COVERING ARRAYS ON GRAPHS: QUALITATIVE INDEPENDENCE GRAPHS AND EXTREMAL SET PARTITION THEORY

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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Abstract

There has been a good deal of research on covering arrays over the last 20 years. Most of this work has focused on constructions, applications and generalizations of covering arrays. The main focus of this thesis is a generalization of covering arrays, covering arrays on graphs. The original motivation for this generalization was to improve applications of covering arrays to testing systems and networks, but this extension also gives us new ways to study covering arrays.

Two vectors v, w in $\mathbb{Z}_k^n$ are *qualitatively independent* if for all ordered pairs $(a, b) \in \mathbb{Z}_k \times \mathbb{Z}_k$ there is a position i in the vectors where $(a, b) = (v_i, w_i)$. A *covering array* is an array with the property that any pair of rows are qualitatively independent. A *covering array on a graph* is an array with a row for each vertex of the graph with the property that any two rows which correspond to adjacent vertices are qualitatively independent. A covering array on the complete graph is a covering array. A covering array is *optimal* if it has the minimum number of columns among covering arrays with the same number of rows.

The addition of a graph structure to covering arrays makes it possible to use methods from graph theory to study these designs. In this thesis, we define a family of graphs called the *qualitative independence graphs*. A graph has a covering array, with given parameters, if and only if there is a homomorphism from the graph to a particular qualitative independence graph. Cliques in qualitative independence graphs relate to covering arrays and independent sets are connected to intersecting partition systems.

It is known that the exact size of an optimal binary covering array can be determined using Sperner's Theorem and the Erdős-Ko-Rado Theorem. In this thesis,

we find good bounds on the size of an optimal binary covering array on a graph. In addition, we determine both the chromatic number and a core of the binary qualitative independence graphs. Since the rows of general covering arrays correspond to set partitions, we give extensions of Sperner's Theorem and the Erdős-Ko-Rado Theorem to set-partition systems. These results are part of a general framework to study extremal partition systems.

The core of the binary qualitative independence graphs can be generalized to a subgraph of a general qualitative independence graph called the *uniform qualitative independence graph*. Cliques in the uniform qualitative independence graphs relate to *balanced* covering arrays. Using these graphs, we find bounds on the size of a balanced covering array. We give the spectra for several of these graphs and conjecture that they are graphs in an association scheme.

We also give a new construction for covering arrays which yields many new upper bounds on the size of optimal covering arrays.

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☆Karen Meagher
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Contents

List of Tables

Chapter 1

Introduction

Covering arrays, also known as qualitatively independent families and surjective arrays, have been the focus of much research for the last twenty years. They are natural generalizations of orthogonal arrays and Sperner systems. Covering arrays are computationally difficult to find and useful in multiple applications, particularly for designing test suites that test interactions of parameters in a system [15, 16, 22, 74, 75].

For positive integers n, r, k, a covering array $CA(n, r, k)$ is an $r \times n$ array with entries from $\mathbb{Z}_k$ and the property that in any pair of rows, each of the k^2 ordered pairs from $\mathbb{Z}_k \times \mathbb{Z}_k$ appear in at least one column. This property is called *qualitative independence*, and thus, the set of rows in a covering array is a qualitatively independent family. The parameter n is the *size* of the covering array and $\mathbb{Z}_k$ is its *alphabet*. For most applications, it is desirable to have covering arrays with smallest possible size.

Much of the work on covering arrays focuses on developing constructions for them [21, 49, 65, 71], finding bounds on their minimum size [70] and more recently, on improving heuristic searches for small covering arrays [13, 18, 59, 68]. As a result of this work, there are many bounds on the size of covering arrays with specific parameters. For the special case of binary covering arrays (covering arrays with $k = 2$), the exact sizes of the smallest array are known [45, 47]. For a fixed k, the asymptotic growth, as n increases, of the largest r such that a $CA(n, r, k)$ exists is known [28, 29, 30].

The first result in this thesis is a new construction that produces some of the smallest known covering arrays for specific parameters. This construction, called *the group construction of covering arrays*, is presented in Section 2.4.1 and published in [55] and is also used in [51]. Appendix A contains two tables of new upper bounds on the minimal size of covering arrays.

To improve applications, generalizations of orthogonal arrays and of covering arrays have been considered [17, 39, 57]. In this thesis, we examine covering arrays with a graph structure. Prior to this thesis, there have been few studies of covering arrays on graphs. Seroussi and Bshouty [63] prove that finding the smallest binary covering array on a graph is an NP-hard problem and Stevens gave some basic results on covering arrays on graphs in his Ph.D. thesis [69].

For positive integers n and k, a *covering array on a graph* G, denoted $CA(n, G, k)$, is a $|V(G)| \times n$ array, whose entries are from $\mathbb{Z}_k$, with the property that each row corresponds to a vertex in G and rows that correspond to adjacent vertices in G are qualitatively independent. A covering array on a complete graph is a standard covering array.

Adding a graph structure to covering arrays adds much more than just an improvement for applications of covering arrays to designing test suites. It gives us a new way to study covering arrays and provides interesting results and research problems in graph theory and extremal set theory.

The goal of this thesis is to develop a new framework to study covering arrays and covering arrays on graphs. The first way this is done is by rephrasing a design theory problem as a graph theory problem. This is achieved with a family of graphs called the *qualitative independence graphs*. The second way this is done is by generalizing the results from extremal set theory that are used to find the exact size of an optimal binary covering array. To this end, we give extensions of Sperner's Theorem and the Erdős-Ko-Rado Theorem to partition systems and we lay the foundation for a theory of extremal partition systems.

The new results in this thesis are mostly contained in Chapters 5–8. Next, we give an overview of the major results from each of these four chapters.

The Qualitative Independence Graphs

We start with covering arrays on graphs with a particular focus on binary covering arrays. We give an upper bound for the size of a covering array on a graph based on the chromatic number of the graph and a lower bound based on the size of the maximum clique of the graph. The upper bound is of particular interest because it gives a method to construct covering arrays on graphs from standard covering arrays and it raises a question that partially motivates this work: can we find a graph for which this bound is not tight?

In Section 5.2, we define a family of graphs, called the *qualitative independence graphs*, denoted $QI(n,k)$, for positive integers n and k. It has become evident that the qualitative independence graphs are an important family of graphs. First, for infinitely many of these graphs, the upper bound on the size of covering array on the graph from the chromatic number is not tight. Second, this family of graphs gives a good characterization of covering arrays for all graphs, namely that for a graph G and positive integers k and n, a $CA(n,G,k)$ exists if and only if there is a graph homomorphism G to $QI(n,k)$. Finally, for all positive integers k and n, a clique of size r in the graph $QI(n,k)$ corresponds to a $CA(n,r,k)$. This rephrases a design theory question to a graph theory question. In this thesis, we use techniques from graph theory, particularly algebraic graph theory, to analyze the graphs $QI(n,k)$.

We give formulae for both the size of the maximum clique and the chromatic number of the graphs $QI(n,2)$. These two formulae are used to establish good bounds on the minimal size of a binary covering array on a given graph. Also, using these formulae, we determine the core of $QI(n,2)$. The structure of these cores implies that for a graph G, if there exists a $CA(n,G,2)$, then there exists a $CA(n,G,2)$ in which each row has exactly $\lfloor \frac{n}{2} \rfloor$ ones.

Finally, in Section 5.4, for a positive integer k, we find upper bounds on the size of the maximum clique and the chromatic number of $QI(k^2,k)$ and we prove that this graph is a $(k!)^{k-1}$-regular graph.

Eigenvalues and Association Schemes

For positive integers n and k, where k divides n, it is clear how to generalize the core we found for the binary qualitative independence graphs to a subgraph of $QI(n,k)$. These subgraphs are called the *uniform qualitative independence graphs* and denoted $UQI(n,k)$. It is not at all clear if $UQI(n,k)$ is a core for $QI(n,k)$, nevertheless, the uniform qualitative independence graphs are an intriguing family of graphs.

In Section 6.2, we give bounds on the size of the maximum cliques in a uniform qualitative independence graph. These bounds are not new, but the proof given here is new and uses properties of the uniform qualitative independence graphs.

The qualitative independence graph $QI(9,3)$, which is also a uniform qualitative independence graph, has previously appeared in the literature in a different context. Mathon and Rosa [50] prove that $QI(9,3)$ is part of an association scheme and give its eigenvalues and their multiplicities.

We show in Section 6.4.3 that the method used to find the eigenvalues in [50] can be generalized to the uniform qualitative independence graphs. In particular, in Chapter 6, we give an equitable partition on the vertices of the uniform qualitative independence graphs that can be used to find their eigenvalues. The eigenvalues and multiplicities for the graphs $UQI(3c,3)$ for $c = 3,4,5,6$ and $QI(16,4)$ are stated.

It is interesting to ask if other qualitative independence graphs are also classes in an association scheme and in particular, if the scheme used in [50] can be generalized. In Section 6.5, two sets of graphs are given that are generalizations of the scheme from [50]. We conjecture that these sets of graphs form association schemes and we give their modified matrix of eigenvalues.

Extremal Partition Theory

Since each row of a binary covering array corresponds to a set, two results from extremal set theory, Sperner's Theorem and the Erdős-Ko-Rado Theorem, can be used to find the exact size of the smallest binary covering array, $CA(n,r,2)$ for all values of r. The famous Erdős-Ko-Rado Theorem [24] is concerned with the maximal cardinality of intersecting set systems as well as with the structure of such maximal systems. The equally well-known Sperner's Theorem is concerned with the cardinality and structure of the largest system of incomparable sets.

Since the rows of a non-binary covering array correspond to partitions, it is desirable to have versions of Sperner's Theorem and the Erdős-Ko-Rado Theorem for partition systems. In general, this motivates the study of *extremal partition systems.* There is an extensive theory of extremal finite set systems but there are almost no similar results for systems of partitions. One goal of this thesis is to lay down the foundation for the study of extremal partition systems.

In Section 7.1, we give a new generalization of Sperner's Theorem for partition systems. For this generalization we give an exact result for a class of partition systems, a bound and an asymptotic result for all partition systems. Our extension of Sperner's Theorem is different from other Sperner-type theorems for systems of collections of sets that have been examined [28, 29, 30, 43].

In Section 7.2 of this thesis, an extension of the Erdős-Ko-Rado Theorem to intersecting set partitions is proven. We consider two different notions of intersecting partitions. First, we consider two partitions to be intersecting if they both contain a common class. With this type of intersection, we prove a version of the Erdős-Ko-Rado Theorem for partition systems. In particular, we prove that the largest intersecting partition system is a trivially intersecting system. Another notion of intersecting partitions is *partial intersection.* Two partitions are partially t-intersecting if they have classes with intersection of cardinality at least t. We conjecture a version of the Erdős-Ko-Rado Theorem for partially intersecting partitions, that is, a trivially partially intersecting partition system is the largest partially intersecting partition system. This conjecture is proven in certain restricted cases in Section 7.5. Different extensions of the Erdős-Ko-Rado Theorem to systems of intersecting partitions have also been considered by P. L. Erdős, Székely and László [25].

General extremal partition systems

Higher order extremal problems are extremal problems in which the elements in the system are disjoint families of subsets of a finite set called *clouds*, rather than sets. Ahlswede, Cai and Zhang [3] present a framework for several different types of higher order extremal problems.

In Chapter 8, we examine higher order extremal problems in which the elements of the system are partitions, rather than clouds. We use a framework, similar to the one defined by Ahlswede, Cai and Zhang, to present a variety of different types of extremal problems for partition systems. For many of these problems we give an exact solution and for others we give bounds on the cardinality of the maximum partition system. These results include our extension of Sperner's Theorem to partition systems and our Erdős-Ko-Rado Theorem for partition systems. This chapter includes many open problems and provides a unified framework for problems considered in previous chapters.

Overview of the Document

In Chapter 2, we introduce covering arrays, define three closely related designs (orthogonal Latin squares, transversal designs and orthogonal arrays) and describe three constructions for covering arrays. In Chapter 3, we show three common types of set systems and state results from extremal set theory that are related to covering arrays (Sperner's Theorem, Bollobás's Theorem and the Erdős-Ko-Rado Theorem), and we give an application of Sperner's Theorem and the Erdős-Ko-Rado Theorem to covering arrays. In Chapter 4, we review some basic graph theory and algebraic graph theory, including spectral theory of graphs and association schemes. We use these to study the qualitative independence graphs. In Chapter 5, we introduce the object which motivates much of this work: covering arrays on graphs. We also introduce the qualitative independence graphs. This chapter focuses on the binary qualitative independence graphs. We define the uniform qualitative independence graphs in Chapter 6. Using an equitable partition on the vertices of this graph it is possible to find the spectrum of these graphs — the spectrum of several instances of these graphs is given in this chapter. Then, we consider whether these graphs are part of an association scheme. In Chapter 7, extensions of Sperner's Theorem and Erdős-Ko-Rado Theorem are given for partition systems. In Chapter 8, a general framework for extremal partition systems is described. In Chapter 9, we list open questions and conjectures that arise from this study. Finally, in Appendix A we present a table of the new best bounds on the size of covering arrays. These new bounds come from our new construction given in Chapter 2.

Chapter 2

Covering Arrays and Related Designs

Covering arrays, which are also known as qualitatively independent families and surjective arrays, are a generalization of the well-known and well-studied orthogonal arrays [40]. They are a mathematically rich design with many applications.

Orthogonal arrays were first developed by Rao in the 1940s for designing statistical experiments. Experiments based on orthogonal arrays are particularly effective; such experiments allow for independent estimation of each factor in the system and the effect of different pairs of factors can be compared with the same accuracy (for more information, see Section 11.4 of [40]). But, for many parameters, it is not possible to build an orthogonal array. Covering arrays are a relaxation of orthogonal arrays (of index 1) that still provide good test design. Orthogonal arrays are closely related to two other designs: Latin squares and transversal designs. In this chapter, we give an overview of covering arrays and related designs. The only new result in this chapter is a group construction for covering arrays given in Section 2.4.1.

2.1 Mutually Orthogonal Latin Squares

Latin squares are a very old and well-studied design. It is believed that these were first studied by Euler around 1782. Latin squares have also appeared in works of art, math puzzles and even game shows.

Definition 2.1.1 (*Latin Square*). Let n be a positive integer. A *Latin square of order* n is an $n \times n$ array on n symbols with the property that each symbol occurs exactly once in each row and each column.

Unless otherwise stated, in this thesis we use the set $\{0, 1, \dots, n-1\}$ as the symbols in a Latin square of order n. A Latin square is in *standard form* if the symbols in the first row occur in their natural order.

> **Example 2.1.2.** Below are two Latin squares. The first square, L, is a Latin square of order 4 in standard form. The second Latin square, M, is of order 5 and is also in standard form. We remark that it is the addition table for $\mathbb{Z}_5$.
>
> $$L = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

For every n, there exists a Latin square of order n; the addition table for $\mathbb{Z}_n$ is always a Latin square of order n. Any permutation of the rows or the columns of a Latin square is still a Latin square. Also, any relabelling of the symbols in a Latin square is still a Latin square. Thus any Latin square can be put in standard form by permuting the columns or relabelling the symbols.

Definition 2.1.3 (*Orthogonal Latin Squares*). Let n be a positive integer and let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ be Latin squares of order n. Consider the $n \times n$ array with entries $[(a_{i,j}, b_{i,j})]$; if, in this array, each of the n^2 ordered pairs of symbols occurs exactly once, then the squares A and B are *orthogonal*.

If A and B are a pair of orthogonal Latin squares, then any relabelling of the symbols in either A or B, produces another pair of Latin squares which are orthogonal. So for any pair of orthogonal Latin squares, both can be put in standard form.

