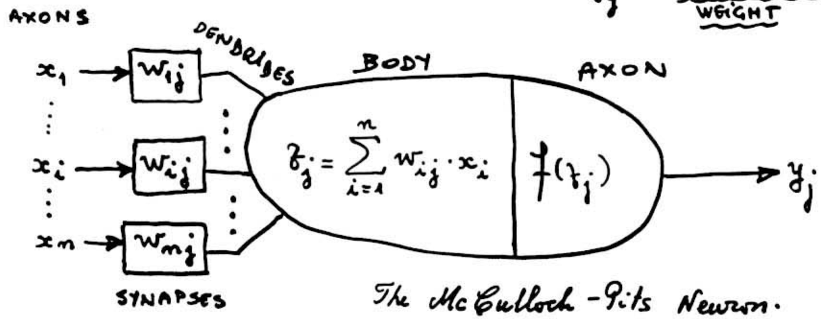
# NEURAL NETWORKS PRIMER

▶ **BIOLOGICAL NEURON** : BODY, ONE AXON, MULTITUDE OF DENDRITES, MULTITUD OF SYNAPSES (CONNECTING DENDRIDES OF DIFFERENT NEURONS). THE STRENGTH OF A GIVEN SIGNAL IS DETERMINED BY THE EFFICIENCY OF THE SYNAPTIC TRANSMIS. A NEURON SENDS AN IMPULSE DOWN TO ITS AXON IF IT IS RECEIVING ENOUGH SIGNALS FROM OTHER NEURONS IN A WINDOW TIME INTERVAL CALLED THE PERIOD OF LATENT SUMMATION. THE INCOMING SIGNAL TO A DENDRIDE MAY BE EITHER INHIBITATORY OR EXCITATORY. A NEURON WILL FIRE AN IMPULSE DOWN ITS AXON, IF ECITATION EXCEED ITS INHIBITION BY A CRITICAL AMOUNT CALLED THE THRESHOLD OF THE NEURON.

▶ **ARTIFICIAL NEURON**

$w_{ij}$ ← INTERCONNEXION WEIGHT




*The McCulloch-Pitts Neuron.*

DIFFERENT N.N. USE DIFFERENT $f(z_j)$ FUNCTIONS

PERCEPTRON = A McC-P NEURON WITH A BINARY THRESHOLD :


N.N.
- WITH FIXED WEIGHTS : THE TASK TO BE ACCOMPLISHED MUST BE WELL DEFINED "A PRIORI". $w_{ij}$ WILL BE EXPLICITLY DETERMINED FROM THE DESCRIPTION OF THE PROBLEM.
- WITH ADAPTABLE WEIGHTS: → LEARNING LAWS ARE USED TO ADJUST $w_{ij}$
  WHEN IT IS NOT KNOWN "A PRIORI" WHAT $w_{ij}$ VALUES SHOUD BE
  - SUPERVISED LEARNING N.N IS SUPPLIED WITH BOTH THE INPUT AND THE CORRECT OUTPUT VALUES
  - UNSUPERVISED LEARNING ONLY THE INPUT VALUES ARE PROVIDED.

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■ **NEURONAL DYNAMICAL SYSTEMS**

NEURONAL ACTIVATIONS CHANGE WITH TIME. (DIFFERENTIAL EQUATIONS)

$\dot{x}_i = g_i(X, Y, \ldots)$ $\Big\{$ HOW THE ACTIVATION (MEMBANE POTENTIALS) OF THE $i$TH NEURON IN FIELD $F_i$ CHANGE AS A FUNCTION OF NETWORK PARAMETERS.

■ **ADDITIVE NEURONAL DYNAMICS**

? WHAT AFFECTS A NEURON
- EXTERNAL INPUTS;
- OTHER NEURONS;
- NOTHING → ACTIVATION DECAY TO ITS RESTING VALUE:

$$\dot{x}_i = -x_i \Rightarrow x_i(t) = x_i(0) \cdot e^{-t}$$

▶ **PASSIVE MEMBRANE DECAY** : $\dot{x}_i = -A_i \cdot x_i$

PASSIVE DECAY RATE: $A_i > 0$

$$\Rightarrow x_i(t) = x_i(0) \cdot e^{-A_i t}$$

↳ IT VARIES INVERSELY WITH THE RESISTANCE $R_i$ OF THE CELL MEMBRANE : $A_i = \frac{1}{R_i}$

↳ INTERPRETING $x_i$ AS A VOLTAGE → $A_i$ IS THE CONDUCTANCE OF THE CELL MEMBRANE MEASURING THE PERMEABILITY OF MEMBRANE, AXONAL, OR SYNAPTIC IONS.

