']\!]_D : I \longrightarrow \otimes[\![\Gamma']\!]_D$ with $\Gamma' = \Gamma'_1, A_i, A_{i+1}, \Gamma'_2$, $\Gamma'_1 = A_1, \cdots, A_{i-1}$, $\Gamma'_2 = A_{i+2}, \cdots, A_n$, and $[\![A_i]\!]_D = (V_i, V_i)$ for all i. We define

$[\![\Pi]\!]_D = (1_{\Gamma_1'} \otimes s_{V_i,V_{i+1}} \otimes 1_{\Gamma_2'})[\![\Pi']\!]_D$. When we translate this into $\mathbb{C}$ we get: given $[\![\Pi']\!]_D : V_1 \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes \cdots \otimes V_n \longrightarrow V_1 \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes \cdots \otimes V_n$, $[\![\Pi]\!]_D = \rho^{-1}[\![\Pi']\!]_D\rho$ where $\rho = 1_{\Gamma_1'} \otimes s_{V_{i+1},V_i} \otimes 1_{\Gamma_2'}$.

— *Tensor*: Given $[\![\Pi']\!]_D = f : I \longrightarrow \otimes[\![\Gamma']\!]_D \otimes [\![A]\!]_D$ and $[\![\Pi'']\!]_D = g : I \longrightarrow \otimes[\![\Gamma'']\!]_D \otimes [\![B]\!]_D$, then we define

$$[\![\Pi]\!]_D : I \longrightarrow \otimes[\![\Gamma']\!]_D \otimes [\![\Gamma'']\!]_D \otimes [\![A]\!]_D \otimes [\![B]\!]_D = \rho(f \otimes g)$$

where ρ is the permutation $\rho : \otimes[\![\Gamma']\!]_D \otimes [\![\Gamma'']\!]_D \otimes [\![A]\!]_D \otimes [\![B]\!]_D \longrightarrow [\![\Gamma']\!]_D \otimes [\![A]\!]_D \otimes [\![\Gamma'']\!]_D \otimes [\![B]\!]_D$. Let $\Gamma' = A_1', \cdots, A_n'$ and $\Gamma'' = A_1'', \cdots, A_m''$, $[\![A_i']\!]_D = (V_i', V_i')$ and $[\![A_j'']\!]_D = (V_j'', V_j'')$, $[\![A]\!]_D = (V, V)$, and $[\![B]\!]_D = (W, W)$. When translated into $\mathbb{C}$ we get $f : V_1' \otimes \cdots \otimes V_n' \otimes V \longrightarrow V_1' \otimes \cdots \otimes V_n' \otimes V$ and $g : V_1'' \otimes \cdots \otimes V_m'' \otimes W \longrightarrow V_1'' \otimes \cdots \otimes V_m'' \otimes W$. Then

$$[\![\Pi]\!]_D = \rho^{-1}(f \otimes g)\rho$$

where ρ is the permutation, $\rho = 1_{\Gamma'} \otimes s_{\Gamma'',V} \otimes 1_W$

— *Par*: Given $[\![\Pi']\!]_D : I \longrightarrow \otimes[\![\Gamma']\!]_D \otimes [\![A]\!]_D \otimes [\![B]\!]_D$, define $[\![\Pi]\!]_D = [\![\Pi']\!]_D$.

Proposition 6.2. *Let* Π *be an* **MLL** *proof of* $\vdash [\Delta], \Gamma$ *and* σ *model* Δ. *Suppose* $[\![p]\!]_D = ([\![p]\!], [\![p]\!])$ *for all atomic formulas p. Then,*

1. $[\![A]\!]_D = ([\![A]\!], [\![A]\!])$, *for any formula A.*
2. $EX(\theta(\Pi), \sigma) = [\![\Pi]\!]_D$. *In particular, if* Π *is cut-free (i.e.,* $|\Delta| = 0, \sigma = 1_I$ *), then* $\theta(\Pi) = [\![\Pi]\!]_D$.

Proof.
(1). By induction on formulas. (2). By induction on proofs.