> **Example 2.1.4.** Below are two orthogonal Latin squares of order 4. Note that both Latin squares are in standard form.

$$L = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \end{bmatrix}, \quad L' = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{bmatrix}$$

$$[(L_{i,j}, L'_{i,j})] = \begin{bmatrix} (0,0) & (1,1) & (2,2) & (3,3) \\ (3,2) & (2,3) & (1,0) & (0,1) \\ (1,3) & (0,2) & (3,1) & (2,0) \\ (2,1) & (3,0) & (0,3) & (1,2) \end{bmatrix}$$

A set of Latin squares with the property than any two distinct squares from the set are orthogonal is called a set of *mutually orthogonal Latin squares* (MOLS). Let $N(n)$ denote the cardinality of the largest set of MOLS of order n. For all n,

$$1 \leq N(n) \leq n - 1. \tag{1}$$

To derive this upper bound, consider a set of $N(n)$ Latin squares in standard form. For each of the Latin squares, the $(1, 1)$ entry is 0 and the $(1, i + 1)$ entry is i. Consider the symbol which occurs in the $(2, 1)$ position of each Latin square in this set. This symbol must be one of $n - 1$ letters from the set $\{1, \ldots, n - 1\}$. Assume two Latin squares A and B from the set have the entry $i \in \{1, \ldots, n-1\}$ in the $(2, 1)$ position. Then $(A_{1,i+1}, B_{1,i+1}) = (i, i) = (A_{2,1}, B_{2,1})$, so the Latin squares A and B are not be orthogonal. Thus each symbol in $\{1, \ldots, n - 1\}$ corresponds to at most one Latin square in the set and there can be at most $n - 1$ MOLS of order n.

A set of $n - 1$ MOLS of order n is called a *complete set of MOLS*. When n is a prime power, that is, $n = p^a$, where p is a prime number and a is a positive integer, there exists a complete set of MOLS. We do not give a proof of this theorem here; rather in Section 2.4.1, we give a construction of an equivalent design.

Theorem 2.1.5 (see [73]). *For a prime power n, $N(n) = n - 1$.*

Example 2.1.6. The following is a complete set of MOLS of order 4:

$$A_1 = \begin{array}{|cccc|}\hline 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ \hline\end{array} \quad A_2 = \begin{array}{|cccc|}\hline 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ \hline\end{array} \quad A_3 = \begin{array}{|cccc|}\hline 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ \hline\end{array}$$

It is possible to construct orthogonal Latin squares from smaller orthogonal Latin squares. This construction provides a lower bound on $N(n)$.

Theorem 2.1.7 ([73]). *For all positive integers m, n, we have*

$$N(mn) \geq \min\{N(m), N(n)\}.$$

Proof. Let $l = \min\{N(m), N(n)\}$. Let $M_1, M_2, \ldots, M_l$ be the first l orthogonal Latin squares of order m, and let $N_1, N_2, \ldots, N_l$ be the first l orthogonal Latin squares of order n. For $0 \leq r \leq m-1$, let N_i^r be the Latin square obtained from N_i by adding rn to each entry of N_i. In each Latin square M_i, for $i = 1, \ldots, l$, replace each occurrence of the symbol r by the square N_i^r. This set of l squares will be a set of mutually orthogonal Latin squares of order mn. ✰

For $n = 6$ the previous theorem states that $N(6) \geq \min\{N(2), N(3)\} = 1$. Since $N(n) \geq 1$ for all n, this is not a useful result. It is non-trivial to conclude that it is impossible to build a pair of MOLS of order 6 (indeed, $N(6) = 1$, see Section 6.4 [73]). Euler conjectured that $N(n) = 1$ for all $n \equiv 2 \pmod 4$. This conjecture was proven false by Bose, Shrikhande and Parker [12]. The methods they used were generalized (in the following article in the same issue of the journal!) by Chowla, Erdős and Straus [14] to show $\lim_{n\to\infty} N(n) = \infty$. Currently, it is known that $N(2) = 1$, $N(3) = 2$, $N(6) = 1$ and $N(10) \geq 2$, and for all other values of n, $N(n) \geq 3$ [73].

2.2 Transversal Designs

Transversal designs are a generalization of mutually orthogonal Latin squares. Often, results for Latin squares are more conveniently rephrased in terms of transversal designs (for example, the proof of the non-existence of a pair of orthogonal Latin squares of order 6 in [73]).

Definition 2.2.1 (*Transversal Design*). Let r, λ, k be positive integers. A *transversal design*, $T[r, \lambda; k]$, is a triple $(X, \mathcal{G}, \mathcal{B})$ with the following properties:

1. X is a set of rk symbols, called *varieties.*

2. $\mathcal{G}$ is a partition of the set X, $\mathcal{G} = \{G_1, G_2, \ldots, G_r\}$, where each G_i is a k-subset of X. The sets G_i are called *groups.*

3. $\mathcal{B}$ is a collection of r-sets of X, called *blocks*, with the property that each block intersects each group G_i, for $i = 1, \ldots, r$, in exactly one variety.

4. Any pair of varieties from different groups occur in the same number of blocks. This number is called the *index* and is denoted by λ.

Example 2.2.2. The following is a transversal design, $T[4, 1; 3]$.

$$X = \{0,\ a,\ \alpha,\ \heartsuit,\ 1,\ b,\ \beta,\ \clubsuit,\ 2,\ c,\ \gamma,\ \diamondsuit\}$$

$$\mathcal{G} = \left\{ \begin{array}{llll} G_1 = & \{0 & 1 & 2\} \\ G_2 = & \{a & b & c\} \\ G_3 = & \{\alpha & \beta & \gamma\} \\ G_4 = & \{\heartsuit & \clubsuit & \diamondsuit\} \end{array} \right\} \quad \mathcal{B} = \left\{ \begin{array}{lllll} B_1 = & \{0 & a & \alpha & \heartsuit\} \\ B_2 = & \{0 & b & \beta & \clubsuit\} \\ B_3 = & \{0 & c & \gamma & \diamondsuit\} \\ B_4 = & \{1 & a & \gamma & \clubsuit\} \\ B_5 = & \{1 & b & \alpha & \diamondsuit\} \\ B_6 = & \{1 & c & \beta & \heartsuit\} \\ B_7 = & \{2 & a & \beta & \diamondsuit\} \\ B_8 = & \{2 & b & \gamma & \heartsuit\} \\ B_9 = & \{2 & c & \alpha & \clubsuit\} \end{array} \right\}$$

Theorem 2.2.3 ([73]). *Let r, k be positive integer with $r \geq 3$. A transversal design $T[r, 1; k]$ exists if and only if there are $r - 2$ MOLS of order k.*

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be a $T[r, 1; k]$. For each block $B \in \mathcal{B}$ in the transversal design, use the first pair of symbols in the block to index the position in a Latin square. Then use the remaining $r - 2$ elements in the block B as the entry in the given position for each of the $r - 2$ Latin squares. Similarly, a transversal design can be constructed from a set of MOLS. ☆

Example 2.2.4. For the transversal design in Example 2.2.2 the corresponding pair of MOLS of order 3 are:

$$\begin{array}{c|ccc|} & 0 & 1 & 2 \\ \cline{2-4} a & \alpha & \gamma & \beta \\ b & \beta & \alpha & \gamma \\ c & \gamma & \beta & \alpha \\ \cline{2-4} \end{array} \qquad \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \cline{2-4} a & \heartsuit & \clubsuit & \diamondsuit \\ b & \clubsuit & \diamondsuit & \heartsuit \\ c & \diamondsuit & \heartsuit & \clubsuit \\ \cline{2-4} \end{array}$$

Two well-known generalizations of transversal designs are *transversal covers* and *point-balanced transversal covers*. A transversal cover is a triple $(X, \mathcal{G}, \mathcal{B})$ with the same properties as a transversal design $T[r, 1; k]$ except that the property that any pair of varieties from different groups occur in *exactly one* block is relaxed to any pair of varieties from different groups occur in *at least one* block. Transversal covers are denoted by $TC[r, k]$. The fewest blocks possible in a $TC[r, k]$ is denoted by $tc[r, k]$.

A point-balanced transversal cover is a transversal cover $TC[r, k]$ with the additional property that every variety appears in the same number of blocks. Point-balanced transversal covers are denoted by $PBTC[r, k]$ and the fewest blocks possible in a $PBTC[r, k]$ is denoted by $pbtc[r, k]$.

Stevens, Moura and Mendelsohn [71] prove several bounds for transversal covers and point-balanced transversal covers. Four of these bounds are stated here. These bounds are not stated in full generality; there are refinements that can be made which are not included here.

Lemma 2.2.5 ([71]). *Let r, k be positive integers, then*

1. $\frac{pbtc(r,k)}{k} + k(k-1) \leq tc(r, k) \leq pbtc(r, k)$;
2. $tc(r, k) \geq \left\lceil \frac{k \log_2 r}{2} \right\rceil$;
3. $tc(r, k) \geq k^2 + 2$ *for all* $r \geq k + 2$ *and* $k \geq 3$;
4. *if there exists a* $PBTC(r, k)$ *with* b *blocks, then*

$$r \leq \left\lfloor \frac{\binom{b}{\frac{b}{k}-(k-2)}}{k\binom{\frac{b}{k}}{k-2}} \right\rfloor .$$

2.3 Orthogonal Arrays

This section introduces a third design, orthogonal arrays. The results in this section (and many more which are not included here) can be found in Hedayat, Sloane and Stufken's text, *Orthogonal Arrays* [40].

Definition 2.3.1 (*Orthogonal Array*). Let n, r, k, t be positive integers with $t \leq r$. An *orthogonal array*, $OA(n, r, k, t)$, of index λ, with strength t and alphabet size k, is an $r \times n$ array with entries from $\{0, 1, \ldots, k-1\}$ and the property that any $t \times n$ subarray has all k^t possible t-tuples occurring as columns exactly λ times.

The parameter λ is not included in the notation since in any $OA(n, r, k, t)$, $\lambda = n/k^t$.

The rows of an orthogonal array are also called *factors*. Like MOLS and transversal designs, constructing general orthogonal arrays is difficult. Often the best results for orthogonal arrays are bounds on the sizes of either n or r when all other parameters are fixed.

A strength-2 orthogonal array, $OA(n, r, k, 2)$, is equivalent to a transversal design $T[r, \lambda; k]$, where $\lambda = n/k^2$. For each row r_i of an $OA(n, r, k, 2)$ with $i \in \{0, \ldots, r-1\}$ rewrite each letter a in the row as a_i. Then, the kr varieties in the transversal design are a_i for $a \in \{0, 1, \ldots, k-1\}$ and $i \in \{0, 1, \ldots, r-1\}$. Define the r groups of the transversal design by $G_i = \{a_i \ : \ a = 0, 1, \ldots, k-1\}$ for $i = 0, \ldots, r-1$. Finally, the blocks in the transversal design are the columns in the covering array. Similarly, it is possible to construct an $OA(n, r, k, 2)$ from a $T(r, \lambda; k)$ where $n = k^2\lambda$.

Example 2.3.2. The following $OA(9, 4, 3, 2)$ is equivalent to the transversal design $TD[4, 1; 3]$ from Example 2.2.2 in Section 2.2.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \end{bmatrix}$$

This equivalence between transversal designs and strength-2 orthogonal arrays implies that the existence of an $OA(k^2, r, k, 2)$ is equivalent to the existence of $r - 2$

MOLS of order k. The equivalence between orthogonal arrays, transversal designs and MOLS are summarized in the theorem below.

Theorem 2.3.3 (see [72]). *Let r, k be positive integers with $k \geq 2$ and $r \geq 3$. Then the existence of any one of the following designs implies the existence of the other two designs:*

1. *an $OA(k^2, r, k, 2)$,*
2. *a $T[r, 1; k]$,*
3. *$r - 2$ $MOLS(k)$.*

Results on the largest set of MOLS can be translated to results for orthogonal arrays (for a table of largest known MOLS see Section II.2 of [20]). For example, Inequality (1) from Section 2.1 implies the following corollary.

Corollary 2.3.4. *For k a positive integer, if an $OA(k^2, r, k, 2)$ exists, then $r \leq k+1$.*

Further, Theorem 2.1.5 can be translated to the following corollary. Again, the proof of this is delayed until Section 2.4.1 where a construction for an equivalent design is given.

Corollary 2.3.5. *For k a prime power, there exists an $OA(k^2, k+1, k, 2)$.*

The requirement that k be a prime power can not be dropped from Corollary 2.3.5. For example, since $N(6) = 1$, it is not possible to build an $OA(36, 4, 6, 2)$ so no $OA(36, r, 6, 2)$ for $r \geq 4$ exists.

2.4 Covering Arrays

For fixed positive integers n, k and t, and for some values r, it is impossible to build an orthogonal array $OA(n, k, r, t)$. For example, for an $OA(9, r, 3, 2)$, Inequality (4) from Lemma 2.2.5 shows that

$$r \leq \frac{\binom{9}{\frac{9}{3}-1}}{3\binom{3}{1}} = 4.$$

Thus, it is impossible to build an orthogonal array with $k = 3, t = 2$ and $\lambda = 1$ with more than four rows. It may be possible to build an orthogonal array $OA(9\lambda, 5, 3, 2)$ for some λ sufficiently large, but for many applications increasing the λ produces an array with too many columns. Instead, the condition on the array that each t-tuple occurs *exactly* once can be relaxed to be that each t-tuple occurs *at least* once. These arrays are called *covering arrays*, as all t-tuples are covered in the array.

Definition 2.4.1 (*Covering Array*). Let n, r, k, t be positive integers with $t \leq r$. A *covering array*, t-$CA(n, r, k)$, with strength t and alphabet size k is an $r \times n$ array with entries from $\{0, 1, \ldots, k-1\}$ and the property that any $t \times n$ subarray has all k^t possible t-tuples occurring at least once.

Example 2.4.2. The following array is a 2-$CA(11, 5, 3)$

$$\begin{bmatrix} 0 & 0 & 2 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 1 & 1 & 2 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 2 & 1 & 2 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}.$$

It is possible to construct an $OA(18, 5, 3)$ which has every pair occurring exactly twice, but this would require 7 more columns than the above covering array.

The number of columns, n, in a t-$CA(n, r, k)$ is the *size* of the covering array. For many applications it is best to use the covering array with the smallest size. Thus, the smallest possible size of a covering array is denoted by t-$CAN(r, k)$, that is

$$t\text{-}CAN(r, k) = \min_{l \in \mathbb{N}} \{l : \exists\ t\text{-}CA(l, r, k)\}.$$

A covering array t-$CA(n, r, k)$ with $n = t$-$CAN(r, k)$ is said to be *optimal*.

It is often useful to consider the maximum number of rows possible in a covering array with n columns on a given alphabet. This is denoted by t-$N(n, k)$, that is

$$t\text{-}N(n, k) = \max_{r \in \mathbb{N}} \{r : \exists\ t\text{-}CA(n, r, k)\}.$$

Throughout this thesis only strength-2 covering arrays are considered. Thus, the t will be dropped from the notation, so $CA(n, r, k)$, $CAN(r, k)$ and $N(n, k)$ will be used to denote 2-$CA(n, r, k)$, 2-$CAN(r, k)$ and 2-$N(n, k)$.

Just as strength-2 orthogonal arrays correspond to transversal designs, covering arrays correspond to transversal covers. The bounds from Lemma 2.2.5 apply to covering arrays. In particular, from Inequality (3) of Lemma 2.2.5, $CAN(k+2, k, 2) \geq k^2 + 2$, so the covering array in Example 2.4.2 is an optimal $CA(11, 5, 3)$.

There is a different way to consider the rows of a covering array which we also use throughout this thesis.