▶ **MEMBRANE TIME CONSTANTS**

THE MEMBRANE TIME CONSTANT $C_i > 0$ SCALES THE TIME VARIABLE OF THE ACTIVATION DYNAMICL SYST. ← ACTS AS THE MEMBRANE CAPACITANCE

$$C_i \dot{x}_i = -A_i x_i$$

$C_i \dot{x}_i = I_i$ THE MEMBRANE CURREN

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## ▶ MEMBANE RESTING POTENTIAL $P_i$ $(>,=,<:0)$

IS THE ACTIVATION VALUE TO WHICH THE MEMBANE POTENTIAL EQUILIBRATES IN THE ABSENCE OF EXTERNAL OR NEURONAL INPUTS:

$$C_i \dot{x}_i = -A_i x_i + P_i$$

$P_i$ —?— MEMBRANE RESTING CURRENT?

$$x_i(t) = x_i(0)\, e^{-(A_i/C_i)t} + \frac{P_i}{A_i}\left(1 - e^{-(A_i/C_i)t}\right)$$

$P_i/A_i$ = STEADY STATE SOLUTION.

if $A_i = 1$, $P_i$ = LARGE ENOUGH ("SUPRATHRESHOLD") THE NEURON WILL SPONTANEOUSLY GENERATE SIGNALS OR PULSE TRAINS IN THE ABSENCE OF EXTERNAL OR NEURONAL STIMULI (AN OBSERVED BIOLOGICAL EFFECT)

if $A_i = 1$, $P_i = 0$, AND THE LOGISTIC SIGNAL FUNCTION $S_i(x_i) = (1+e^{-cx})^{-1}$ THEN AT STEADY STATE $S_i = 1/2$ ⇒ MAY CORRESPOND TO A MODERATELY DENSE PULSE TRAIN.

IN GENERAL $A_i$, $C_i$, AND $P_i$ VARY SLOWLY WITH TIME, BUT SLOWLY ENOUGH TO BE CONSIDERED CONSTANT DURING A NEURONAL EPOCH OF EQUILIBRATION.

## ▶ ADDITIVE EXTERNAL INPUT

$$\dot{x}_i = -x_i + I_i$$

THIS DENOTES AN INPUT NOT A CURRENT

$A_i = 1$, $C_i = 1$, $P_i = 0$

$$x_i(t) = x_i(0) \cdot e^{-t} + I_i(1 - e^{-t}) \xrightarrow[t\to\infty]{} I_i$$

i.e.: THE MEMBANE POTENTIAL ADAPTS ITS BEHAVIOR TO THE APPLIED EXTERNAL STIMULUS.

$I_i$ CAN REPRESENT THE MAGNITUTE OF DIRECTLY EXPERIENCED SENSORY INFORMATION OR DIRECTLY APPLIED CONTROL INFORMATION

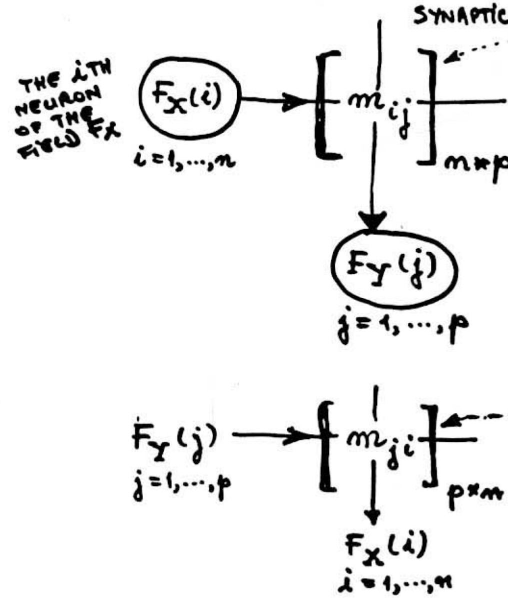
MATHEMATICALLY IS REQUIRED THAT $I_i$ CHANGE SLOWLY ALOWING TO BE ASSUMED CONSTANT.



## ■ ADDITIVE NEURONAL FEEDBACK

N.N. "COMPUTATION": ASYNCHRONOUS, NONLINEAR, MASSIVELY FED BACK, FAST, AND CEASELESS.

### ▶ SYNAPTIC CONNECTION MATRICES




SYNAPTIC (CONNECTION) MATRIX $\mathbf{M}_{m\times p}$ IT DESCRIBES THE FORWARD PROJECTION FROF $F_X$ TO $F_Y$.

$m_{ij}$ ARE SYNAPTIC EFFICACIES

$m_{ij} > 0$ ⇒ $ij$ SYNAPSE IS EXCITATORY

$m_{ij} < 0$ ⇒ $ij$ SYNAPSE IS INHIBITORY

$\frac{dm_{ij}}{dt} = 0$ ⇒ NO LEARNING

BACWARD PROJECTION MATRIX $\mathbf{N}_{p\times m}$

- N.N. SPECIFIED BY THE 4-TUPLE $(F_X, F_Y, M, N)$.