— Axiom: Let Π be $\vdash A, A^\perp$ and $[\![A]\!]_D = (V, V)$, then from above,

$$[\![\Pi]\!]_D = s_{V,V} = \theta(\Pi).$$

— Cut: Let Π be the cut of Π' and Π'' on cut formulas $A, A^\perp$, $[\![\Pi']\!]_D = f$ and $[\![\Pi'']\!]_D = g$, $[\![A]\!]_D = (V, V)$. We know that $\theta(\Pi) = Tr^{V \otimes V}((1_{\Gamma'} \otimes 1_{\Gamma''} \otimes s_{V,V})\rho^{-1}(\Lambda^{-1}(f) \otimes \Lambda^{-1}(g))\rho)$, but for f and g as above $\Lambda^{-1}(f) = f$ and $\Lambda^{-1}(g) = g$ as morphisms in $\mathbb{C}$. Hence we get that

$$\theta(\Pi) = Tr^{V \otimes V}((1_{\Gamma'} \otimes 1_{\Gamma''} \otimes s_{V,V})\rho^{-1}(f \otimes g)\rho).$$

Assume that the cuts Δ' and Δ'' are represented by σ' and σ'' respectively. Then we have:

$$\begin{array}{rcl} EX(\theta(\Pi), \sigma' \otimes \sigma'' \otimes s_{V,V}) & = & EX(EX(\theta(\Pi), s_{V,V}), \sigma' \otimes \sigma'') \text{ assoc. of cut} \\ & = & EX(EX(\tau^{-1}(\theta(\Pi') \otimes \theta(\Pi''))\tau, s_{V,V}), \sigma' \otimes \sigma'') \\ & & \text{(by MGoI interpretation)} \\ & = & EX(\rho^{-1}(EX(\theta(\Pi'), \sigma') \otimes EX(\theta(\Pi''), \sigma''))\rho, s_{V,V}) \\ & & \text{(Naturality and dinaturality)} \\ & = & EX(\rho^{-1}(f \otimes g)\rho, s_{V,V}) \\ & = & [\![\Pi]\!]_D \end{array}$$

— *Exchange*: Let Π be obtained from Π' by exchanging A_i, A_{i+1} and σ represent Δ.

We know that $[\![\Pi]\!]_D = \rho^{-1}[\![\Pi']\!]_D\rho$ where $\rho = 1_{\Gamma_1'} \otimes s_{V_{i+1},V_i} \otimes 1_{\Gamma_2'}$, where $[\![A_i]\!]_D = (V_i, V_i)$, for all i. Moreover,

$$\begin{array}{rcl} EX(\theta(\Pi),\sigma) & = & EX(\tau^{-1}\theta(\Pi')\tau,\sigma),\ \text{MGoI int.} \\ & = & \rho^{-1}EX(\theta(\Pi'),\sigma)\rho,\ \text{naturality of trace} \\ & = & \rho^{-1}[\![\Pi']\!]_D\rho,\ \text{inductive hyp.} \\ & = & [\![\Pi]\!]_D. \end{array}$$

— Tensor: Suppose Π is obtained from Π' and Π'' and that Δ' and Δ'' are represented by σ' and σ'' respectively.

$$\begin{array}{rcl} EX(\theta(\Pi),\sigma'\otimes\sigma'') & = & EX(\tau^{-1}(\theta(\Pi')\otimes\theta(\Pi''))\tau,\sigma'\otimes\sigma'') \\ & = & \rho^{-1}(EX(\theta(\Pi'),\sigma')\otimes EX(\theta(\Pi''),\sigma''))\rho \\ & = & \rho^{-1}([\![\Pi']\!]_D\otimes[\![\Pi'']\!]_D)\rho \\ & = & [\![\Pi]\!]_D. \end{array}$$

— Par: As $EX(\theta(\Pi'),\sigma) = [\![\Pi']\!]_D$ and $\theta(\Pi) = [\![\Pi']\!]_D$, the result follows.

$\square$

Example 6.3. Observe that even at the simple level of two axioms joined by a cut, the MGoI and denotational interpretations differ. For let Π be the proof obtained by applying the cut rule to two axioms $\vdash A, A^\perp$, and suppose $[\![A]\!]_D = (V,V)$ with $V = [\![A]\!]$ an object of $\mathbb{C}$. Then the denotational semantics of Π, $[\![\Pi]\!]_D = s_{V,V}$. On the other hand, for the MGoI semantics of Π, $\theta(\Pi) = s_{V\otimes V,V\otimes V}$. However, observe that (as in Proposition above, in the case of Cut)

$$EX(s_{V\otimes V,V\otimes V}, s_{V,V}) = s_{V,V}.$$

Thus, as we have seen above, the two semantics differ in how they interpret the formulas, and the proof interpretations are related by the Proposition 6.2 above. Note that denotational semantics (in the compact Int-categories above) is not set up to explicitly keep track of the cuts nor for modelling the cut-elimination process. The removal of cuts is hidden in the composition in the model category.