Definition 2.4.3 (*Qualitatively Independent Vectors*). Let k, n be positive integers. Two vectors $u, v \in \mathbb{Z}_k^n$ are *qualitatively independent* if for each one of the possible k^2 ordered pairs $(a, b) \in \mathbb{Z}_k \times \mathbb{Z}_k$, there is an index i so $(u_i, v_i) = (a, b)$. A set of vectors is qualitatively independent if any two distinct vectors in the set are qualitatively independent.

The set of rows in a covering array $CA(n, r, k)$ is a set of r pairwise qualitatively independent vectors from $\mathbb{Z}_k^n$.

In Section 3.4.1, qualitatively independent *sets* are considered; these are defined in Definition 3.4.1. Qualitatively independent sets are equivalent to qualitatively independent binary vectors — that is, qualitatively independent vectors from $\mathbb{Z}_2^n$ (see Section 3.4.1). Qualitative independence can also be extended to partitions (Definition 5.2.1), and, for all positive integers k and n, qualitatively independent k-partitions are equivalent to qualitatively independent vectors in $\mathbb{Z}_k^n$ (see Section 5.2.1).

The most commonly cited application of covering arrays is for testing systems (usually software or networks). Assume that the system to be tested has r parameters and that each parameter can take k different values. Each row in the array corresponds to a parameter or a variable in the system. The letters in a k-alphabet, $\{0, 1, \ldots, k-1\}$, correspond to the different values the parameters can be assigned. Each column of the covering array describes the values for the parameters in a test run for the system. The test suite described by all the columns in a covering array will, for every pair of parameters, test all k^2 possible values of those two parameters. This means, any two parameters will be tested completely against one another. There are several references that support the practical effectiveness of test suites that guarantee this pairwise coverage [15, 16, 22, 74, 75].

2.4.1 Constructions for Covering Arrays

In light of this application, there has been much research to find constructions of covering arrays with the fewest possible columns [13, 66, 70]. Colbourn [19] gives a comprehensive survey of results for covering arrays. Determining the exact value of $CAN(r, k)$, for a given r and k, is, in general, a difficult problem. A construction for covering arrays gives an upper bound on $CAN(r, k)$. In this section, three constructions will be given: the finite field construction, the block-size recursive construction and the group construction. This last construction is new and improves many of the upper bounds on the size of covering arrays.

Finite Field Construction

The construction given in this section is the very well-known *finite field construction for orthogonal arrays*. It is essentially this construction that is used to construct a complete set of MOLS of order k, where k is a prime power (Theorem 2.1.5). This construction builds the $OA(k^2, k+1, k, 2)$ from Corollary 2.3.5 which is also a $CA(k^2, k+1, k)$.

Lemma 2.4.4. *Let k be a prime power, then $CAN(k+1, k) = k^2$.*

Proof. We will prove this by constructing a $CA(k^2, k+1, k)$, call this C. In this proof, we will index the rows and columns of C starting from 0.

Let $GF[k]$ be the finite field of order k. Fix an ordering of the elements of $GF[k]$, $\{f_0, f_1, f_2, \ldots, f_{k-1}\}$ with $f_0 = 0$ and $f_1 = 1$.

Let the first row of C be each element in the field repeated k times in the fixed order, so the entry in column c of the first row is f_l, where $l = \lfloor c/k \rfloor$. For $i = 0, \ldots, k-1$, set the entry in row $i+1$ and column c of the covering array to be $f_i f_l + f_j$ where $l = \lfloor c/k \rfloor$ and $j \equiv c \pmod{k}$.

Any row of C is qualitatively independent from any other row of C. First, it is not hard to see that the first row is qualitatively independent from all other rows. Next, assume rows i_0+1 and i_1+1 (for any distinct $i_0, i_1 \in \{0, 1, \ldots, k-1\}$) are not qualitatively independent. Then for columns r and s, a pair is repeated between two distinct rows i_0+1 and i_1+1. In particular, $(C_{i_0+1,r}, C_{i_1+1,r}) = (C_{i_0+1,s}, C_{i_1+1,s})$. Thus,

$$f_{i_0} f_{l_r} + f_{j_r} = f_{i_0} f_{l_s} + f_{j_s},$$

and

$$f_{i_1} f_{l_r} + f_{j_r} = f_{i_1} f_{l_s} + f_{j_s},$$

where $l_r = \lfloor r/k \rfloor$ and $j_r \equiv r \pmod{k}$, and $l_s = \lfloor s/k \rfloor$ and $j_s \equiv s \pmod{k}$.

As $GF[k]$ is a field, $f_{l_r} = f_{l_s}$, or $l_r = l_s$. From this it also follows that $f_{j_r} = f_{j_s}$ and $j_r = j_s$. These together imply that $f_{i_0} = f_{i_1}$ which contradicts $i_0 + 1 \neq i_1 + 1$.

☆

Inequality (1) from Section 2.1 and Lemma 2.4.4 can be restated in terms of $N(n, k)$.

Lemma 2.4.5. *For any positive integer k, $N(k^2, k) \leq k+1$, and for k a prime power, $N(k^2, k) = k + 1$.*

Example 2.4.6. The following array is a $CA(16, 5, 4)$ from the finite field construction on the finite field with four elements, labelled $\{0, 1, 2, 3\}$. This covering array corresponds to the complete set of MOLS of order four in Example 2.1.6 in Section 2.1.

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 \\
0 & 1 & 2 & 3 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 \\
0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1
\end{bmatrix}$$

Note that for each row, other than the first row, the first four entries are (in order) $0, 1, 2, 3$. This leads to the concept of *disjoint columns*. In a covering array, two columns are disjoint if, for each row, the two columns have different entries. An example of two columns that are disjoint is a column of all 0s and a column of all 1s. A covering array has m disjoint columns if there is a set of at least m columns that are pairwise disjoint.

The term "disjoint columns" comes from the correspondence between covering arrays and transversal covers — in particular, each column of a covering array corresponds to a block in a transversal cover. If two columns in the covering array are disjoint, then the two blocks in the transversal cover corresponding to these columns are disjoint sets.

Corollary 2.4.7. *For any prime power k, there exists a covering array $CA(k^2, k, k)$ with k disjoint columns.*

Proof. Let C be the covering array $CA(k^2, k+1, k)$ built by the finite field construction. Let C' be the covering array formed by removing the first row of C. By the finite field construction, for columns $j = 0, \ldots, k-1$ the entry on row i of C' is $f_i 0 + f_j = f_j$. Thus the first k columns of C' are disjoint. ✰

Example 2.4.8. The following array is a $CA(16, 4, 4)$ with four disjoint columns.

$$\left[\begin{array}{|cccc|cccccccccccc}
\hline
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 \\
0 & 1 & 2 & 3 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 \\
0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 \\
\hline
\end{array}\right]$$

Block-size Recursive Construction

The next construction is known as the *block-size recursive construction*; it appears in [60] and [70]. This construction uses two covering arrays with the same alphabet, A a $CA(n, r, k)$ and B a $CA(m, s, k)$, to build a $CA(m+n, rs, k)$. Let a_i for $i = 0, 1, \ldots, r-1$ denote the rows of A, and let b_j for $j = 0, 1, \ldots, s-1$ denote the rows of B. So,

$$A = \overbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{r-1} \end{bmatrix}}^{n \text{ columns}} \quad \text{and} \quad B = \overbrace{\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{s-1} \end{bmatrix}}^{m \text{ columns}}.$$

Construct a $CA(n+m, rs, k)$ as follows. The first s rows of the $CA(n+m, rs, k)$ are row a_0 of A concatenated with row b_j of B for $j = 0, \ldots, s-1$. The next s rows of $CA(n+m, rs, k)$ are row a_1 of A concatenated with row b_j of B for $j = 0, \ldots, s-1$. In general, row t of $CA(n+m, rs, k)$ is row a_i, where $i = \lfloor t/s \rfloor$, concatenated with row b_j, where $j \equiv t \pmod{s}$. Thus,

$$CA(n+m, rs, k) = \overbrace{\left[\begin{array}{c|c} a_0 & b_0 \\ a_0 & b_1 \\ \vdots & \vdots \\ a_0 & b_{s-1} \\ \hline a_1 & b_0 \\ a_1 & b_1 \\ \vdots & \vdots \\ a_1 & b_{s-1} \\ \hline \vdots & \vdots \\ \hline a_{r-1} & b_0 \\ a_{r-1} & b_1 \\ \vdots & \vdots \\ a_{r-1} & b_{s-1} \end{array}\right]}^{n+m \text{ columns}}.$$

This is indeed a covering array. Any two distinct rows t_0 and t_1 of this $CA(m+n, rs, k)$ are of the form $a_{i_0}b_{j_0}$ and $a_{i_1}b_{j_1}$. If $i_0 \neq i_1$ then all possible pairs in the k-alphabet occur in the first n columns between a_{i_0} and a_{i_1}. If $i_0 = i_1$, then $j_0 \neq j_1$ and all possible pairs in the k-alphabet occur in the last m columns between b_{i_0} and b_{i_1}.

Theorem 2.4.9 ([60],[70]). *If there exists a $CA(m, s, k)$ and a $CA(n, r, k)$, then there exists a $CA(m + n, sr, k)$.*

Let C be a $CA(m + n, sr, k)$ built by the block-size recursive construction. For any two rows of C, for all $i \in \{0, \dots, k-1\}$, the pairs $(i, i) \in \mathbb{Z}_k \times \mathbb{Z}_k$ must occur in the first n columns of C. This is clear because each pair of rows has either the first n columns the same or the first n columns are a pair of distinct rows from a covering array. Either way, any pair (i, i) is covered. Similarly, for any two rows of C, the pairs (i, i), for all $i \in \{0, \dots, k-1\}$, also occur in the last m columns.

It is not necessary to cover these pairs twice; if we could remove some of these pairs we could improve the block-size recursive construction. In particular, if one of the covering arrays in the construction has a set of disjoint columns, the letters in each row of this array can be relabelled so that the disjoint columns are constant columns; then these columns can be removed from the final covering array. For example, if the block-size recursive construction is used with the covering array $CA(k^2, k+1, k)$ from the finite field construction (Lemma 2.4.4) and the $CA(k^2, k, k)$ with k disjoint columns (Corollary 2.4.7), all pairs $(i, i) \in \mathbb{Z}_k \times \mathbb{Z}_k$ are covered in the first k^2 columns, so the first k columns of $CA(k^2, k, k)$ do not need to be included in the final covering array.

Example 2.4.10. Let k be a prime power. Denote the elements of the finite field of order k by $\{0, 1, ..., k-1\}$. The following covering array is constructed with the block-size recursive construction using the covering array $CA(k^2, k+1, k)$ from the finite field construction, Lemma 2.4.4, and the $CA(k^2, k, k)$ with k disjoint columns from Corollary 2.4.7. Note that columns $k^2 + 1$ through $k^2 + k$ can be removed so this is a $CA(k^2 - k, k+1, k)$,

$$\left[\begin{array}{llll|llll}
\text{00... 0} & \text{11... 1} & \dots & \text{k-1k-1...k-1} & \cancel{\text{01...k-1}} & \text{01...k-1} & \dots & \text{0 1 ...k-1} \\
\text{00... 0} & \text{11... 1} & \dots & \text{k-1k-1... k-1} & \cancel{\text{01...k-1}} & \text{12... 0} & \dots & \text{k-1 0 ...k-2} \\
\vdots & & & & & & & \\
\text{00... 0} & \text{11... 1} & \dots & \text{k-1k-1... k-1} & \cancel{\text{01...k-1}} & & & \dots \\
\hline
\text{01...k-1} & \text{01...k-1} & \dots & \text{0 1 ... k-1} & \cancel{\text{01... k-1}} & \text{01...k-1} & \dots & \text{0 1 ...k-1} \\
\text{01...k-1} & \text{01...k-1} & \dots & \text{0 1 ... k-1} & \cancel{\text{01...k-1}} & \text{12... 0} & \dots & \text{k-1 0 ...k-2} \\
\vdots & & & & & & & \\
\text{01...k-1} & \text{01...k-1} & \dots & \text{0 1 ... k-1} & \cancel{\text{01...k-1}} & & & \dots \\
\hline
\text{01...k-1} & \dots & & & \cancel{\text{01...k-1}} & \text{01...k-1} & \dots & \text{0 1 ...k-1} \\
\text{01...k-1} & \dots & & & \cancel{\text{01...k-1}} & \text{12... 0} & \dots & \text{k-1 0 ...k-2} \\
\vdots & & & & & & & \\
\text{01...k-1} & \dots & & & \cancel{\text{01...k-1}} & & & \dots
\end{array}\right].$$

The result of this construction is that it is possible to build small covering arrays with a large number of rows when the alphabet is a prime power.

Lemma 2.4.11 ([60], [70]). *For a prime power k, there exists a $CA(2k^2-k, k(k+1), k)$. Equivalently, for any prime power k and any integer $r \leq k(k+1)$,*

$$CAN(r,k) \leq 2k^2 - k.$$

Further, the block-size recursive method applied to the covering array $CA(2k^2-k, k(k+1), k)$, from Lemma 2.4.11 and the covering array $CA(k^2, k, k)$ with k disjoint columns produces a covering array $CA(3k^2-2k, k^2(k+1), k)$. Applying the block-size recursive construction with this new covering array and the array $CA(k^2, k, k)$ with k disjoint columns i times produces a $CA(k^2+i(k^2-k), k^i(k+1), k)$. Moreover, in each row of this covering array, each letter occurs exactly $k+i(k-1)$ times.

A covering array $CA(n, r, k)$ with the property that each letter occurs exactly n/k times in every row is a *balanced covering array.*

Lemma 2.4.12. *For k a prime power and any positive integer i, there exists a balanced covering array $CA(k^2+i(k^2-k), k^i(k+1), k)$. Thus,*

$$CAN(k^i(k+1), k) \leq k^2 + i(k^2-k).$$

Two extensions of this construction are given by Colbourn, Martirosyan, Mullen, Shasha, Sherwood and Yucus [21].

Group Construction of Covering Arrays

For a prime power k we know from Lemma 2.4.4 that there exists $CA(k^2, k+1, k)$ and from Lemma 2.4.11 that there exists a $CA(2k^2-k, k(k+1), k)$. We give here a new construction that can be used to build covering arrays $CA(n, r, k)$, where $k^2 \leq n \leq 2k^2-k$ and $k+1 \leq r \leq k(k+1)$. This construction appeared in [51, 55].

This construction involves selecting a subgroup of the symmetric group on k elements, $G < Sym(k)$, and finding a *starter vector*, $v \in \mathbb{Z}_k^r$, (which depends on the group G). The vector is used to form a circulant matrix M. The group acting on the matrix M produces several matrices which are concatenated to form a covering array. Often it will be necessary to add a small matrix, C, to complete the covering conditions.

This group construction extends the method of Chateauneuf, Colbourn and Kreher [13]. This method has several parameters that depend on each other; the group G, the vector v and the small array C. We first choose a specific group G which will determine C and then, knowing G, we find v by a computer search. To motivate the choice for G, an example of how this method works is given.

Example 2.4.13. Let

$$G = \{e, (12)\} < Sym(3) \text{ and } v = (0, 1, 1, 1, 2) \in \mathbb{Z}_3^5.$$

Build the following circulant matrix taking v as the first column,

$$M = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The elements of G acting on the matrix M produce

$$M_e = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad M_{(12)} = \begin{bmatrix} 0 & 1 & 2 & 2 & 2 \\ 2 & 0 & 1 & 2 & 2 \\ 2 & 2 & 0 & 1 & 2 \\ 2 & 2 & 2 & 0 & 1 \\ 1 & 2 & 2 & 2 & 0 \end{bmatrix}.$$

The vector $C = [0\ 0\ 0\ 0\ 0]^\top$ is needed to ensure the coverage of all pairs.