[KOHONEN]: ONE-LAYER N.N. → AUTOASSOCIATIVE;
TWO-LAYER N.N → HETEROASSOCIATIVE;

- **M** AND **N** MATRICES DIFFER IN STRUCTURE (IN GENERAL)

↓

FIXED POINT EQUILIBRUM BEHAVIOR TENDS NOT TO OCCUR. THE EQUILIBRUM BEHAVIOR IS OCCILATORY OR APERIODIC.

## ▶ BIDIRECTIONAL AND UNIDIRECTIONAL CONNECTION TOPOLOGIES

- **BIDIRECTIONAL NETWORKS** = THE FORWARD AND BACKWARD SYNAPTIC PROJECTIONS, $\mathbf{M}$ AND $\mathbf{N}$, HAVE THE SAME STRUCTURE; $M = N^T$, $N = M^T$.
- **BIDIRECTIONAL ASSOCIATE MEMORY (BAM)** = BIDIRECTIONAL NETWORK WITH AN OVERALL STABLE BEHAVIOR (DUE TO THE ACTIVATION DYNAMICS OF $F_X$ AND AND $F_Y$
- **UNIDIRECTIONAL NETWORK** = A NEURON FIELD SYNAPTICALLY INTRACONNECTS TO ITSELF, IT IS A BIDIRECTIONAL NETWORK WITH $F_X \equiv F_Y$.

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- IN "non-BAM" INTERPRETATION $m_{ij} = m_{ji}$ ON THE SAME AXON-SYNAPTIC FIBER
  IN "BAM" INTERPRETATION (TWO-FIELD NETWORK) $m_{ij} = m_{ji}$ BUT $\mathbf{M}$ AN $\mathbf{N}$ REPRESENT DIFFERENT SYNAPTIC WEBS.

- CONCATENATION OF $F_X$ AND $F_Y$, $F_Z = [F_X | F_Y]$.

  $F_Z$ INTERCONNECTS TO ITSELF BY THE MATRIX $B = \begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix}$ i.e. HETEROASSOCIATION IS EXTENDED TO AUTOASSOCIATION.

  IN THE "BAM" CASE: $B = B^T \Rightarrow$ "BAM" SYMMETRIZES AN ARBITRARY RECTANGULAR MATRIX $\mathbf{M}$.

- $\mathbf{M}$ AND $\mathbf{N}$ INTERCONNECT $F_X$ AND $F_Y$; $\mathbf{P}$ AND $\mathbf{Q}$ INTRACONNECT $F_X$ AND $F_Y$. $\Rightarrow C = \begin{pmatrix} P & M \\ N & Q \end{pmatrix}$ INTRACONNECTS $F_Z$.

IF $P$ (OR $Q$) IS SYMETRIC THEN THERE IS A LATERAL INHIBITION OR COMPETITIVE CONNECTION TOPOLOGY. THE STRENGTH OF THE INHIBITORY CONNECTIONS IS OFTEN DISTANCE DEPENDENT, TYPICALLY DECREASING WITH PHYSICAL DISTANCE.



## ■ ADDITIVE ACTIVATION MODELS

THE ADDITIVE ACTIVATION MODEL OF THE N.N. $(F_X, F_Y, M, N)$

$$\begin{cases} \dot{x}_i = -A_i \cdot x_i + \sum_{j=1}^{p} S_j(y_j)\cdot m_{ji} + I_i & i = 1,2,\ldots,n \\ \dot{y}_j = -A_j \cdot y_j + \sum_{i=1}^{n} S_i(x_i)\cdot m_{ij} + J_j & j = 1,2,\ldots,p \end{cases}$$

THE ADDITIVE AUTOASSOCIATIVE $(F_X = F_Y)$ MODEL

$$\left\{ \dot{x}_i = -A_i \cdot x_i + \sum_{j=1}^{n} S_j(x_j)\cdot m_{ji} + I_i \quad \middle|\; i = 1,\ldots,n \right.$$

**HOPFIELD CIRCUIT**: ADDITIVE AUTOASSOCIATIVE MODEL, EACH NEURON HAVING A STRICTLY INCREASING BOUNDED SIGNAL FUNCTION ($S' > 0$), AND SYMMETRIC SYNAPTIC CONNECTION MATRIX ($M = M^T$).