In our paper (HS04b) we discussed a "natural" noncompact *-autonomous category closely connected with GoI. In the next section we discuss a related construction for MGoI.

6.2. *The paracategory of types*

As we saw above, Girard's Geometry of Interaction interprets formulas as types (i.e. biorthogonally closed sets of morphisms with respect to an orthogonality $\perp$). One may ask: is there a natural $*$-autonomous category of such "types" whose arrows are induced from the GoI interpretation of proofs?

In our paper (HS04b) we introduced such a category, called $\mathcal{O}(\mathbb{C})$, based on a GoI situation $(\mathbb{C}, T, U)$, where U was a reflexive object and $\mathbb{C}$ was a *traced Unique Decomposition Category*. These latter categories, which are Σ-monoid enriched, are useful in discussing sum-style total traces (AHS02; Hagh00a; HS04a).

In what follows, we generalize this construction to the case of a "partial" category, also denoted $\mathcal{O}(\mathbb{C})$, but now arising from a general GoI category $(\mathbb{C}, T, \perp)$ in the sense of this paper. The intuition behind this construction is to use the MGoI interpretation for formulae to define the objects in $\mathcal{O}(\mathbb{C})$, and to use the MGoI interpretation of a cut-free proof of $\vdash A^{\perp}, B$ to define a morphism $f : A \longrightarrow B$ in $\mathcal{O}(\mathbb{C})$. Here, the situation is complicated by the fact that traces are now partial, so we need a notion of *partial category*, i.e. structures like categories but for which composition of morphisms is only partially defined (even between morphisms of composable form).

There are various notions of partial category in the literature. We shall use a slightly modified version of Freyd's *paracategories.* The latter theory has been exposed in work of Hermida and Mateus (HerMat03). We shall call our notion a *Kleene precategory*, or simply a *precategory.*

Definition 6.4. A *Kleene precategory* $\mathbb{C}$ consists of a class of objects and for any two objects A and B a set of arrows from A to B, denoted $\mathbb{C}(A, B)$. Every homset $\mathbb{C}(A, A)$ has an identity morphism 1_A and there is a partially defined composition operation on homsets

$$\circ : \mathbb{C}(A, B) \times \mathbb{C}(B, C) \longrightarrow \mathbb{C}(A, C)$$

such that $\circ(f, g) : A \longrightarrow C$ for any $f : A \longrightarrow B$ and $g : B \longrightarrow C$ whenever it is defined. We shall use gf instead of $\circ(f, g)$ as is common. These data need to satisfy the following axioms:

(i) For any $f : A \longrightarrow B$, $1_B f$ and $f 1_A$ are defined and $1_B f = f$ and $f 1_A = f$.
(ii) For any $f : A \longrightarrow B, g : B \longrightarrow C$, and $h : C \longrightarrow D$, $h(gf) \sim (hg)f$.

In the second item above $\sim$ stands for *Kleene equality*, meaning: the left side of the equality is defined iff the right side is, and in either case the two sides are equal.

Let $\mathbb{C}$ and $\mathbb{D}$ be two precategories; a *Kleene functor* $F : \mathbb{C} \longrightarrow \mathbb{D}$ consists of two maps:

— $F : ob(\mathbb{C}) \longrightarrow ob(\mathbb{D})$ and
— For every A and B objects in $\mathbb{C}$, $F_{AB} : \mathbb{C}(A, B) \longrightarrow \mathbb{D}(FA, FB)$ such that
  (i) For every $A \in ob(\mathbb{C})$, $F_{AA}(1_A) = 1_{FA}$,
  (ii) For every $f : A \longrightarrow B$ and $g : B \longrightarrow C$, $F(gf) \sim F(g)F(f)$.

Given a GoI category $(\mathbb{C}, T, \perp)$, we define the precategory $\mathcal{O}(\mathbb{C})$ as follows. **Note:** we will only use the $(\mathbb{C}, \perp)$ structure here; in particular we will not be using the functor T as we will not discuss modeling the exponentials of Linear Logic.