From this, a $CA(11, 5, 3)$ is built by concatenating the matrices C, M_e and $M_{(1,2)}$

$$\left[\begin{array}{c|ccccc|ccccc} 0 & 0 & 2 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 1 & 1 & 2 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 2 & 1 & 2 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 2 & 0 \end{array}\right].$$

Consider the group action of G on the pairs from $\mathbb{Z}_k$, let $\mathcal{O}$ be the set of orbits from this action. For the group action of G on the matrix M to produce a covering array, it is necessary that for any orbit $O \in \mathcal{O}$, and for any two rows r_0, r_1 of M, there must be at least one column c of M such that $(M_{r_0,c}, M_{r_1,c}) \in O$.

To determine which vectors $v = (v_0, v_1, \ldots, v_{r-1})$ are starter vectors, consider the sets d_i, for $i = 1, \ldots, r-1$,

$$d_i = \{(v_j, v_{j+i})|\ j = 0, 1, \ldots, r-1\}, \tag{2}$$

where the subscripts are taken modulo r.

For v to be a starter vector, each set d_i (for $i = 1, \ldots, r-1$) must contain at least one element from each orbit of the group action of G on the ordered pairs from $\mathbb{Z}_k$.

To produce small covering arrays, the group G should be small and have few orbits. These two properties of the group action are connected by Burnside's Theorem (also known as Cauchy-Frobenius Theorem). This theorem states that if a group G acting on a set X has N orbits, then

$$N = \frac{1}{|G|} \sum_{g \in G} \text{fix}(g)$$

where fix(g) is the number of elements $x \in X$ for which the action of $g \in G$ fixes x (this means $g(x) = x$).

For this construction, it is best to use a group that has $N|G| = \sum_{g \in G} \text{fix}(g)$ small. The group used in both [51] and [55] is the $(k-1)$-element group $G = \langle(1\ 2\ \ldots\ k-1)\rangle < Sym(k)$. This group was chosen because each element, other than the identity, fixes exactly one pair from $\mathbb{Z}_k$, the pair $(0, 0)$.

Consider the action of the subgroup $G = \langle(1\ 2\ \ldots\ k-1)\rangle < Sym(k)$ on the pairs from $\mathbb{Z}_k$. There are $k+2$ orbits generated by this group action:

1. $\{(0, 0)\}$,

2. $\{(0, i)\ :\ i = 1, \ldots, k-1\}$,

3. $\{(i, 0)\ :\ i = 1, \ldots, k-1\}$,

4. $\{(a, b) : a, b \in \mathbb{Z}_k \backslash \{0\} \text{ and } a - b \equiv j \pmod{k-1}\}$ for $j = 0, \ldots, k-2$.

Consider a vector $v = (0, v_1, v_2, \ldots, v_{r-1})$ from $\mathbb{Z}_k^r$ where only the first entry of the vector is 0. Then consider the sets of pairs

$$d_i = \{(v_j, v_{j+i}) | \ 1 \leq j \leq r-1 \text{ and } i + j \neq 0\} \text{ for } 1 \leq i \leq r-1,$$

(where the subscripts are taken modulo r). Let v be a vector in $\mathbb{Z}_k^r$ with the property that the set of differences $v_j - v_{j+i}$ for all $(v_j, v_{j+i}) \in d_i$ covers $\mathbb{Z}_{k-1}$ for every $i = 1, \ldots, r-1$. Then the set

$$d_i^* = \{(v_j, v_{j+i}) | \ j = 0, \ldots, r-1\}, \quad i = 1, \ldots, r-1$$

intersects all the orbits except $\{(0,0)\}$.

Set M to be the $r \times r$ circulant matrix generated from the vector v. Then, for any pair of rows in M, at least one element from each of the orbits of G (except $\{(0,0)\}$) occurs in some column. For each $g \in G$, let M_g be the matrix formed by the action of g on the elements of M. As any two rows in M have a representative from each orbit of G, for any two rows all pairs from $\mathbb{Z}_k$ (except $\{(0,0)\}$) will occur in M_g for some $g \in G$. Finally let C be the $r \times 1$ matrix with all entries equal to 0. Then the array formed by concatenating C and M_g, for all $g \in G$, is a covering array.

Starter vectors exist for many values of r and k. Table 1 is an exhaustive list of all values of r and k with $k \leq 10$ with $k+1 \leq r \leq 2(k+1)$ for which a starter vector exists for the given group G. These were found by an exhaustive search for all values with $k = 3, \ldots, 10$ and all r with $k+1 \leq r \leq 2(k+1)$; if values in this range are not listed in Table 1 then a starter vector does not exist for those values.

Many of these starter vectors give an upper bound on the size of a covering array which is better than the previous best known bound. For example, when $k = 6$ and $r = 9$, a $CA(46, 9, 6)$ can be constructed with this method. Previously, the smallest known covering array with $r = 9$ and $k = 6$ was a $CA(48, 9, 6)$. A table of the new upper bounds obtained with this construction is given in Appendix A.

For values of $k \geq 11$ the exhaustive search was found to be inefficient. Replacing the exhaustive search, a simple hill-climbing algorithm has improved bounds for covering arrays with $11 \leq k \leq 20$ (these bounds are in [51]). These new upper bounds

k	r
3	5,8
4	5–10
5	7–12
6	9–14
7	10–16
8	9, 11–18
9	13–20
10	15–22

Table 1: Exhaustive list of values for which a starter vector exists with $k \leq 10$ and $k+1 \leq r \leq 2(k+1)$.

k	r
11	17
12	18–19
13	21–27
14	23–29
15	27–31
16	29–33
17	33–35
18	35–37

Table 2: A list of values (not exhaustive) for which a starter vector exists with $11 \leq k \leq 18$ and $k+1 \leq r \leq 2(k+1)$.

are listed in also Appendix A. Table 2 gives an non-exhaustive list of values for which a starter vector exists with $11 \leq k \leq 18$ and $k+1 \leq r \leq 2(k+1)$.

The covering arrays that we construct with the group construction using our group G have the property that the k columns that have 0 in the first row are mutually disjoint, except for the first row. Removing the first row from the covering arrays, $CA(n, r, k)$, that we constructed with the group construction method produces a $CA(n, r-1, k)$ with k disjoint columns. This method not only produces small covering arrays, it also produces arrays that can be used in the block-size recursive construction.

For example, if $k = 4$, from Lemma 2.4.12, it is possible to construct a $CA(16+$

$12i, 5(4^i), 4)$, for all positive integers i. From the group construction, there exist covering arrays with four disjoint columns with the following parameters: $CA(19, 5, 4)$, $CA(22, 6, 4)$, $CA(25, 7, 4)$, $CA(28, 8, 4)$ and $CA(31, 9, 4)$. Using these covering arrays with the block-size recursive construction and the $CA(16, 5, 4)$, from Example 2.4.6, Section 2.4.1, it is possible to construct the following covering arrays for all positive integers i: $CA(16 + 22i, 5(6^i), 4)$, $CA(16 + 25i, 5(7^i), 4)$, $CA(16 + 28i, 5(8^i), 4)$ and $CA(16 + 31i, 5(9^i), 4)$.

2.4.2 Asymptotic Results

With the results we have seen so far, it is possible to show one bound on the asymptotic growth of $N(n, k)$. From Lemma 2.4.12, for k a prime power and i a positive integer, there exists a covering array $CA(k^2 + i(k^2 - k), k^i(k + 1), k)$. For any integer i, set $n = k^2 + i(k^2 - k)$, then,

$$N(n, k) \geq k^i(k + 1) > k^{i+1} = k^{(n-k)/(k^2-k)}.$$

To motivate the next result, a short digression on Shannon theory is needed. Shannon theory was developed by C. Shannon [64] and has resulted in many interesting and difficult combinatorial problems. One of the goals of this theory was to develop methods to identify encoded messages that may have some of their bits changed. To do this, we need to be able to identify when two messages are "really different", and not just the same message with some bits changed.

For a graph G with vertex set V, two vectors $v, w \in V^n$ are considered "really different" if for some $i \in \{1, \ldots, n\}$, $(v_i, w_i) \in E(G)$, where $E(G)$ is the edge set of G. Denote by $N(G, n)$ the size of the largest set of vectors from V^n that are pairwise "really different." Then, the *zero error capacity* of a graph G is defined to be

$$C(G) = \limsup_{n\to\infty} \frac{\log_2 N(G, n)}{n}.$$

We consider the same asymptotic growth of $N(n, k)$, that is,

$$\limsup_{n\to\infty} \frac{\log_2 N(n, k)}{n}.$$

When k is a prime power,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\log_2 N(n,k)}{n} &\geq \limsup_{n\to\infty} \frac{\log_2(k^{(n-k)/(k^2-k)})}{n} \\
&= \limsup_{n\to\infty} \frac{n\log_2 k - k\log_2 k}{n(k^2-k)} \\
&= \frac{\log_2 k}{k^2-k}.
\end{aligned}$$

Poljak and Tuza [61] show that for a prime power k, we have the following bounds:

$$\frac{\log_2 k}{k^2-k} \leq \limsup_{n\to\infty} \frac{\log_2 N(n,k)}{n} \leq \frac{2}{k}. \tag{3}$$

In fact, the upper bound holds for all k (this is shown in Section 3.2.1).

Gargano, Körner and Vaccaro show that

$$\limsup_{n\to\infty} \frac{\log_2 N(n,k)}{n} = \frac{2}{k}. \tag{4}$$

The proof of Equation (4) is very complex, non-constructive and is contained in the three papers [28, 29, 30].

Chapter 3

Extremal Set Theory

One of the central problems in extremal set theory is the problem of finding a system of sets with the largest cardinality given some restriction on the sets of the system. Usually the restriction is of the form that any two distinct sets in the system satisfy some constraint.

In this chapter, two such problems are discussed. The problem considered in Section 3.2 is to find the maximum cardinality of a set system, over a finite ground set, satisfying the constraint that any two distinct sets in the system are incomparable (one is not contained in the other). The major result for this type of restriction is Sperner's Theorem. The problem that we describe in Section 3.3 is to find the set system with the largest cardinality satisfying the constraint that any two distinct sets in the system are intersecting. The main result for this type of problem is the Erdős-Ko-Rado Theorem.

Some design theory problems can be rephrased as extremal set theory problems. For example, the minimal size of a binary covering array can be found by using Sperner's Theorem and the Erdős-Ko-Rado Theorem. General covering arrays are not related to set systems, instead they correspond to systems of partitions. In Chapter 7, a theory of extremal partitions is introduced, and several of the results presented in this chapter are extended to include partition systems; such extensions are also known as *higher order* extremal problems.

3.1 Set Systems

For n a positive integer, let $X = \{1, 2, \ldots, n\}$ be an n-set. The *power set* of X, denoted $P(X)$, is the collection of all subsets of X. A *set system* on an n-set is a collection of sets from $P(X)$. For a positive integer $k \leq n$, a *k-set* is a set $A \in P(X)$ with $|A| = k$. The collection of all k-sets of an n-set is denoted by $\binom{[n]}{k}$. A *k-uniform set system* on an n-set is a collection of sets from $\binom{[n]}{k}$. It is possible to arrange $P(X)$ in a poset ordered by inclusion. In this poset, for every positive integer $k \leq n$, the sets $\binom{[n]}{k}$ are the *level sets*. In this section, three types of set systems are defined: t-designs, set partitions and hypergraphs.

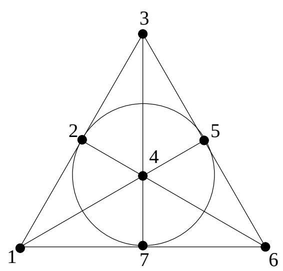
3.1.1 t-Designs

In this section, we introduce t-designs and give some basic results. For more on t-designs, see [8, 72, 73].

Definition 3.1.1 (t-(n, k, λ) *Design*). Let t, n, k and λ be positive integers such that $2 \leq t \leq k < n$. A t-(n, k, λ) *design* is a k-uniform set system $\mathcal{A}$ on an n-set with the property that any t-set of the n-set is contained in exactly λ sets in $\mathcal{A}$.

A t-(n, k, λ) design with $t = 2$ is also known as a (n, k, λ)-BIBD (BIBD stands for *balanced incomplete block design*). The sets in a design are also called the *blocks* of the design.

Example 3.1.2. The most famous t-design is the ubiquitous Fano plane. The Fano plane is a 2-$(7, 3, 1)$ design (and thus a $(7, 3, 1)$-BIBD). The Fano plane is represented in the picture below. The vertices are the elements of the 7-set and each of the blocks is represented by a line in the plane.



That is, the set system $\mathcal{A}$ below is a 2-$(7, 3, 1)$ design,

$$\mathcal{A} = \{\{123\}, \{145\}, \{167\}, \{246\}, \{257\}, \{347\}, \{356\}\}.$$

Theorem 3.1.3 (see [72]). *In a t-(n, k, λ) design there are exactly*

$$b = \frac{\lambda\binom{n}{t}}{\binom{k}{t}}$$

blocks. Moreover, every element of the n-set is contained in exactly

$$r = \frac{kb}{n} = \frac{\lambda\binom{n-1}{t-1}}{\binom{k-1}{t-1}}$$

sets.

Clearly, if a t-(n, k, λ) design exists, then b and r in the above theorem must be integers. This gives a necessary condition for the existence of a t-(n, k, λ) design.

Definition 3.1.4 (*Resolvable Design*). Suppose $\mathcal{A}$ is a t-(n, k, λ) design. A *parallel class* in $\mathcal{A}$ is a collection of pairwise disjoint sets from $\mathcal{A}$ whose union is the entire n-set. A partition of $\mathcal{A}$ into $r = \frac{\lambda\binom{n-1}{t-1}}{\binom{k-1}{t-1}}$ parallel classes is called a *resolution.* A t-(n, k, λ) design is *resolvable* if a resolution exists.

For many values of t, n, k, λ, it is not possible to construct a t-(n, k, λ) design. On the other hand, it is always possible to construct a design called a *packing.*

Definition 3.1.5 (t-(n, k, λ) *packing*). Let t, n, k and λ be positive integers such that $2 \leq t \leq k < n$. A t-(n, k, λ) *packing* is a k-uniform set system $\mathcal{A}$ on an n-set with the property that any t-set of the n-set is contained in at most λ sets in $\mathcal{A}$.

The definition of a *resolvable packing* follows.

Definition 3.1.6 (*Resolvable Packing*). Suppose $\mathcal{A}$ is a t-(n, k, λ) packing. A *parallel class* in $\mathcal{A}$ is a collection of pairwise disjoint sets from $\mathcal{A}$ whose union is the entire n-set. A partition of $\mathcal{A}$ into parallel classes is called a *resolution.* A t-(n, k, λ) packing is *resolvable* if a resolution exists.

3.1.2 Set Partitions

A *set partition* of an n-set is a set of disjoint non-empty subsets (called classes) of the n-set whose union is the n-set. Throughout this thesis, set partitions will be referred to simply as *partitions*. A partition P is called a *k-partition* if it contains k classes, that is $P = \{P_1, P_2, \ldots, P_k\}$, or, equivalently, $|P| = k$. For positive integers k, n, let $\mathcal{P}_k^n$ denote the set of all k-partitions of an n-set. Let $S(n,k) = |\mathcal{P}_k^n|$; the values $S(n,k)$ are the *Stirling numbers of the second type*.