↓ GLOBALLY STABLE FEEDBACK N.N. (CONVERGE EXPONENTIALLY QUICKLY TO FIXED POINTS FOR ALL INPUTS.

$$C_i \cdot \dot{x}_i = -\frac{x_i}{R_i} + \sum_{j=1}^{n} S_j(x_j)\cdot m_{ij} + I_i \quad \Big|\; i = 1,\ldots,n$$

## ■ ADDITIVE BIVALENT (BINARY) MODELS

NEURONS WITH THRESHOLD SIGNAL FUNCTIONS ("ON" - "OFF").

### ▶ BIVALENT ADDITIVE BAM

DISCRETE ADDITIVE "BAM" WITH THRESHOLD SIGNAL FUNCTIONS:

$$\begin{cases} x_i^{k+1} = \sum_{j=1}^{p} S_j(y_j^k)\cdot m_{ji} + I_i & i = 1,\ldots,n \\ y_j^{k+1} = \sum_{i=1}^{n} S_i(x_i^k)\cdot m_{ij} + J_j & j = 1,\ldots,p \end{cases}$$

("k" ARE DISCRETE TIME STEPS)

$$S_j(y_j^k) = \begin{cases} 1 & \text{if } y_j^k > V_j \\ S_j(y_j^{k-1}) & \text{if } y_j^k = V_j \\ 0 & \text{if } y_j^k < V_j \end{cases}$$

WITH THRESHOLD (BINARY) SIGNAL FUNCTIONS, $S_j(y_j^k)$ AND $S_i(x_i^k)$.

**EXAMPLE** ANY REAL-VALUED MATRIX CAN BE USED TO ILLUSTRATE THE "BAM".

$$(y_1, y_2, y_3) = S(x_1, x_2, x_3, x_4) \cdot \begin{pmatrix} -3 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 3 & 2 \\ -2 & 1 & -1 \end{pmatrix}$$

$U_i = V_j = 0 \Leftarrow$ THE THRESHOLD VALUES FOR THE BINARY SIGNAL FUNCTION

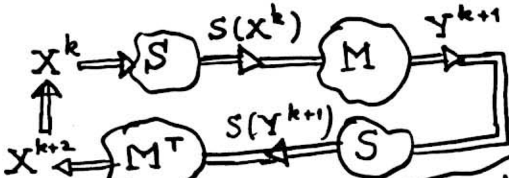
$$X^k = (x_1^k, x_2^k, x_3^k, x_4^k) = (5, -2, 3, 1) \Rightarrow S(X^k) = (1\ 0\ 1\ 1)$$

$$Y^{k+1} = S(X^k) \cdot M = (1\ 0\ 1\ 1) \cdot \begin{pmatrix} -3 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 3 & 2 \\ -2 & 1 & -1 \end{pmatrix} = (-5\ 4\ 3)$$

$$S(Y^{k+1}) = (0\ 1\ 1)$$

$$X^{k+2} = S(Y^{k+1}) \cdot M^T = (0\ 1\ 1) \cdot \begin{pmatrix} -3 & 1 & 0 & -2 \\ 0 & -2 & 3 & 1 \\ 2 & 0 & 2 & -1 \end{pmatrix} = (2\ -2\ 5\ 0)$$

$$S(X^{k+2}) = (1\ 0\ 1\ 1)$$




$$S(X^{k+2}) = S(X^k)$$

$$\Downarrow$$

$$S(Y^{k+3}) = S(Y^{k+1})$$

BIDIRECTIONAL EQUILIBRUM FOR EVER

ASYNCHRONOUS STATE CHANGES MAY LEAD TO DIFFERENT BIDIRECTIONAL EQUILIBRIA

"NEARBY" INITIAL CONDITIONS TEND TO CONVERGE TOWARD THE SAME FIXED-POINT EQUILIBRIUM, THOUGH THE TENDENCY FALLS OFF WITH THE DISTANCE

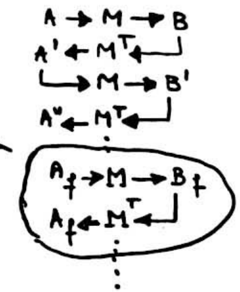
"CONTENT-ADDRESSABLE MEMORY" PROPERTY CHARACTERISTIC OF N.N.



## ▶ BIDIRECTIONAL STABILITY

A "BAM" N.N. $(F_X, F_Y, M)$ is BIDIRECTIONALLY STABLE IF ALL INPUTS CONVERGE TO FIXED-POINT EQUILIBRIA

BIDIRECTIONAL STABILITY IS A DYNAMIC EQUILIBRIUM: i.e. THE SAME SIGNAL INFORMATION FLOWS BACK AND FORTH IN A BIDIRECTIONAL FIXED POINT.




THE BIDIRECTIONAL EQUILIBRIUM CAN BE VIEWED AS A RESONANT STATE.