• **Objects:** An object A of $\mathcal{O}(\mathbb{C})$ is a bi-orthogonally closed subset of $\mathbb{C}(U, U)$ for some object U in $\mathbb{C}$. The definition of objects is motivated by the notion of a *type*, the MGoI interpretation of formulas. In fact, if A is a formula of **MLL** with $[\![A]\!] = U$, then A, as an object of $\mathcal{O}(\mathbb{C})$, is the type we called $\theta(A)$.

• **Arrows:** Let A and B be objects in $\mathcal{O}(\mathbb{C})$ where A is a bi-orthogonally closed subset of $\mathbb{C}(U, U)$ and B is a bi-orthogonally closed subset of $\mathbb{C}(V, V)$. Then a morphism $f : A \longrightarrow B$ in $\mathcal{O}(\mathbb{C})$ is a morphism $f : U \otimes V \longrightarrow U \otimes V$ in $\mathbb{C}$ such that (1) for any $a \in A$, $f{\bullet}a := Tr^U_{V,V}(s_{U,V}(a \otimes 1_V) f s_{V,U})$ exists and is in B, and (2) For any $b \in B^{\perp}$,

$f\hat{\cdot}b := Tr^V_{U,U}((1_U \otimes b)f)$ exists and is in $A^\perp$. Intuitively we think of f as the MGoI denotation of a cut-free proof of the sequent $\vdash A^\perp, B$. Note that we do not require that f actually be the denotation of a proof: this just motivates the definition.

• **Identity:** The identity morphism on $A \subseteq \mathbb{C}(U,U)$, denoted 1_A, is given by $s_{U,U}$, the symmetry morphism in $\mathbb{C}$ on U. Note that for any $a \in A$, $1_A \cdot a = Tr^U_{U,U}(s_{U,U}(a \otimes 1_U)s_{U,U}s_{U,U}) = Tr^U_{U,U}(s_{U,U}(a \otimes 1_U)) = a \in A$. The latter equality is known as *generalized yanking* in TMC's (see (Hagh00; Hagh00a).) Similarly for any $a \in A^\perp$, $1_A \hat{\cdot} a = Tr^U_{U,U}((1_U \otimes a)s_{U,U}) = a \in A^\perp$. This definition of identity morphisms is motivated by the MGoI interpretation of the cut-free proof of $\vdash A, A^\perp$.

• **Composition:** Composition is defined as follows: given $f : A \longrightarrow B$ and $g : B \longrightarrow C$ in $\mathcal{O}(\mathbb{C})$, with $A \subseteq \mathbb{C}(U,U)$, $B \subseteq \mathbb{C}(V,V)$, and $C \subseteq \mathbb{C}(W,W)$,

$$gf = Tr^{V\otimes V}_{U\otimes W, U\otimes W}((1_U \otimes 1_W \otimes s_{V,V})\tau^{-1}(f \otimes g)\tau).$$

where $\tau = (1_U \otimes 1_V \otimes s_{W,V})(1_U \otimes s_{W,V} \otimes 1_V)$. First, note that this trace may not exist as we are in a partially traced category and thus the composition operation is a *partially* defined one.

The intuition behind this definition is the following: we think of f and g as denotations of cut-free proofs Π_1 and Π_2 of $\vdash A^\perp, B$ and $\vdash B^\perp, C$ respectively. We then apply the MGoI interpretation for the cut rule applied to $\theta(\Pi_1)$ and $\theta(\Pi_2)$, which yields the interpretation of the proof Π of $\vdash [B, B^\perp], A^\perp, C$ obtained from Π_1 and Π_2. But we need a cut-free proof, so we normalize the proof by applying the execution formula to $\theta(\Pi)$. This yields the MGoI interpretation of the cut-free proof of $\vdash A^\perp, C$ obtained from Π_1 and Π_2. As before, the f and g here are *not* assumed to be denotations of the proofs above, we just think of them as such to motivate our definition.

Note that the above composition is the same as the formula for composition in the Int-category $\mathcal{G}(\mathbb{C})$ (called *symmetric feedback* in (Abr96)), see (Hagh00; AHS02; HS04b); of course this is no surprise, as the definition of composition in $\mathcal{G}(\mathbb{C})$ is also motivated by the execution formula applied to the cut of two proofs. Composition is illustrated in Figure 1 below.

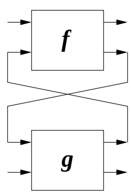

Fig. 1. Composition.