A partition $P \in \mathcal{P}_k^n$ is *uniform* if every class $P_i \in P$, $i = 1, \ldots, k$, has the same cardinality, that is, $|P_i| = n/k$ for all $P_i \in P$. For positive integers n, c, k with $n = ck$, denote by $\mathcal{U}_k^n$ the set of all uniform k-partitions in $\mathcal{P}_k^n$. Analogous to the Stirling numbers of the second type, denote $U(n,k) = |\mathcal{U}_k^n|$, that is, for $n = ck$,

$$U(n,k) = \frac{1}{k!}\binom{n}{c}\binom{n-c}{c}\cdots\binom{n-(k-1)c}{c}.$$

If k does not divide n, it is not possible for a partition in $\mathcal{P}_k^n$ to be uniform, in this case *almost-uniform partitions* are considered. For positive integers n, k, c with $n = ck + r$ where $0 \leq r < k$, a partition $P \in \mathcal{P}_k^n$ is *almost-uniform* if every class of P has cardinality c or $c+1$. Note that in an almost-uniform partition, there are r classes of cardinality $c+1$ and $k-r$ classes of cardinality c. Denote by $\mathcal{AU}_k^n$ the set of all almost-uniform partitions in $\mathcal{P}_k^n$. Denote $AU(n,k) = |\mathcal{AU}_k^n|$. For $n = ck + r$,

$$\begin{aligned} AU(n,k) \quad &= \quad \frac{1}{r!(k-r)!}\binom{n}{c}\binom{n-c}{c}\cdots\binom{n-(k-r-1)c}{c} \\ &\qquad \binom{n-(k-r)c}{c+1}\binom{n-(k-r)c-(c+1)}{c+1}\cdots\binom{c+1}{c+1}. \end{aligned}$$

If k divides n, then $\mathcal{AU}_k^n = \mathcal{U}_k^n$ and $U(n,k) = AU(n,k)$.

3.1.3 Hypergraphs

Hypergraphs are generalizations of graphs, these will be used in Section 7.1. A *hypergraph* on n vertices is a collection of subsets of an n-set. The elements of the n-set are the *vertices* of the hypergraph and the subsets are the *edges* of the hypergraph. A hypergraph is *uniform* if all the edges in the hypergraph have the same cardinality. A uniform hypergraph with all edges of cardinality two is a graph. For positive integers n and c with $c \leq n$, the *complete c-uniform hypergraph*, denoted $K_n^{(c)}$, has vertex set $\{1, 2, \dots, n\}$ and edge set all c-sets from the n-set, that is, $\binom{[n]}{c}$.

The *degree* of a vertex in a hypergraph is the number of edges that contain the vertex. If every vertex in a hypergraph has degree d, then the hypergraph is *d-regular*.

A *1-factor* of a hypergraph on an n-set is a set of disjoint edges whose union is the entire n-set. A *1-factorization* of a hypergraph is a partition of its edges into a collection of pairwise disjoint 1-factors. The *size of a 1-factorization* is the number of 1-factors in the 1-factorization. For positive integers c, n, if c divides n, then a 1-factorization of $K_n^{(c)}$ is possible.

Theorem 3.1.7 (1-factorization of hypergraphs [10]). *Let c, n be positive integers such that c divides n, then the hypergraph $K_n^{(c)}$ has a 1-factorization.*

Let c, k, n be positive integers such that $n = ck$, the size of a 1-factorization of $K_n^{(c)}$ is $\frac{1}{k}\binom{n}{c} = \binom{n-1}{c-1}$.

Assume for positive integers k, c, n that $n = ck$. A 1-factor of the complete uniform hypergraph $K_n^{(c)}$ is equivalent to a uniform k-partition of an n-set. Further, for any $t \leq c$, a 1-factorization of the complete hypergraph $K_n^{(c)}$ corresponds to a resolvable t-(n, c, λ) design with $\lambda = \binom{n-t}{c-t}$. In particular, it is a resolvable (n, c, λ)-BIBD, where $\lambda = \binom{n-2}{c-2}$.

3.2 Sperner Theory

An important class of problems in extremal set theory deals with the maximum cardinality of a set system with some restriction on the sets in the system. The first restriction that we consider is that any two distinct sets from the system must be *incomparable*. Two subsets A and B of an n-set are *comparable* if $A \subseteq B$ or $B \subseteq A$. If A and B are not comparable then they are *incomparable*.

Definition 3.2.1 (*Sperner Set System*). Let n be a positive integer. A *Sperner set system* $\mathcal{A}$ on an n-set is a set system on an n-set with the property that any two distinct sets in $\mathcal{A}$ are incomparable.

In 1928, Sperner proved that a Sperner set system cannot have more sets than are in the largest level set [67]. We give the proof of this here since a similar proof is used in Lemma 3.3.2 and Theorem 5.3.3.

Theorem 3.2.2 (Sperner's Theorem [67]). *Let n be a positive integer. If $\mathcal{A}$ is a Sperner set system on an n-set, then $|\mathcal{A}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Proof. A chain in a poset is a collection of sets in the poset with the property that any two distinct sets in the chain are ordered in the poset. We will show that the poset formed by $P(\{1, 2, \ldots, n\})$ ordered by inclusion can be decomposed into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ disjoint chains. Any Sperner system can intersect such a chain in at most one set, and thus, has cardinality no more than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Let r be a positive integer with $r < \lfloor \frac{n}{2} \rfloor$. We will show that every r-set can be matched to an $(r+1)$-set such that the r-set is a subset of the $(r+1)$-set. Construct a bipartite graph as follows. For every r-set of the n-set there is a corresponding vertex in the first part of the graph. For each $(r+1)$-set in the n-set there is a corresponding vertex in the second part of the graph. Two vertices are adjacent in this bipartite graph if and only if one of the corresponding sets is contained in the other set. All the vertices in the first part of the graph have degree $n - r$ and the vertices in the second part all have degree $r + 1$. Let S be a set of vertices from the first part of the graph. Let $N(S)$ be the set of vertices in the second part of the graph adjacent to any vertex in S. Since every vertex in the first part of the graph has degree $n - r$ and every vertex in the second part of the graph has degree $r + 1$, counting the number of edges between S and $N(S)$ we have that $|S|(n - r) \le |N(S)|(r + 1)$. Since $\frac{r+1}{n-r} \le 1$, we have that $|S| \le |N(S)|$ and by Hall's Theorem, there is a one-to-one matching from the first part of this graph to the second part. Thus, there is a one-to-one matching from $\binom{[n]}{r}$ to $\binom{[n]}{r+1}$ for $r \le \lfloor \frac{n}{2} \rfloor$.

Similarly, for any positive integer r with $r > \lfloor \frac{n}{2} \rfloor$, there is a one-to-one matching from $\binom{[n]}{r}$ to $\binom{[n]}{r-1}$. Finally, by the same argument, if n is odd, there is one-to-one matching from $\binom{[n]}{\frac{n-1}{2}}$ to $\binom{[n]}{\frac{n+1}{2}}$.

Two sets are in the same chain if they are matched in one of these matchings. Then these matching define $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ disjoint chains which partition the poset, since each chain has exactly one set in the set $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$. ✰

Moreover, $|\mathcal{A}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ if and only if $\mathcal{A} = \binom{[n]}{k}$ where $k = \lfloor \frac{n}{2} \rfloor$ or $k = \lceil \frac{n}{2} \rceil$.

A sharper inequality is the *LYM Inequality* named after Lubell [48], Meshalkin [56] and Yamamoto [77], who each independently established the result.

Theorem 3.2.3 (LYM Inequality). *Let n be a positive integer and $\mathcal{A}$ be a Sperner set system on an n-set. Then*

$$\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \leq 1.$$

The LYM Inequality implies Sperner's Theorem. Since for all $k \leq n$, $\binom{n}{k} \leq \binom{n}{\lceil \frac{n}{2} \rceil}$, we have that for a Sperner set system $\mathcal{A}$ on an n-set

$$|\mathcal{A}| \binom{n}{\lceil \frac{n}{2} \rceil}^{-1} \leq \sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \leq 1.$$

The LYM Inequality can be rearranged into another form. Let p_i denote the number of subsets in $\mathcal{A}$ of size i, then

$$\sum_{i=1}^{n} \frac{p_i}{\binom{n}{i}} \leq 1. \tag{5}$$

3.2.1 Bollobás's Theorem

An important theorem on set systems is Bollobás's Theorem. This theorem implies Sperner's Theorem and the LYM Inequality. Further, it is an example of a *higher order* Sperner Theorem, meaning that it gives a Sperner-like result for a system of families of sets rather than systems of sets.

Theorem 3.2.4 (Bollobás's Theorem [9]). *Let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ be sequences of sets such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Then*

$$\sum_{i=1}^{m} \binom{a_i + b_i}{a_i}^{-1} \leq 1,$$

where $a_i = |A_i|$ and $b_i = |B_i|$.

In [43] it is noted that the natural extension of this theorem to three families of sets is not true. In Section 7.1, we give a different higher order version of Sperner's Theorem.

Recall that for given positive integers n and k, $N(n,k)$ is the largest r such that a $CA(n,r,k)$ exists. Bollobás's Theorem was used by Stevens, Moura and Mendelsohn [71] and by Poljak and Tuza [61] (Equation (3), Section 2.4.2) to find an upper bound on $N(n,k)$.

Let C be a $CA(n,r,k)$. For each i, $1 \le i \le r$, let k_i^1 be a letter which occurs least often in row i of C and k_i^2 be a letter which occurs second least often in row i of C, where ties are broken arbitrarily. Define two subsets A_i^1 and A_i^2 by $j \in A_i^1$ if and only if k_i^1 occurs in column j and $j \in A_i^2$ if and only if k_i^2 occurs in column j.

Define sets A_i and B_i for $i = 1, \dots, 2r$ by $A_i = A_i^1$ for $1 \le i \le r$ and $A_i = A_{i-r}^2$ for $r+1 \le i \le 2r$ and $B_i = A_i^2$ for $1 \le i \le r$ and $B_i = A_{i-r}^1$ for $r+1 \le i \le 2r$.

For all $i \in \{1, \dots, 2r\}$, $|A_i| \le \lfloor n/k \rfloor$, $|A_i| + |B_i| \le \lfloor 2n/k \rfloor$ and $A_i \cap B_i = \emptyset$. For all distinct $i, j \in \{1, \dots, 2r\}$, $A_i \cap B_j \ne \emptyset$. By Bollobás's Theorem, we have $2r \le \binom{\lfloor 2n/k \rfloor}{\lfloor n/k \rfloor}$, and using Stirling's formula, $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, we obtain

$$N(n,k) \le \frac{1}{2}\binom{\lfloor 2n/k \rfloor}{\lfloor n/k \rfloor} = O\left(\frac{4^{n/k}}{n^{1/2}}\right).$$

Then

$$\limsup_{n\to\infty} \frac{\log_2 N(n,k)}{n} \le \limsup_{n\to\infty} \frac{1}{n}\log_2\left(\frac{4^{n/k}}{n^{1/2}}\right) = \limsup_{n\to\infty} \frac{1}{n}\left(\frac{2n}{k} - \frac{1}{2}\log_2 n\right) = \frac{2}{k}.$$

As stated in Section 2.4.2, Gargano, Körner and Vaccaro [28, 29, 30] show that for all $k \ge 2$,

$$\limsup_{n\to\infty} \frac{\log_2 N(n,k)}{n} = \frac{2}{k}.$$

3.3 The Erdős-Ko-Rado Theorem

The Erdős-Ko-Rado Theorem is a key result in extremal set theory. It is a theorem that has many proofs, applications and extensions. We state this theorem without a proof, a particularly nice proof can be found in [44]. There are many generalizations of this theorem; two such results will be stated in Section 3.3.3. In Section 3.4, we show how the Erdős-Ko-Rado Theorem, with Sperner's Theorem, can be used to find the exact value of $CAN(r, 2)$ for all r.

The Erdős-Ko-Rado Theorem has been extended to combinatorial objects other than sets. Rands in 1982 [62] gives a similar result for intersecting blocks of t-(v, k, λ) designs. In Section 7.2, we give an extension of the Erdős-Ko-Rado Theorem to uniform partition systems.

3.3.1 Intersecting Set Systems

For fixed positive integers t, k, n, let $I(t, k, n)$ denote the collection of all set systems $\mathcal{A}$ on an n-set with the following properties: for all $A \in \mathcal{A}$, $|A| \leq k$; and for all distinct $A, B \in \mathcal{A}$, $A \not\subset B$; for all $A, B \in \mathcal{A}$, $|A \cap B| \geq t$.

The set systems in $I(t, k, n)$ are also known as t*-intersecting set systems* and if $t = 1$, these are also called *intersecting set systems*. The requirement that for all distinct sets A, B in a system that $A \not\subset B$ forces all set systems in $I(t, k, n)$ to be Sperner set systems. If $2k - t \geq n$, then any two k-sets from the n-set have at least t elements in common.

Definition 3.3.1 (k-*Uniform* t-*Intersecting Set System*). For positive integers n, k, t a k*-uniform* t*-intersecting set system* is a k-uniform set system, $\mathcal{A}$, on an n-set with the property that for all distinct $A, B \in \mathcal{A}$, we have $A \not\subset B$; and for all $A, B \in \mathcal{A}$, we have $|A \cap B| \geq t$.

For n sufficiently large, if there exists a t-intersecting set system on an n-set, then there exists a k-uniform t-intersecting set system that has at least the same cardinality. In particular, it is possible to replace each set of size less than k in a t-intersecting set system on an n-set by a set of size k, which is t-intersecting with all the sets in the system. The proof of this is not included since it is similar to the proof of Sperner's Theorem.

Lemma 3.3.2 ([24]). *If $2k \leq n$, then for any $\mathcal{A} \in I(t, k, n)$ there exists a k-uniform t-intersecting set system $\mathcal{A}' \in I(t, k, n)$ with $|\mathcal{A}| \leq |\mathcal{A}'|$.*

For positive integers n, k, t with $t \leq k \leq n$, a set system on an n-set is a *k-uniform trivially t-intersecting set system* if it is equal, up to a permutation on $\{1, \ldots, n\}$, to

$$\mathcal{A} = \left\{ A \in \binom{[n]}{k} : \{1, 2, \ldots, t\} \subseteq A \right\}.$$

The cardinality of a k-uniform trivially t-intersecting set system is $\binom{n-t}{k-t}$. If $t = 1$ then a k-uniform trivially t-intersecting set system is simply called a *k-uniform trivially intersecting set system.*

3.3.2 The Erdős-Ko-Rado Theorem

The Erdős-Ko-Rado Theorem proves that the set system in $I(t, k, n)$ with the largest cardinality, provided that n is sufficiently large, is a k-uniform trivially t-intersecting set system.

Theorem 3.3.3 (Erdős, Ko and Rado [24]). *Let k and t be positive integers, with $0 < t < k$. There exists a function $f(t, k)$ such that if n is a positive integer with $n > f(t, k)$, then for any $\mathcal{A} \in I(t, k, n)$*

$$|\mathcal{A}| \leq \binom{n-t}{k-t}.$$

Moreover, equality holds if and only if $\mathcal{A}$ is a k-uniform trivially t-intersecting set system.

Erdős, Ko and Rado, in 1961, proved that $f(t, k) \leq t + (k-t)\binom{k}{t}^3$ and stated this bound was certainly not the best possible. In 1984, Wilson [76] improved this to the exact bound $f(t, k) = (t+1)(k-t+1)$. It is interesting to note that Wilson's proof is very algebraic; it uses the eigenvalues of the Johnson scheme (see Example 4.2.9, Section 4.2.3) and ratio bounds (see Section 4.2.4).

3.3.3 Generalizations of the Erdős-Ko-Rado Theorem

In this section, we state two extensions of the Erdős-Ko-Rado Theorem. The first, by Hilton and Milner [42], gives the largest non-trivial intersecting set system for

$n \geq 2k$. For the proof of Proposition 7.4.3, we use a partition system similar to the non-trivial set system given by Hilton and Milner. The second extension of the Erdős-Ko-Rado Theorem is by Ahlswede and Khachatrian [4]. In 1997, they gave a complete description of all the maximal k-uniform t-intersecting set systems for all values of n, k and t. Conjecture 7.4.2 in Section 7.4 is a generalization of this result by Ahlswede and Khachatrian to partition systems.