WHEN BOTH THE CHANGING NEURONS (STM) AND THE CHANGING SYNAPSES (LTM) EQUILIBRATE, AN ADAPTIVE "BAM" FIXE POINT OCCUR. [GROSSBERG]: ADAPTIVE RESONANCE (JOINT EQUILIBRATION OF NEURONS AND SYNAPSES).

## ▶ LYAPUNOV FUNCTIONS

ALLOW TO PROVE THE GLOBAL STABILITY OF A DYNAMICAL SYSTEM WITHOUT SOLVING THE DIFFERENTIAL EQ. OF THAT SYST.

A DYNAMICAL SYSTEM IS STABLE IF SOME LYAPUNOV FUNCTION "L" DECREASES ALONG TRAJECTORIES: $\dot{L} \leq 0$. A DYNAMICAL SYSTEM IS ASYMPTOTICALLY STABLE IF $\dot{L} < 0$. IN A STABLE EQUILIBRIUM THE TRAJECTORY MAY HOVER ARBITRARILY CLOSE TO THE EQUILIBRIUM POINT WITHOUT REACHING IT. IN AN ASYMPTOTICALLY STABLE SYST. THE STATE TRAJECTORY REACHES (IN GENERAL EXPONENTIALLY FAST) THE EQUILIBRIUM.

THE MONOTONICITY OF L PROVIDES A SUFFICIENT (NOT NECESSARY) CONDIT. FOR STABILITY. $\Rightarrow$ INABILITY TO PRODUCE L PROVES NOTHING.

FOR A SYMMETRIC MATRIX "A" AND A SQUARE MATRIX "B", THE QUADRATIC FORM OF LYAPUNOV FUNCTION: $L = XAX^T$ IS STRICTLY DECREASING FOR ANY LINEAR DYNAMICAL SYSTEM $\dot{X} = XB$ IF AND ONLY IF THE MATRIX $AB^T + BA$ IS NEGATIVE DEFINITE

INDEED
$$\dot{L} = \dot{X}AX^T + XA\dot{X}^T = XBAX^T + XAB^TX^T = X[BA + AB^T]X^T$$

EXAMPLE A DYNAMICAL (PASSIVE DECAY) SYSTEM: $\boxed{\dot{x}_i = -x_i}$

(HAVING THE SOLUTION $x_i(t) = x_i(0)\cdot e^{-t}$ WHICH $\rightarrow 0$)

$$L = \frac{1}{2} X \dot{I} X^T = \frac{1}{2}\sum_{i=1}^{m} x_i^2$$

$X = (x_1, \dots, x_i, \dots x_m)$

$\dot{I}$ = m-BY-m IDENTITY MATRIX

$$\dot{L} = \sum_{i=1}^{m} \frac{\partial L}{\partial x_i}\dot{x}_i = -\sum_{i=1}^{m} x_i^2 < 0$$

OR

$$\dot{L} = -\sum_{i=1}^{m} \dot{x}_i^2$$

ALONG THE SYSTEM TRAJECTORIES i.e. THE SYSTEM IS ASYMPTOTICALLY STABLE AS LONG AS AT LEAST ONE STATE VARIABLE CHANGES ($\dot{x}_i \neq 0$) OR DIFFERS FROM ZERO ($x_i \neq 0$)

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A DYNAMICAL SYSTEM IS ASYMPTOTICALLY STABLE IF AND ONLY IF ITS JACOBIAN MATRIX HAS EIGENVALUES WITH NEGATIVE REAL PARTS.

A LYAPUNOV FUNCTION OFTEN MEASURES THE ENERGY OF A NEURAL SYSTEM

FOR N.N. PURPOSES A LYAPUNOV FUNCTION NEED ONLY DECREASE AND BE BOUNDED.

A BOUNDED DECREASING "L" MUST COME TO A STOP. THE STOPPING POINT CORRESPONDS TO AN EQUILIBRIUM POINT (POINT OF PROGRAMMABILITY).

"L" APPROACH REVEALS ONLY THE EXISTENCE OF STABLE POINTS, NOT THEIR NUMBER OR NATURE.



# "BAM" CONNECTION MATRICES

BIPOLAR HEBBIAN (OUTER-PRODUCT) LEARNING – THE MOST POPULAR METHOD FOR CONSTRUCTING THE MATRIX "M" (SCULPTING THE SYSTEM ENERGY SURFACE) IN THE "BAM" (BIDIRECTIONAL ASSOCIATIVE MEMORIES).

$(A_i, B_i)$ I/O ASSOCIATION OF BINARY [0,1] VECTORS $A_i$ (OF LENGTH $m$) AND $B_i$ (OF LENGTH $p$).

$(X_i, Y_i)$ ASSOCIATION OF BIPOLAR [-1,1] VECTORS $X_i$ (OF LENGTH $m$) AND $Y_i$ (OF LENGTH $p$).