Proposition 6.5. *Let* $(\mathbb{C}, T, \perp)$ *be a GoI category, then,* $\mathcal{O}(\mathbb{C})$ *is a Kleene precategory.*

Proof. Clearly identity morphisms exist. We shall use a graphical calculus for the proofs that follow. Figure 2 below shows the proofs for $h(gf) \sim (hg)f$, where the dashed box shows definedness, and $f1_A = f$. The proof of $1_B f = f$ is similar.

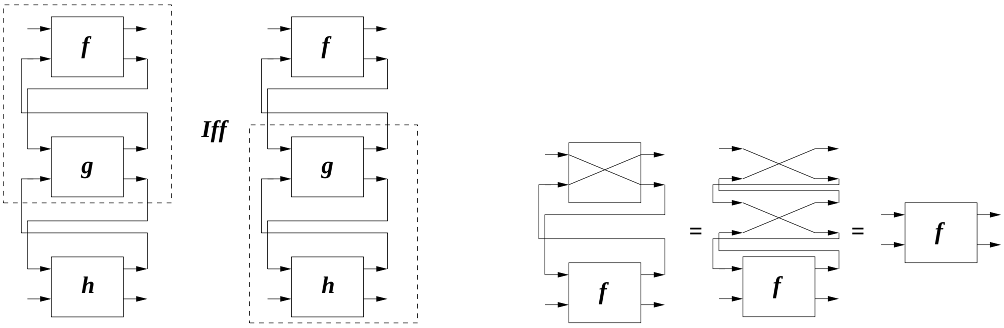

Fig. 2. Graphical proof that $h(gf) \sim (hg)f$ and $f1_A = f$

$\square$

Remark 6.6. Note by restricting to genuine formulas and proofs of **MLL**, we obtain a *subcategory* of "definable" types and arrows in $\mathcal{O}(\mathbb{C})$. Since the definitions of morphisms are motivated directly by the MGoI interpretation of cut-free proofs, they will be well-defined. This is because by Theorem 4.6, the MGoI denotations of proofs are algorithms, and thus in the cut-free case are data of appropriate type, which implies well-definedness of morphisms. More generally, it is clear for morphisms of $\mathcal{O}(\mathbb{C})$ constructed following the logical rules of **MLL**, we can follow the lines of the inductive proof of Theorem 4.6 to conclude their well-definedness.

How do we extend $\mathcal{O}(\mathbb{C})$ to the $*$-autonomous level?

6.2.1. *Towards a $*$-autonomous structure of Types*

Unfortunately, there is no universally agreed-upon definition of $*$-autonomous paracategories into which to fit the above paracategory $\mathcal{O}(\mathbb{C})$ of types. The thesis of Malherbe (Mal10) discusses strict compact paracategories, but the question of even defining non-strict $*$-autonomous pre- or paracategory structure is still an open problem. Rather than do so, as a final remark, we sketch below the appropriate modification of the definitions in (HS04b) which we suggest will be relevant to the ultimate theory.

- **Tensor**: Given $A \subseteq \mathbb{C}(U, U)$ and $B \subseteq \mathbb{C}(V, V)$, objects in $\mathcal{O}(\mathbb{C})$, define:

$$A \otimes B = \{a \otimes b \,|\, a \in A, b \in B\}^{\perp\perp} \subseteq \mathbb{C}(U \otimes V, U \otimes V).$$

Given $f : A \longrightarrow B$ and $g : A' \longrightarrow B'$, with $A' \subseteq \mathbb{C}(U', U')$ and $B' \subseteq \mathbb{C}(V', V')$, we define

$$f \otimes g = (1_U \otimes s_{V,U'} \otimes 1_{V'})(f \otimes g)(1_U \otimes s_{U',V} \otimes 1_{V'}).$$

Notice that the tensor product used on the righthand side is the one in $\mathbb{C}$. Here is the formal proof that motivates this definition (ignoring the exchange rule):

$$\dfrac{\dfrac{\vdash A^{\perp}, B \;\; , \vdash A'^{\perp}, B'}{\vdash A^{\perp}, A'^{\perp}, B \otimes B'}\; times}{\vdash A^{\perp} \mathbin{⅋} A'^{\perp}, B \otimes B'}\; par$$

• **Tensor Unit**: The unit of tensor is given by $I = \{1_I\}^{\perp\perp} \subseteq \mathbb{C}(I, I)$.