Theorem 3.3.4 (Hilton and Milner [42]). *Let k and n be positive integers with $2 \leq k \leq \frac{n}{2}$. Let $\mathcal{A} \in I(1, k, n)$ such that $\cap_{A \in \mathcal{A}} A = \emptyset$. Then*

$$|\mathcal{A}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

It is not difficult to construct an intersecting k-set system $\mathcal{A}$ with cardinality $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ and $\cap_{A \in \mathcal{A}} A = \emptyset$. For $2 \leq k \leq \frac{n}{2}$, define a set system $\mathcal{A}'$ by $A \in \mathcal{A}'$ if $1 \in A$ and $A \cap \{2, 3, \ldots, k+1\} \neq \emptyset$. Define $\mathcal{A} = \mathcal{A}' \cup \{\{2, 3, \ldots, k+1\}\}$. Then $|\mathcal{A}| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ and $\cap_{A \in \mathcal{A}} A = \emptyset$.

Theorem 3.3.5 (Ahlswede and Khachatrian [4]). *Let t, k, n be positive integers with $1 \leq t \leq k \leq n$ and r be an integer with $r \leq k - t$. Define*

$$\mathcal{A}_r = \left\{ A \in \binom{[n]}{k} : |A \cap \{1, 2, \ldots, t + 2r\}| \geq t + r \right\}.$$

If

$$(k - t + 1)\left(2 + \frac{t-1}{r+1}\right) < n < (k - t + 1)\left(2 + \frac{t-1}{r}\right),$$

then $\mathcal{A}_r$ is the unique (up to a permutation on $\{1, \ldots, n\}$) system in $I(t, k, n)$ with maximal cardinality. By convention, $\frac{t-1}{r} = \infty$ for $r = 0$.

If

$$n = (k - t + 1)\left(2 + \frac{t-1}{r+1}\right),$$

then $|\mathcal{A}_r| = |\mathcal{A}_{r+1}|$ and these two systems are the unique (up to a permutation on $\{1, \ldots, n\}$) systems with maximal cardinality in $I(t, k, n)$.

Note that $\mathcal{A}_0$ is a trivially t-intersecting set system. Thus, when $r = 0$, Theorem 3.3.5 gives the Erdős-Ko-Rado Theorem with the exact lower bound for n.

3.4 Application of the Erdős-Ko-Rado Theorem

Sperner's Theorem and the Erdős-Ko-Rado Theorem can be used to find the exact value of $CAN(r, 2)$ for all r. In 1973, both Katona [45] and Kleitman and Spencer [47] gave similar proofs of this. The proof given here is closer to the proof given by Kleitman and Spencer.

3.4.1 Qualitatively Independent Subsets

Definition 3.4.1 (*Qualitatively Independent Subsets*). Two subsets A and B of an n-set are *qualitatively independent subsets* if

$$A \cap B \neq \emptyset, \quad A \cap \overline{B} \neq \emptyset, \quad \overline{A} \cap B \neq \emptyset, \quad \overline{A} \cap \overline{B} \neq \emptyset.$$

The term "qualitative independence" comes from probability theory: if two sets A and B are qualitatively independent, then the information that $x \in A$ gives no information about whether or not $x \in B$.

Definition 3.4.1 for qualitatively independent sets is equivalent to Definition 2.4.3 for binary vectors. To see this, let A and B be qualitatively independent subsets of an n-set. Define the vector corresponding to A to be $u \in \mathbb{Z}_2^n$, with $u = (u_1, u_2, \ldots, u_n)$, $u_k = 1$ if $k \in A$ and $u_k = 0$ otherwise. Similarly let $v \in \mathbb{Z}_2^n$ be the vector corresponding to the set B. Since A and B are qualitatively independent, $A \cap B \neq \emptyset$. So there exists some $i \in A \cap B$, and so $(u_i, v_i) = (1, 1)$. Further, since $A \cap \overline{B} \neq \emptyset$, there exists some $j \in A \cap \overline{B}$, and so $(u_j, v_j) = (1, 0)$. Similarly, for $k \in \overline{A} \cap B$ we get $(u_k, v_k) = (0, 1)$ and for $l \in \overline{A} \cap \overline{B}$ we get $(u_l, v_l) = (0, 0)$. Thus the vectors u and v are qualitatively independent.

Conversely, if $u, v \in \mathbb{Z}_2^n$ are qualitatively independent, then the sets A and B, defined by $i \in A$ if and only if $u_i = 1$ and $i \in B$ if and only if $v_i = 1$, are also qualitatively independent.

Theorem 3.4.2 ([45, 47]). *If $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ is a qualitatively independent set system of an n-set, then*

$$|\mathcal{A}| \leq \binom{n-1}{\lfloor n/2 \rfloor - 1}.$$

Further, this bound is attained by a $\lfloor n/2 \rfloor$-uniform trivially 1-intersecting set system.

Proof. First, assume n is even. Define

$$\mathcal{A}^* = \{A_i, \overline{A_i} \ : \ A_i \in \mathcal{A}\}.$$

The system $\mathcal{A}^*$ is a Sperner set system so, $|\mathcal{A}^*| \leq \binom{n}{n/2}$. Hence $|\mathcal{A}| \leq 1/2\binom{n}{n/2} = \binom{n-1}{n/2-1}$. This bound is attained by the set system

$$\mathcal{A} = \{A \in \binom{[n]}{\frac{n}{2}} : 1 \in A\}.$$

Next, assume n is odd. If $A_i \in \mathcal{A}$ and $|A_i| \geq n/2$ then replace A_i with $\overline{A_i}$. This does not affect the pairwise qualitative independence of $\mathcal{A}$. So, we can assume that each $A_i \in \mathcal{A}$ has $|A_i| \leq \lfloor n/2 \rfloor$.

By the definition of qualitative independence, $\mathcal{A}$ is a 1-intersecting set system and by the Erdős-Ko-Rado Theorem, $|\mathcal{A}| \leq \binom{n-1}{\lfloor n/2 \rfloor -1}$. This bound is attained by the set system

$$\mathcal{A} = \{A \in \binom{[n]}{\frac{n-1}{2}} : 1 \in A\}.$$

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From Theorem 3.4.2, the exact size of the optimal binary covering array with r rows can be found for all r.

Theorem 3.4.3 ([47]). *Let r be a positive integer, then*

$$CAN(r, 2) = \min \left\{ n : \binom{n-1}{\lfloor n/2 \rfloor - 1} \geq r \right\}.$$

Chapter 4

Graph Theory

As stated in Section 2.4, covering arrays can be used to design test suites for systems and networks. To improve efficiency in such applications, several variations on covering arrays have been considered, such as mixed-level covering arrays and variable-strength covering arrays [17, 39, 57]. The generalization considered in this thesis adds a graph structure to the covering array, obtaining a *covering array on a graph*. These are defined in Chapter 5, where we also define a family of graphs called the *qualitative independence graphs*. These graphs are used to find bounds both on covering arrays on graphs and covering arrays. For this reason, we need to introduce graph concepts that will be used throughout this thesis.

The first half of this chapter is a review of basic graph theory. The second half is a review of algebraic graph theory, which includes spectral theory of graphs and association schemes. These will be used in Chapter 6 in the study of the qualitative independence graphs. Association schemes at first look seem to be purely algebraic objects, but they are very closely related to designs. In fact, they were first defined by Bose and Shimamoto in 1952 [11] to generalize block designs. For more on the history of block designs and association schemes see Chapter 13 of Bailey's *Association Schemes: Designed Experiments, Algebra and Combinatorics* [6].

4.1 Basic Graph Theory

Throughout this chapter, G will denote a simple graph, unless otherwise stated. The vertex set of G will be denoted by $V(G)$, and the edge set will be denoted by $E(G)$.

4.1.1 Graph Homomorphism

This section introduces graph homomorphism; for more information on graph homomorphism see [34, 37, 41].

Definition 4.1.1 (*Graph Homomorphism*). Let G and H be graphs. A mapping ϕ from $V(G)$ to $V(H)$ is a *graph homomorphism* from G to H (or simply a *homomorphism*) if for all $v, w \in V(G)$, the vertices $\phi(v)$ and $\phi(w)$ are adjacent in H whenever v and w are adjacent in G.

For graphs G and H, if there exists a homomorphism from G to H, then we write $G \to H$.

For graphs G and H, a map ϕ from $V(G)$ to $V(H)$ is a *graph isomorphism* (or simply an *isomorphism*) if ϕ is a bijection such that $x, y \in V(G)$ are adjacent in G if and only if $\phi(x)$ and $\phi(y)$ are adjacent in H. If there exists an isomorphism between two graphs, then we say the graphs are *isomorphic*.

A homomorphism from a graph G to itself is a *graph endomorphism* (or an *endomorphism*). An isomorphism from a graph G to itself is a *graph automorphism* (or an *automorphism*). The *automorphism group* for a graph G is the group of all automorphisms of G; it is denoted $Aut(G)$.

A *fibre* of a homomorphism $\phi : G \to H$ is the preimage $\phi^{-1}(w)$ of some vertex $w \in V(H)$, that is,

$$\phi^{-1}(w) = \{v \in V(G) \ : \ \phi(v) = w\}.$$

4.1.2 Colourings, Cliques and Independent Sets

The *complete graph on n vertices*, K_n, is the graph with n vertices and with an edge between any two distinct vertices.

A *proper colouring* of G with n colours is a map from $V(G)$ to a set of n colours such that no two adjacent vertices are assigned the same colour. A proper colouring of a graph G with n colours is equivalent to a homomorphism from G to K_n. The *chromatic number* of a graph G, denoted $\chi(G)$, is the smallest n such that $G \to K_n$.

A *clique* in a graph G is a set of vertices from $V(G)$ in which any two distinct vertices are adjacent in G. The size of a maximum clique in G is denoted by $\omega(G)$. If G has a clique of size n, then there is a homomorphism $K_n \to G$, and the size of a maximum clique in G is the largest n for which $K_n \to G$. An *n-clique* is a clique of size n.

For graphs G and H, if there is a homomorphism $G \to H$, then $\chi(G) \leq \chi(H)$ and $\omega(G) \leq \omega(H)$. Also, for all graphs G, $\omega(G) \leq \chi(G)$.

An *independent set* in a graph G is a set of vertices from $V(G)$ in which no two vertices are adjacent in G. The size of a largest independent set in a graph G is denoted by $\alpha(G)$.

The vertices which are assigned the same colour in a proper colouring form an independent set. In fact, a proper colouring on a graph G partitions the vertices of G into independent sets called *colour classes*. A proper colouring corresponds to a binary function on the independent sets of a graph: each independent set that is a colour class in the proper colouring is assigned a value of 1 and all other independent sets are assigned a value of 0 by the function. Further, each vertex is in exactly one independent set which has an assigned value of 1. The fractional relaxation of this binary function is used in Section 4.1.5 to define a fractional colouring.

4.1.3 Vertex-Transitive Graphs

The *degree* of a vertex $v \in V(G)$ is the number of vertices in G which are adjacent to v. If every vertex in G has the same degree, then we say that G is *regular*. Specifically, if every vertex in G has degree k, we say G is *k-regular*.

A graph is *vertex transitive* if its automorphism group acts transitively on the set of vertices. This means that for any two distinct vertices, there is an automorphism on the graph that maps one vertex to the other. If a graph is vertex transitive, then it must also be regular.

There are many other types of transitivity. One example is *arc transitivity*. An *arc* in a graph is an ordered pair of adjacent vertices. A graph is arc transitive if its automorphism group acts transitively on the arcs of the graph. This means for any two arcs in the graph there is an automorphism that maps one arc to the other.

4.1.4 Cores of a Graph

Definition 4.1.2 (*Core Graph*). A graph G is a *core* if any endomorphism on G is an automorphism.

If a graph G is a core, then there is no homomorphism from G to a proper subgraph of G. Indeed, for any positive integer n, the complete graph K_n is a core.

Definition 4.1.3 (*Core of a Graph*). A *core of a graph* G is a subgraph $G^\bullet$ of G such that $G^\bullet$ is a core and there is a homomorphism $G \to G^\bullet$.

If $G_1^\bullet$ and $G_2^\bullet$ are both cores of a graph G, then $G_1^\bullet$ and $G_2^\bullet$ are isomorphic (see Lemma 6.2.2, [34]). Further, a core of a graph preserves many important properties of the graph. For example,

$$\chi(G) = \chi(G^\bullet) \text{ and } \omega(G) = \omega(G^\bullet).$$

Any automorphism on G induces an automorphism on $G^\bullet$, so a core of a graph also preserves vertex transitivity.

Theorem 4.1.4 ([37]). *If G is a vertex-transitive graph, then $G^\bullet$ is also vertex transitive.*

If G is a vertex-transitive graph, then for any homomorphism from G to $G^\bullet$ each fibre of the homomorphism has the same cardinality. This means that the preimage of every vertex in $G^\bullet$ has the same cardinality. From this we can conclude the following theorem.

Theorem 4.1.5 ([37]). *If G is a vertex-transitive graph, then $|V(G^\bullet)|$ divides $|V(G)|$.*

4.1.5 Kneser Graphs

In this section, a family of vertex-transitive graphs, the Kneser graphs, is defined. The vertex set of a Kneser graph is the set of all r-subsets of an n-set, so this graph is closely related to set systems. In Section 5, we introduce a similar family of graphs, called the *qualitative independence graphs*, whose vertex set is a set of partitions rather than sets.

Definition 4.1.6 (*Kneser Graphs*). For positive integers r, n with $r \leq n$, the *Kneser graph* $K_{n:r}$ is the graph whose vertex set is $\binom{[n]}{r}$ and r-subsets are adjacent if and only if they are disjoint.

Kneser graphs are related to a generalization of proper colouring called *fractional colouring*. From Section 4.1.2, a proper colouring of a graph is usually considered to be a homomorphism from G to a complete graph, but it can also be considered a binary function on the independent sets of the graph. A fractional colouring is also a function on the independent sets of a graph, but the requirement that the function be binary is relaxed to that the function be non-negative.

For a graph G with $v \in V(G)$, define $\mathcal{I}(G, v)$ to be the set of all independent sets in G that contain the vertex v, and define $\mathcal{I}(G)$ to be the set of all independent sets in G.

Definition 4.1.7 (*Fractional Colouring*). A *fractional colouring* of a graph G is a non-negative function f on the independent sets of G with the property that for any vertex $v \in V(G)$,

$$\sum_{S \in \mathcal{I}(G,v)} f(S) \geq 1.$$

Definition 4.1.8 (*Fractional Chromatic Number*). Let G be a graph and f a fractional colouring on G. The *weight* of f is the sum of the values of f over all independent sets in G, that is, $\sum_{S \in \mathcal{I}(G)} f(S)$. The *fractional chromatic number* of G, denoted $\chi^*(G)$, is the minimum weight over all fractional colourings of G.

The connection between fractional chromatic number and Kneser graphs is that, like the chromatic number, the fractional chromatic number can be determined by a homomorphism. While the chromatic number is determined by homomorphisms to complete graphs, the fractional chromatic number is determined by homomorphisms to Kneser graphs.

Theorem 4.1.9 (see Theorem 7.4.5, [34]). *For any graph G,*

$$\chi^*(G) = \min\left\{\frac{n}{r} : G \to K_{n:r}\right\}.$$

For any graph G, the fractional chromatic number satisfies the following bounds:

$$\omega(G) \leq \chi^*(G) \leq \chi(G). \tag{6}$$

The fractional chromatic number, like the chromatic number and maximum clique size, is preserved by cores, that is, for any graph G,

$$\chi^*(G) = \chi^*(G^\bullet). \tag{7}$$

For the special case when G is a vertex-transitive graph, there is another formula for $\chi^*(G)$. This formula, is derived from the fact that there exists a fractional colouring with the minimum weight that assigns the same value to each maximum independent set in G.