$A_i = (X_i + \underline{1})/2$ ; $X_i = 2\cdot A_i - \underline{1}$ ; [$\underline{1} = (1,1,\dots,1)$ m-BIT VECTOR

▶ m – ASSOCIATIONS

THE BIPOLAR OUTER-PRODUCT: $M = \sum_{k=1}^{m} \underbrace{X_k^T Y_k}_{\text{m-BY-p CORRELATION MATRIX}}$ ;

THE BINARY OUTER-PRODUCT: $M = \sum_{k=1}^{m} \underbrace{A_k^T B_k}_{\text{m-BY-p CORRELATION MATRIX}}$ ;

THE BOOLEAN OUTER-PRODUCT: $M = \bigoplus_{k=1}^{m} A_k^T B_k$ ;

THE WEIGHTED OUTER-PRODUCT: $M = \sum_{k=1}^{m} \underbrace{w_k X_k^T Y_k}_{\text{SCALED m-BY-p BIPOLAR CORRELATION MATRIX}}$ ;

$w_k$ COEFFICIENTS CAN BE SELECTED TO PRODUCE A FADING-MEMORY EFFECT: e.g. TIME-ORDERING THE WEIGHTS: $w_1 < w_2 < \dots < w_m$ WHICH GIVES MORE WEIGHT TO MORE RECENT ASSOCIATIONS (WITH LARGER $k$); another example is EXPONENTIAL FADING $w_k = c^{m-k}$ $(0<c<1)$

AN APPROPRIATELY WEIGHTED FADING-MEMORY N.N. YIELDS A "MOVING WINDOW" NONLINEAR FILTER FOR AN ARBITRARY SEQUENCE OF ASSOCIATION TRAINING SAMPLES.

# ▶ OPTIMAL LINEAR ASSOCIATIVE MEMORY (OLAM) MATRIX

(→ KOHONEN'S WORKS) $\hat{M} = X^* Y$

DATA-DEPENDENT "BAM" CONNECTION MATRIX:

$X^*$ = PSEUDO-INVERSE OF $X$;

- FOR A SCALAR $x$, $x^* = 1/x$ OR $x^* = 0$ IF $x = 0$;
- FOR A VECTOR $x$, $x^* = \frac{x^T}{x x^T}$ OR $x^* = 0$ IF $x = \underline{0}$;
- FOR A RECTANGULAR MATRIX $X$, $X^* = X^T (X X^T)^{-1}$ IF $(X X^T)^{-1}$ EXISTS

*) THE "OLAM" MATRIX $\hat{M}$ MINIMIZES THE MEAN-SQUARE ERROR OF FORWARD RECALL IN ONE-SHOT, SYNCHRONOUS LINEAR NETWORK.

**) IF THE SET OF VECTORS $\{X_1, \dots, X_m\}$ IS ORTHONORMAL ($X_i X_j^T = 1$ if $i = j$; $= 0$ if $i \neq j$) THEN THE "OLAM" MATRIX REDUCES TO THE LINEAR ASSOCIATIVE MEMORY (LAM):

$$\boxed{\hat{M} = X^T Y}$$

THIS ASSUMPTION (OF LINEARLY INDEPENDENCE) IS CRITICAL FOR THE "LAM" RECALL ACCURACY

## ▶ AUTOASSOCIATIVE "OLAM" FILTERING

AUTOASSOCIATIVE "OLAM" ASSOCIATES $m$ KNOWN SIGNAL VECTORS $x_1, \dots, x_m$ TO THEMSELF. $M = X^* X$.

$M$ LINEARLY "FILTERS" INPUT MEASUREMENT $x$ VECTOR TO THE OUTPUT VECTOR $x'$ BY MATRIX MULTIPLICATION $\boxed{x M = x'}$

*Kohonen* CALLS $\mathbf{I} - X^* X$ THE NOVELTY DETECTOR ON $R^n$.
IT WAS APPLIED TO IMAGE-SUBSTRACTION PROBLEMS.

$X X^*$ AND $\mathbf{I} - X^* X$ UNIQUELY DECOMPOSE EVERY $R^n$ VECTOR $x$ INTO A SIGNAL VECTOR $\hat{x}$ AND AN ORTHOGONAL NOVELTY VECTOR $\tilde{x}$.

$$x = x X^* X + x(\mathbf{I} - X^* X) = \hat{x} + \tilde{x}.$$



## BAM CORRELATION ENCODING EXAMPLE

UNWEIGHTED ($w_1 = w_2 = 1$) BINARY ASSOCIATIONS:

$A1 = (101010)$, $B1 = (1100)$
$A2 = (111000)$, $B2 = (1010)$

CORRESPONDENT BIPOLAR ASSOCIATIONS

$X_1 = (1\ {-1}\ 1\ {-1}\ 1\ {-1})$, $Y_1 = (1\ 1\ {-1}\ {-1})$
$X_2 = (1\ 1\ 1\ {-1}\ {-1}\ {-1})$, $Y_2 = (1\ {-1}\ 1\ {-1})$

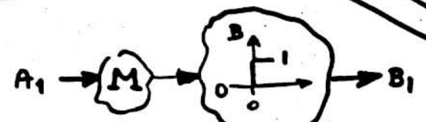
$A_i$ AND $X_i$ ARE CONSIDERED (FOR CONVENIENCE) TO BE SIGNAL FUNCTIONS.