• **Symmetry**: The symmetry $s_{A,B} : A \otimes B \longrightarrow B \otimes A$ with $A \subseteq \mathbb{C}(U, U)$ and $B \subseteq \mathbb{C}(V, V)$ is defined as

$$s_{A,B} = (s_{V,U} \otimes 1_V \otimes 1_V)(1_V \otimes s_{V,U} \otimes 1_U)(s_{V,V} \otimes s_{U,U})(1_V \otimes s_{U,V} \otimes 1_U)(s_{U,V} \otimes 1_V \otimes 1_U).$$

Here is the formal proof that motivates this definition:

$$\dfrac{\dfrac{\dfrac{\vdash B^\perp, B \quad \vdash A^\perp, A}{\vdash B^\perp, A^\perp, B \otimes A}\ times}{\vdash A^\perp, B^\perp, B \otimes A}\ exchange}{\vdash A^\perp \mathbin{⅋} B^\perp, B \otimes A}\ par$$

• **Duality**: Given $A \subseteq \mathbb{C}(U, U)$ define

$$A^\perp = \{f \in \mathbb{C}(U, U) \,|\, g \in A \,\text{implies}\, f \perp g\} \subseteq \mathbb{C}(U, U).$$

Given $f : A \longrightarrow B$, with $A \subseteq \mathbb{C}(U, U)$ and $B \subseteq \mathbb{C}(V, V)$, we define$f^\perp : B^\perp \longrightarrow A^\perp$ as $f^\perp = s_{U,V} f s_{V,U}$.

• **Par product**: This arises via de Morgan duality from $\otimes$. Thus given A and B objects of $\mathcal{O}(\mathbb{C})$ with $A \subseteq \mathbb{C}(U, U)$ and $B \subseteq \mathbb{C}(V, V)$, we define

$$A \mathbin{⅋} B = \{a \otimes b \,|\, a \in A^\perp, b \in B^\perp\}^\perp.$$

Given $f : A \longrightarrow B$ and $g : A' \longrightarrow B'$, with $A' \subseteq \mathbb{C}(U', U')$ and $B' \subseteq \mathbb{C}(V', V')$we define $f \mathbin{⅋} g = (f^\perp \otimes g^\perp)^\perp$.

• **Par Unit**, defined by $\perp = \{1_I\}^\perp \subseteq \mathbb{C}(I, I)$.

One may show that $\otimes$ and $(-)^\perp$ are (in the appropriate sense) Kleene functors. We then conjecture the following:

Conjecture 6.7. *Let $(\mathbb{C}, T, \perp)$ be a GoI category. Then the Kleene precategory of types $\mathcal{O}(\mathbb{C})$ may be endowed with a $*$-autonomous precategory structure, following the above definitions.*

It is left as an open question how to extend this to the full exponentials of **MELL**, using the functorial properties of T.

7. Conclusion and Future Work

In this paper we give the details of a new categorical semantics, MGOI, for Girard's Geometry of Interaction (Gir89a; Gir95a; AHS02; Gir07; HS10) for both multiplicative and multiplicative exponential linear logic. This semantics, while inspired by GoI, differs from it in significant points. First, MGoI is typed: we do not assume nontrivial reflexive objects in the ambient models. It is based on a new theory of partial traces and trace classes, as well as a version of GoI localized to endomorphism monoids of types. This permits giving a general semantical framework for the solution of feedback equations associated to Girard's execution formula (Gir07; Gir08), as in our tutorial (HS10). As well, we provide an analysis of the critical features needed for analyzing how the execution formula converges (i.e. gives an invariant of cut-elimination).

There are several open questions and directions we believe worth exploring.

1 Is it possible to develop a natural notion of $*$-autonomous pre- or paracategory, which includes the pre-category of types $\mathcal{O}(\mathbb{C})$ in subsection 6.2.1 above? Is there a factorization theorem of the denotational versus the GoI interpretation, analogous to (HS04b)?
2 The problem of extending MGoI to the additives is still very much open.
3 Are there examples of partially traced categorical models of MGoI connected to operator algebras (e.g. von Neumann algebras) as in the recent Girard work (Gir08)? In particular, in such models, discuss the analytic convergence of the Execution formula

$$EX(\theta(\Pi), \sigma) = Tr^{\otimes\Delta}_{\otimes\Gamma,\otimes\Gamma}((1 \otimes \sigma)\theta(\Pi))$$

for proofs Π as well as $EX(f, \sigma)$ more generally.

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