Corollary 4.1.10 (see Chapter 7, [34]). *If G is a vertex-transitive graph, then*

$$\chi^*(G) = \frac{|V(G)|}{\alpha(G)}.$$

For a vertex-transitive graph G, this can be used with Inequality (6) as a bound on the size of a maximum clique in G

$$\omega(G) \leq \chi^*(G) = \frac{|V(G)|}{\alpha(G)}. \tag{8}$$

The next theorem states that for $n > 2r$ the Kneser graph $K_{n:r}$ is a core. The proof of this is included for two reasons: first, it is a nice application of the Erdős-Ko-Rado Theorem; and second, a similar, but more complicated proof is used to prove Lemma 6.4.1, but we do not include it in this thesis. Before we can prove that a Kneser graph is a core, we need a short lemma.

Lemma 4.1.11 (Theorem 7.5.4, [34]). *Let G and H be vertex-transitive graphs with the same fractional chromatic number with a homomorphism $\phi : G \to H$. Then, if S is a maximum independent set in H, the preimage of S, $\phi^{-1}(S) = \{v \in V(G) : \phi(v) \in S\}$, is a maximum independent set in G.*

Theorem 4.1.12 (Theorem 7.9.1, [34]). *For positive integers n, r with $n > 2r$, $K_{n:r}$ is a core.*

Proof. Let S be a maximum independent set in $K_{n:r}$. The vertices of $K_{n:r}$ are r-subsets of an n-set, and S is an r-uniform set system. As S is an independent set, for any $A, B \in S$, $A \cap B \neq \emptyset$. Thus, S is a maximum r-uniform 1-intersecting set system of an n-set. As $n > 2r$, by the Erdős-Ko-Rado Theorem, S must be a trivially 1-intersecting set system of cardinality $\binom{n-1}{r-1}$, and all r-subsets in S contain a common element.

For each $i = \{1, \ldots, n\}$, let $S_i \subset V(K_{n:r})$ be the system of all r-subsets that contain the element i. The sets S_i are all the maximum independent sets in $K_{n:r}$.

Let ϕ be an endomorphism $\phi : K_{n:r} \to K_{n:r}$. By Lemma 4.1.11, the preimage $\phi^{-1}(S_i)$ is also a maximum independent set. This means there is a $j_i \in \{1, \ldots, n\}$ such that $\phi^{-1}(S_i) = S_{j_i}$.

The subset $B = \{1, 2, \ldots, r\}$ is the unique vertex in the intersection of the set systems $S_1 \cap S_2 \cap \cdots \cap S_r$. Then

$$\phi^{-1}(B) = \phi^{-1}(S_1) \cap \phi^{-1}(S_2) \cap \cdots \cap \phi^{-1}(S_r) = S_{j_1} \cap S_{j_2} \cap \cdots \cap S_{j_r}.$$

There is at least one r-subset $A \in V(K_{n:r})$ with $\{j_1, j_2, \ldots, j_r\} \subseteq A$ (there are more subsets if the elements of $\{j_1, j_2, \ldots, j_r\}$ are not distinct). Thus, for every $B \in V(K_{n:r})$, $\phi^{-1}(B) \neq \emptyset$. So every endomorphism ϕ is an onto map and thus, is an automorphism. ✫

4.2 Algebraic Graph Theory

In this section, some concepts from algebraic graph theory are introduced. This theory is used in Chapter 6 to study the qualitative independence graphs. This section follows Godsil and Royle's *Algebraic Graph Theory* [34] and Godsil's *Algebraic Combinatorics* [31].

4.2.1 Spectral Theory of Graphs

Let G be a simple graph on n vertices labelled by $1, 2, \ldots, n$. The *adjacency matrix* of G is the $n \times n$ matrix with a 1 in the i, j position if vertices i and j are adjacent in G, and 0 if vertices i and j are not adjacent in G. The adjacency matrix of a graph G is denoted by $A(G)$. The *characteristic polynomial* of a graph G is the characteristic polynomial of the matrix $A(G)$. The eigenvalues and eigenvectors of the graph G are the eigenvalues and eigenvectors of the matrix $A(G)$. The *spectrum* for a graph is the set of all the eigenvalues with their multiplicities.

For any undirected graph G on n vertices, $A(G)$ is a real symmetric matrix. Thus, the eigenvalues of G are real numbers and the eigenvectors for G span $\mathbb{R}^n$. If two graphs are isomorphic, they have the same spectrum. But, two graphs can have the same spectrum and not be isomorphic.

The eigenvalues of a graph G can give information about G. In particular, we will consider cases where eigenvalues can be used to find upper bounds on $\alpha(G)$ and $\omega(G)$. In general, finding the eigenvalues of a graph is difficult. However, for special classes of graphs we know more, as stated in the following theorems.

Theorem 4.2.1. *The eigenvalues of the complete graph K_n are $n-1$ and -1 with multiplicities 1 and $n-1$, respectively.*

Theorem 4.2.2 (see Theorem 2.4.2, [31]). *If G is a k-regular graph, then k is an eigenvalue of G. Moreover, k is the largest eigenvalue of G.*

In a k-regular graph G, the multiplicity of the eigenvalue k is equal to the number of components in G.

Theorem 4.2.3 (see Lemma 8.5.1, [34]). *If G is a k-regular graph on n vertices with eigenvalues $k, \lambda_2, \ldots, \lambda_n$, then the eigenvalues of the complement $\overline{G}$ of G are $n-k-1, -1-\lambda_2, \ldots, -1-\lambda_n$. Moreover, the graphs G and $\overline{G}$ have the same eigenvectors.*

4.2.2 Equitable partitions

In this section, we introduce a method that reduces the amount of calculation needed to find the eigenvalues of a graph.

For a graph G, a partition π of $V(G)$ with r classes $C_1, C_2, \ldots, C_r$ is an *equitable partition* if the number of vertices in C_j that are adjacent to $v \in C_i$ is a constant b_{ij}, independent of v. For a graph G with an equitable partition π, the *quotient graph of G over π* is the directed multi-graph whose vertices are the r classes C_i with b_{ij} arcs from the i^{th} class to the j^{th} class. This graph is denoted by G/π. The adjacency matrix of G/π is given by

$$(A(G/\pi))_{i,j} = b_{ij}.$$

There are many equitable partitions for a graph G. For any subgroup $X \leq Aut(G)$, the set of orbits formed by the group action of X on the vertices of G is an equitable partition. To see this, consider two vertices v, v' which are in the same orbit under the group action of X. In this case, there is some $x \in X$ such that $x(v) = v'$. For any w adjacent to v, the vertex $x(w)$ is adjacent to v' and is in the same orbit as w. Thus the number of vertices adjacent to v in any orbit is the same as the number of vertices in that orbit adjacent to v'.

Theorem 4.2.4 (see Theorem 9.3.3, [34]). *If π is an equitable partition of a graph G, then the characteristic polynomial of G/π divides the characteristic polynomial of G.*

If λ is an eigenvalue for G/π with multiplicity m, then λ is an eigenvalue for G with multiplicity at least m. In some special circumstances, the eigenvalues of G/π are exactly the eigenvalues of G.

Theorem 4.2.5 (see Theorem 9.4.1, [34]). *Let G be a vertex-transitive graph and π be a partition on $V(G)$ generated by the orbits of some subgroup $X \leq Aut(G)$. If π has a class of size 1 (a singleton class), then every eigenvalue of G is an eigenvalue of G/π.*

This can be used to find all the eigenvalues of the Kneser graphs.

Example 4.2.6. Let r, n be positive integers with $r \leq n$ and let $K_{n:r}$ be a Kneser graph. Fix an arbitrary r-subset $A \in V(K_{n,r})$ of the n-set. For $i = 0, \ldots, r$, define C_i to be the collection of r-subsets of the n-set (the vertices in $K_{n,r}$) that have intersection of size $r - i$ with the subset A. This is an equitable partition; it corresponds to the orbit partition of the group $\text{fix}(A) = \{\sigma \in Sym(n) \ : \ \sigma(A) = A\}$. The class C_0 contains exactly one element, A. It is possible to build the adjacency matrix $A(K_{n:r}/\pi)$ for the quotient graph. Let B be an r-subset that intersects with A in exactly $r - i$ elements, so $B \in C_i$. The i, j entry of $A(K_{n:r}/\pi)$ is the number of r-subsets in C_j that are disjoint from B. This is the number of r-subsets that meet A in exactly $r - j$ positions and are disjoint from B. The number of such subsets is

$$\binom{i}{r-j}\binom{n-r-i}{j}.$$

Since this depends only on i and j, the partition is equitable.

From Theorem 4.2.5 the eigenvalues of $A(K_{n:r}/\pi)$ are exactly the eigenvalues of the Kneser graph $K_{n:r}$. Using some computations with binomial coefficients (Section 9.4, [34]), it is possible to show that the eigenvalues of $A(K_{n:r}/\pi)$, and hence $K_{n:r}$, are

$$(-1)^i \binom{n-r-i}{r-i} \quad \text{for} \quad i = 0, 1, \ldots, r. \tag{9}$$

4.2.3 Association Schemes

Definition 4.2.7 (*Association Scheme*). An *association scheme* on a set X is a set of d graphs $G_1, G_2, \ldots, G_d$, all with vertex set X, which has the following properties:

1. distinct elements $x, y \in X$ are adjacent in exactly one graph G_i;

2. for all $x, y \in X$ and $1 \leq i, j \leq d$, the number of elements z, with x and z adjacent in G_i and y and z adjacent in G_j depends only on i, j, k where G_k is the graph in which x and y adjacent. This number is denoted by $p_{i,j}^k$.

Note that the second requirement implies that each of the graphs in the scheme is regular. The graphs in the scheme are also called the *classes* of the scheme. For any set X, the complete graph on $|X|$ vertices is a scheme with just one class. This scheme is called the *trivial scheme*.

Association schemes can also be defined in terms of matrices. In this case, the classes of the scheme are represented by 01-matrices. The matrix J denotes the matrix with all entries equal to 1.

Definition 4.2.8 (*Association Scheme (definition 2)*). Let n be a positive integer. An *association scheme*, with d classes is a set of $n\times n$ 01-matrices $\{A_0, A_1, \dots, A_d\}$, with the following properties:

1. $A_0 = I$;
2. $\sum_{i=0}^{d} A_i = J$;
3. $A_i^\top = A_i$, for all $i = 1, \dots, d$;
4. for all $i, j \in \{0, \dots, d\}$, A_iA_j is a linear combination of $\{A_0, \dots, A_d\}$;
5. $A_iA_j = A_jA_i$, for all $i, j \in \{0, \dots, d\}$.

These two definitions of association schemes are equivalent. Assume the graphs G_i, for $i = 1, \dots, d$, form an association scheme on a set X with cardinality n. Set $A_i = A(G_i)$ for $i = 1, \dots, d$ and $A_0 = I$. The first three properties of matrices in an association scheme hold since each G_i is a graph and any pair of distinct elements $x, y \in X$ are adjacent in exactly one graph G_i.

Define G_0 to be the graph with vertex set X and a loop at each vertex. Let $x, y \in X$ be adjacent in the graph G_k. Then, for any $i, j \in \{0, \dots, d\}$, the (x, y) entry of the matrix A_iA_j is

$$\begin{aligned}
(A_iA_j)_{x,y} &= \sum_{z\in X}(A_i)_{x,z}(A_j)_{z,y} \\
&= |\{z :\ x \text{ and } z \text{ adjacent in } G_i,\ y \text{ and } z \text{ adjacent in } G_j\}| \\
&= p_{i,j}^k.
\end{aligned}$$

Thus $A_iA_j = \sum_{k=0}^{d} p_{i,j}^k A_k$, and the final two conditions for an association scheme hold.

Conversely, if the 01-matrices A_i, for $i = 0, \dots, d$, form an association scheme, the set of d graphs that have A_i, for $i = 1, \dots, d$, as their adjacency matrices form an association scheme.

The trivial scheme in matrix form is the set of matrices $\{I, J - I\}$. Note that there are two matrices in this scheme, but the scheme only has one class.

In some of the literature ([7, 23]), association schemes as defined here are referred to as *symmetric association schemes*. In these references, a general association scheme is the same as in Definition 4.2.8, but with Condition 3 replaced with the condition that $A_i^T \in \mathcal{A}$. In most recent papers, association schemes are assumed to be symmetric and schemes with Condition 3 replaced with the condition that $A_i^T \in \mathcal{A}$ are call *asymmetric association schemes*. This thesis will follow this trend and use "association scheme" to mean a symmetric association scheme.

Example 4.2.9.(Johnson Scheme). Let i, r, n be positive integers with $i \leq r \leq n$. Define the *generalized Johnson graph* $J(n, r, r - i)$ to be the graph whose vertex set is the set of all r-subsets of an n-set and two r-sets are adjacent if and only if the subsets have exactly i elements in common. The graph $J(n, r, r)$ is isomorphic to the Kneser graph $K_{n:r}$ and the graph $J(n, r, 0)$ has a loop at each vertex and no other edges. If $n < 2r - i$ the graph $J(n, r, r - i)$ is the empty graph.

The r graphs $J(n, r, i)$, for $i = 1, \dots, r$, form an association scheme called the *Johnson scheme*.

If the r-subsets A, B are adjacent in the graph $J(n, r, r-i)$, then $|A \cap B| = i$. For any r-subset C that is adjacent to A in the graph $J(n, r, r-j)$ and adjacent to B in the graph $J(n, r, r - k)$, we have

$$|A \cap C| = j \text{ and } |B \cap C| = k.$$

The total number of such r-subsets C is

$$\sum_{\substack{x+y=j \\ x+z=k}} \binom{n - (2r - i)}{r - (x + y + z)} \binom{i}{x} \binom{r - i}{y} \binom{r - i}{z}.$$

As the number of subsets C depends only on i, j and k, the Johnson scheme is an association scheme.

Let $\mathcal{A}$ be an association scheme with $d+1$ matrices of order n. From Condition 4 in Definition 4.2.8, the span (over $\mathbb{R}$) of the matrices in $\mathcal{A}$ is a $(d+1)$-dimensional algebra. This algebra is called the *Bose-Mesner Algebra*. The matrices in the association scheme are a basis for this algebra, and hence a convenient way to consider the algebra. Often it is more useful to use the matrix definition of association schemes than the graph definition.

Any matrix in an association scheme is a real symmetric matrix; thus, it is diagonalizable and all the eigenvalues are real numbers. Moreover, from Condition 5 in Definition 4.2.8, any two matrices in an association scheme commute; thus, for any $i, j \in \{1, \ldots, d\}$, any eigenspace of the matrix A_i is A_j*-invariant* for any matrix A_j in the scheme. That is, if E_λ is an eigenspace of the matrix A_i, then for any $v \in E_\lambda$ we have that $A_j v \in E_\lambda$ for any matrix A_j in the scheme.

It is possible to find $d+1$ subspaces $U_0, U_1, \ldots, U_d$ of $\mathbb{R}^n$ that partition $\mathbb{R}^n$ with the property that for every $i, j \in \{0, \ldots, d\}$, the space U_j is contained in an eigenspace of A_i. For each $i, j \in \{0, \ldots, d\}$, denote the eigenvalue of A_i on the subspace U_j by $\lambda_i(j)$. Then the *matrix of eigenvalues* for an association scheme is defined to be the matrix $(P)_{j,i} = \lambda_i(j)$. In particular, the i^{th} column of the matrix of eigenvalues contains the list of eigenvalues of the matrix A_i in the scheme.

As the matrix A_0 is the identity matrix, the first column of the matrix of eigenvalues has each entry equal to one. Often, this first column is replaced with a column containing the dimension of the subspace U_i; these values give the multiplicities of the eigenvalues. The matrix of eigenvalues with the first column replaced by the multiplicities of the eigenvalues is called the *modified matrix of eigenvalues* for the association scheme.