THE "BAM" MEMORY MATRIX $M$ IS CONSTRUCTED BY ADDING POINTWISE THE BIPOLAR CORRELATION MATRICES: $X_1^T Y_1$ AND $X_2^T Y_2$.

$$X_1^T Y_1 = (1\ {-1}\ 1\ {-1}\ 1\ {-1})^T (1\ 1\ {-1}\ {-1}) = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix};$$

$$X_2^T Y_2 = (1\ 1\ 1\ {-1}\ {-1}\ {-1})^T (1\ {-1}\ 1\ {-1}) = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix};$$

$$M = X_1^T Y_1 + X_2^T Y_2 = \begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & -2 & 2 & 0 \\ 2 & 0 & 0 & -2 \\ -2 & 0 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix}$$




$A1\,M = (101010) \cdot M = (4\ 2\ {-2}\ {-4}) \xrightarrow{\text{THRESH}} B_1 = (1100)$;

$B_1 M^T = (1100) \cdot M^T = (2\ {-2}\ 2\ {-2}\ 2\ {-2}) \rightarrow A_1 = (101010)$

$(A_1, B_1)$ IS A FIXED POINT OF THE "BAM" DYNAMICAL SYSTEM

THE LYAPUNOV "ENERGY" $L(A_1, B_1) = -A_1 M B_1^T = -(4\ 2\ {-2}\ {-4})(1100)^T = -6$

($-B_1 M^T A_1^T = -6$, THE BACKWARD "L" ENERGY)

SIMILARLY $(A_2, B_2)$ IS A FIXED "BAM" POINT WITH A "L" ENERGY OF $-6$; "BAM" HAS TWO FIXED EQUALLY DEEP ATTRACTORS.

FOR PATTERN RECOGNITION THE "WIDTH" OF AN ATTRACTOR BASIN IS MORE RELEVANT THAN ITS "DEPTH".

[ THE HAMMING DISTANCE OF TWO VECTORS $H(A_i, A_j)$ COUNTS THE SLOTS IN WHICH THE TWO BINARY VECTORS DIFFER.

a) $A = (011000) \rightarrow H(A, A_2) = 1 \;,\; H(A, A_1) = 3\;;$

$AM = (011000)M = (2 -2\ 2 -2) \rightarrow S(AM) = (1010) = B_2$

THE "L" ENERGY: $-AMB_2^T = -4$ — i.e. THE "BAM" SYSTEM RECALLS THE RESONANT PAIR $(A_2, B_2)$.

b) $A = (000110) \rightarrow H(A, A_1) = 3 \;,\; H(A, A_2) = 5\;;$

$AM = (000110)M = (-2\ 2 -2\ 0) \rightarrow S(AM) = (0101) = B_2^C$

$H(A, A_2^C) = 1 \;,\; H(A, A_1^C) = 3$ — THE COMPLEMENT BIT VECTOR OF $B_2$.

$A_2^C M = (000111)M = (-4\ 2 -2\ 4) \rightarrow S(A_2^C M) = (0101) = B_2^C$

{ A IS "ROLLING" INTO THE UNINTENDED FIXED POINT SPURIOUS ATTRACTOR $(A_2^C, B_2^C)$ WITH THE "L" ENERGY $-A_2^C M (B_2^C)^T = -6$

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ON AVERAGE **BIPOLAR** [(−1, 1) INSTEAD OF (0,1)] SIGNAL STATE VECTORS $S(\ldots)$ PRODUCE MORE ACCURATE RECALL THAN BINARY SIGNAL STATE VECTORS.

$$X_1 M = (1 -1\ 1 -1\ 1 -1)\cdot M = (8\ 4 -4 -8) \xrightarrow{S(\ldots)} (1\ 1 -1 -1) = Y_1$$
$$X_2 M = (1\ 1\ 1 -1 -1 -1)\cdot M = (8 -4\ 4 -8) \xrightarrow{S(\ldots)} (1 -1\ 1 -1) = Y_2$$
$$Y_1 M = (1\ 1 -1 -1)\cdot M^T = (4 -4\ 4 -4\ 4 -4) \xrightarrow{S(\ldots)} (1 -1\ 1 -1\ 1 -1) = X_1$$
$$Y_2 M = (1 -1\ 1 -1)\cdot M^T = (4\ 4\ 4 -4 -4 -4) \xrightarrow{S(\ldots)} (1\ 1\ 1 -1 -1 -1) = X_2$$
$$L(X_1, Y_1) \begin{cases} = -X_1 M Y_1^T = -(8\ 4 -4 -8)(1\ 1 -1 -1)^T \\ = -Y_1 M^T X_1^T = \ldots\ldots \end{cases} = -24$$

$(X_1, Y_1)$ AND $(X_2, Y_2)$ ARE FIXED POINTS, WITH "L" ENERGY $= -24$, IN THE BIPOLAR PRODUCT SPACE $\{-1, 1\}^n \times \{-1, 1\}^p$ OF THE "BAM" DYNAMICAL SYSTEM



a) $X = (-1\ 1\ 1 -1 -1 -1) \rightarrow H(X, X_1) = 3 \;,\; H(X, X_2) = 1\;;$

$XM = (4 -4\ 4 -4) \xrightarrow{S(\ldots)} (1 -1\ 1 -1) = Y_2\;;$

i.e. THE 1-BIT NOISY INPUT $X = (-1\ 1\ 1 -1 -1 -1)$ EVOKES THE FIXED POINT $(X_2, Y_2)$ WITH "L" ENERGY $-XMY_2^T = -(4 -4\ 4 -4)\cdot(1 -1\ 1 -1)^T = -16$.

b*) $X = (-1 -1 -1\ 1\ 1 -1) \rightarrow$ CLOSEST TO $X_2^c$ AS $H(X, X_2^c) = 1$ EVOKES THE SPURIOUS (COMPLEMENT) ATTRACTOR $(X_2^c, Y_2^c)$ WITH INITIAL "L" ENERGY $-XMY_2^{cT} = -16$ DECREASING AT THE NEXT ITERATION TO $-X_2^c M Y_2^{cT} = -24$

$$\left[\begin{array}{l} XM = (-4\ 4 -4\ 4) \xrightarrow{S(\ldots)} (-1\ 1 -1\ 1) = Y_2^c; \\ Y_2^c M^T = (-4 -4 -4\ 4\ 4\ 4) \xrightarrow{S(\ldots)} X_2^c; \end{array}\right]$$

## ▶ MEMORY CAPACITY: DIMENSIONALITY LIMITS CAPACITY

e.g. AS MORE BINARY CORRELATION MATRICES $A_k^T B_k$ ARE BOOLEAN ADDED, THE RESULTING MATRIX ELEMENTS ARE MORE FREQUENTLY $m_{ij} = 1$ $\left(M = \bigoplus_{k=1}^{m} A_k^T B_k\right) \Rightarrow$ AFTER A POINT, ADDITIONAL ASSOCIATIONS $(A_k, B_k)$ DO NOT SIGNIFICANTLY CHANGE THE CONNECTION MATRIX.

DIMENSIONALITY/CAPACITY ALWAYS HOLD BUT IN DIFFERENT WAYS

[ THE N.N. TENDS TO EXCEED THE MEMORY CAPACITY AS THE NUMBER $m$ OF PATTERNS APPROACHES $\min(n, p)$ FOR "OLAM" (Optimal Linear Associat. Memory)

[ SUPERVISED N.N., ESPECIALLY FEEDFORWARD MULTILAYER NETWORKS TRAINED WITH AN ESTIMATED-GRADIENT-DESCENT ALGORITHM, CAN USUALLY ACCURATELY STORE MORE PATTERNS $m$ THAN THE NUMBER $n$ OF NETWORK NEURONS. THE PRICE IS ↗ THE NUMBER OF TRAINING ITERATIONS

CAPACITY IMPROVES WITH N.N. SIZE. NETWORK TEND TO LEARN MORE PATTERNS AS THE # OF NEURONS (i.e. SYNAPSES) ↗.

GROSSBERG'S SPARSE CODING THEOREM: [ FOR DETERMINISTIC ENCODING, PATTERN DIMENSIONALITY MUST EXCEED PATERN NUMBER TO PREVENT LEARNING SOME PATTERNS AT THE EXPENSE OF FORGETTING OTHERS.

{ FOR BOOLEAN ENCODING OF BINARY ASSOCIATIONS WITH A BIVALENT ADDITIVE "BAM" THE MEMORY CAPACITY ↗ GREATLY WITH THE UPPER BOUND $\min(2^n, 2^p)$ (INSTEAD OF $\min(n, p)$) IF THE THRESHOLDS $U_i, V_i$ ARE JUDICIOUSLY SELECTED.