4.2.4 Ratio Bounds

There are two bounds using eigenvalues called the *ratio bounds*. The first ratio bound is an upper bound on the size of the maximum independent set in a graph and the second ratio bound is an upper bound on the size of the maximum clique. The ratio bound on independent sets was established for graphs in an association scheme by

Delsarte [23]. The proof given here is from Section 9.6 of [34]. For an interesting examination of several results from the ratio bounds, see Newman's Ph.D. thesis [58].

Lemma 4.2.10 (Ratio Bound for $\alpha(G)$, see Section 9.6, [34]). *Let G be a d-regular graph on n vertices and let τ be the least eigenvalue of G. Then*

$$\alpha(G) \leq \frac{n}{1 - \frac{d}{\tau}}.$$

Any independent set in G with cardinality $\frac{n}{1-\frac{d}{\tau}}$ is called *ratio-tight*. If an independent set is ratio-tight, then more is known about the independent set.

Theorem 4.2.11 ([58]). *Let G be a d-regular graph on n vertices with the least eigenvalue τ. Let $A(G)$ be the adjacency matrix of the graph G. Let S be an independent set in G of size s and $\mathbf{z}$ the characteristic vector of S. If*

$$s = \frac{n}{1 - \frac{d}{\tau}},$$

then

$$A(G)\left(\mathbf{z} - \frac{s}{n}\mathbf{1}\right) = \tau\left(\mathbf{z} - \frac{s}{n}\mathbf{1}\right).$$

Moreover, $s = \frac{n}{1-\frac{d}{\tau}}$ holds if and only if the partition of $V(G)$, $\{S, V(G)\backslash S\}$, is an equitable partition.

A proof of this theorem can be found in [58], where it is attributed to Godsil. This proof uses the fact that the matrix $A - \tau I$ is a positive semi-definite matrix.

From this theorem, the characteristic vector of a ratio-tight independent set is a linear combination of the all-ones vector, $\mathbf{1}$, and an eigenvector corresponding to the smallest eigenvalue. Also, for an independent set S of size s, if the partition $\{S, V(G)\backslash S\}$ of $V(G)$ is equitable, then the smallest eigenvalue of G is $\tau = \frac{d}{1-\frac{n}{s}}$.

There is another ratio bound on the size of the maximum clique in a graph. The set of graphs where the ratio bound on the clique size is valid is more restrictive than for the ratio bound on the size of the maximum independent set.

Theorem 4.2.12 (Ratio Bound for $\omega(G)$, see [32]). *Assume G is a d-regular graph which is either arc transitive or a single graph in an association scheme. Let τ be the least eigenvalue of G, then*

$$\omega(G) \leq 1 - \frac{d}{\tau}.$$

In Chapter 6, we use the ratio bounds to find bounds on the size of the maximum independent sets and the maximum cliques for a family of *qualitative independence graphs* and several *uniform qualitative independence graphs*. These graphs are defined in the next chapter.

Chapter 5

Covering Arrays on Graphs and Qualitative Independence Graphs

In this chapter, we extend the definition of a covering array to include a graph structure. Recall from Definition 2.4.1, for positive integers n, r, k, a covering array, $CA(n, r, k)$, is an $r \times n$ array with entries from $\{0, 1, \ldots, k-1\}$ with the property that any two rows in the arrays are qualitatively independent. One application of covering arrays is to design test suites for systems or networks (see Section 2.4). In such an application, each row of the array corresponds to a parameter in the system, each column corresponds to a test run, and the entries from $\{0, 1, \ldots, k-1\}$ correspond to the values the parameters are assigned in the test run. Such a test suite completely tests any pair of parameters in the system against one another.

If, in a system, it is not necessary to test all pairs of parameters against each other, a graph can be used to describe which pairs of parameters need to be tested. In particular, each vertex in such a graph corresponds to a parameter in the system and vertices are adjacent if and only if the corresponding pairs of parameters need to be tested against each other. Adding such a graph structure to covering arrays makes it possible to build test suites that are more efficient for a specific system and provides a way to use the internal structure of the system to optimize covering arrays.

Definition 5.0.1 (*Covering Array on a Graph*). Let G be a graph and n and k be positive integers. A *covering array on the graph G*, $CA(n, G, k)$, is a $|V(G)| \times n$

array with entries from $\{0, 1, \ldots, k-1\}$ whose rows correspond to the vertices of G and any two rows corresponding to vertices which are adjacent in G are qualitatively independent.

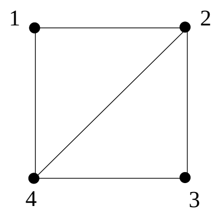
The size of the smallest covering array on a graph G with alphabet k will be denoted by $CAN(G, k)$, that is,

$$CAN(G, k) = \min_{l \in \mathbb{N}} \{l : \exists\, CA(l, G, k)\}.$$

A $CA(n, G, k)$ with $n = CAN(G, k)$ is an *optimal* covering array on G.

Example 5.0.2. An optimal covering array on a graph G with $CAN(G, 2) = 4$.

$G \quad =$



1	0	0	1	1
2	0	1	0	1
3	0	0	1	1
4	0	1	1	0

Covering arrays on graphs are extensions of standard covering arrays; in particular, for K_k the complete graph on k vertices, $CAN(k, g) = CAN(K_k, g)$.

Binary covering arrays on graphs have been studied by Seroussi and Bshouty, who proved that determining the existence of an optimal binary covering array on a graph is an NP-complete problem [63]. General covering arrays on graphs are introduced in the conclusion of Stevens's thesis [69].

In Section 5.1, we show that for all graphs G and all positive integers k,

$$CAN(K_{\omega(G)}, k) \leq CAN(G, k) \leq CAN(K_{\chi(G)}, k).$$

The upper bound is of particular interest because it gives a method to construct covering arrays on graphs (see Section 5.1). This also raises a question that partially motivates this work: can determining $CAN(G, k)$ be reduced to determining $\chi(G)$ and $CAN(\chi(G), k)$? We show that it cannot. In particular, we look for graphs G so that

$$CAN(G, k) < CAN(K_{\chi(G)}, k). \tag{10}$$

In Section 5.2, we define a family of graphs, called the *qualitative independence graphs*. We show that this family of graphs gives a good characterization of covering arrays for all graphs, namely that for a graph G and positive integers k and n, a $CA(n, G, k)$ exists if and only if there is a graph homomorphism $G \to QI(n, k)$. Moreover, for all positive integers k and n, a clique of size r in the graph $QI(n, k)$ corresponds to a $CA(n, r, k)$. This new family converts a problem in combinatorial systems into a question about homomorphisms on graphs [34].

In Section 5.3.1, we show

$$\omega(QI(n,2)) = \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1} \text{ and } \chi(QI(n,2)) = \left\lceil \frac{1}{2} \binom{n}{\lfloor \frac{n}{2} \rfloor} \right\rceil.$$

Using this formula for chromatic number, we show that if n is odd, then the graph $QI(n, 2)$ satisfies Inequality (10). Additionally, we present necessary conditions on a graph G to satisfy Inequality (10) for $k = 2$. These are: if $CAN(K_{\chi(G)}, 2) = c$, then c must be even and

$$\frac{1}{2}\binom{c-2}{\frac{c-2}{2}} < \chi(G) \leq \left\lceil \frac{1}{2}\binom{c-1}{\frac{c}{2}-1} \right\rceil.$$

In Section 5.3.2, we give a graph that is a core of $QI(n, 2)$. The structure of these cores implies that if there exists a $CA(n, r, 2)$ or a $CA(n, G, 2)$, then there exists a covering array with the same parameters in which each row has exactly $\lfloor \frac{n}{2} \rfloor$ ones.

Finally, in Section 5.4, for a positive integer k, we find upper bounds on $\omega(QI(k^2, k))$ and $\chi(QI(k^2, k))$ and prove that $QI(k^2, k)$ is a $(k!)^{k-1}$-regular graph.

Most of the results in this chapter are published in [54].

5.1 Bounds from Homomorphisms

From Section 4.1.2, for graphs G and H, if there is a graph homomorphism $G \to H$, then $\chi(G) \leq \chi(H)$ and $\omega(G) \leq \omega(H)$. We can obtain a similar bound on the optimal size of a covering array.

Theorem 5.1.1. *Let k be a positive integer and G and H be graphs. If there is a graph homomorphism $\phi : G \to H$, then*

$$CAN(G, k) \leq CAN(H, k).$$

Proof. For n a positive integer, assume that there exists a covering array $CA(n, H, k)$. This covering array will be used to construct a covering array $CA(n, G, k)$. For each $i \in \{1, \ldots, |V(G)|\}$, row i of $CA(n, G, k)$ corresponds to a vertex $v_i \in V(G)$. Set row i of $CA(n, G, k)$ to be identical to the row corresponding to the vertex $\phi(v_i) \in V(H)$ in $CA(n, H, k)$. Since ϕ is a homomorphism, any pair of adjacent vertices in G are mapped to adjacent vertices in H. Thus, any pair of rows in $CA(n, G, k)$, which correspond to vertices that are adjacent in G, will be qualitatively independent, since the rows are qualitatively independent in $CA(n, H, k)$. ✫

For any graph G, there are homomorphisms between the following complete graphs

$$K_{\omega(G)} \to G \to K_{\chi(G)}.$$

These homomorphisms can be used to find bounds on $CAN(G, k)$.

Corollary 5.1.2. *For all positive integers k and all graphs G,*

$$CAN(K_{\omega(G)}, k) \leq CAN(G, k) \leq CAN(K_{\chi(G)}, k).$$

5.2 Qualitative Independence Graphs

In Section 5.2.2, we define the *qualitative independence graphs*. The vertex set for these graphs is a set of partitions, so before defining the qualitative independence graphs, we need to extend the definition of qualitative independence to partitions.

We will use the notation for set partitions from Section 3.1.2. In particular, a k-partition of an n-set is a set of k disjoint non-empty classes whose union is the n-set and $\mathcal{P}_k^n$ denotes the set of all k-partitions of an n-set.

5.2.1 Qualitatively Independent Partitions

We have defined qualitative independence for vectors (Definition 2.4.3) and for sets (Definition 3.4.1). There is a third (also equivalent) definition of qualitative independence for partitions.

Definition 5.2.1 (*Qualitatively Independent Partitions*). Let n and k be positive integers with $n \geq k^2$. Let $A, B \in \mathcal{P}_k^n$ be two k-partitions of an n-set. Assume $A = \{A_1, A_2, \ldots, A_k\}$ and $B = \{B_1, B_2, \ldots, B_k\}$. The partitions A and B are *qualitatively independent* if

$$A_i \cap B_j \neq \emptyset \quad \text{for all } i \text{ and } j.$$

If k-partitions $A = \{A_1, \ldots, A_k\}$ and $B = \{B_1, \ldots, B_k\}$ are qualitatively independent, then for each $i \in \{1, \ldots, k\}$, $|A_i| \geq k$ and $|B_i| \geq k$.

For all positive integers k and n, this definition for qualitatively independent k-partitions of an n-set is equivalent to Definition 2.4.3 for qualitatively independent vectors in $\mathbb{Z}_k^n$. Assume vectors $u, v \in \mathbb{Z}_k^n$, with $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$, are qualitatively independent. For the vector u, define a k-partition $P = \{P_1, P_2, \ldots, P_k\}$ by $a \in P_i$ if and only if $u_a = i$. Similarly let Q be the partition corresponding to the vector v. Since u and v are qualitatively independent, for all ordered pairs $(i, j) \in \mathbb{Z}_k \times \mathbb{Z}_k$ there is an index a such that $(u_a, v_a) = (i, j)$. By the definition of P and Q, $a \in P_i$ and $a \in Q_j$. So, for all $i, j \in \{1, \ldots, n\}$, $P_i \cap Q_j \neq \emptyset$. Conversely, if P and Q are qualitatively independent k-partitions, then the vectors $u, v \in \mathbb{Z}_k^n$ corresponding to P and Q are qualitatively independent.

5.2.2 Definition of Qualitative Independence Graphs

Definition 5.2.2 (*Qualitative Independence Graph*). Let n and k be positive integers with $n \geq k^2$. Define the *qualitative independence graph* $QI(n, k)$ to be the graph whose vertex set is the set of all k-partitions of an n-set with the property that every class of the partition has size at least k. Vertices are adjacent if and only if the corresponding partitions are qualitatively independent.

> **Example 5.2.3.** Consider the graph $QI(4, 2)$. There are only three partitions in $\mathcal{P}_2^4$ with the property that every class has size at least two:
>
> $$1\,2 \mid 3\,4, \quad 1\,3 \mid 2\,4, \quad 1\,4 \mid 2\,3.$$
>
> These partitions are all pairwise qualitatively independent, so the graph $QI(4, 2)$ is isomorphic to the complete graph on three vertices, K_3.

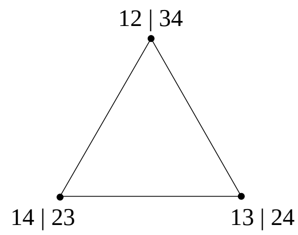


Figure 1: The graph $QI(4, 2)$.

It would be possible to define the vertex set of $QI(n, k)$ to be the set of all k-partitions of an n-set, $\mathcal{P}_k^n$. But then, any k-partition with a class of size smaller than k would be an isolated vertex in $QI(n, k)$. Since the vertex set of $QI(n, k)$ excludes these partitions, the qualitative independence graphs have no isolated vertices, in fact, they are connected and have small diameter.

Lemma 5.2.4. *Let k, n be positive integers with $n \geq k^2$. The graph $QI(n, k)$ has diameter 2.*

Proof. Given any two partitions, $P, Q \in V(QI(n, k))$, it is possible to construct a partition $R \in V(QI(n, k))$ which is qualitatively independent from P and Q.

Define a bipartite multi-graph H as follows: each part of H has k vertices. The k vertices in the first part of H correspond to the classes P_i of P, and the k vertices in the second part of H correspond to the classes Q_i of Q.

For every $a \in \{1, \ldots, n\}$, there exists a unique class $P_{i_a} \in P$ with $a \in P_{i_a}$ and there exists a unique class $Q_{i_a} \in Q$ with $a \in Q_{i_a}$. For each such a, add an edge to H between the vertices corresponding to P_{i_a} and Q_{i_a}. Label this edge by a. The multi-graph H has exactly n edges, one for each $a \in \{1, \ldots, n\}$. Further, each class $P_i \in P$ and each class $Q_i \in Q$ have cardinality at least k. Thus, the degree of each vertex in H is at least k. By a dual of König's Theorem [36], there are k edge-disjoint 1-factors in H.

Use these k 1-factors to build the k-partition R. Each class of R will correspond to a 1-factor of H. In particular, $a \in R_i$ for $i = 1, \ldots, k$ if and only if the edge labelled by a is in the 1-factor that corresponds to the class $R_i \in R$. Place the remaining elements from $\{1, \ldots, n\}$ in classes of R arbitrarily. Each 1-factor has exactly k edges, so the classes of R will have size at least k. Thus, R is a k-partition of an n-set with each class of size at least k, so $R \in V(QI(n,k))$.

Finally, we need to show that the partitions P and R are qualitatively independent. For any $j = 1, \ldots, k$, the class R_j corresponds to a 1-factor of H, this means that for any $i \in \{1, \ldots, k\}$, there is an edge a in the 1-factor incident to P_i. By the definition of R, $a \in R_j$. Since the edge labelled a is incident with the vertex corresponding to P_i, we also know that $a \in P_i$. This means for any $i, j \in \{1, \ldots, n\}$, $a \in P_i \cap R_j$ so $P_i \cap R_j \neq \emptyset$. Thus P and R are qualitatively independent. Similarly, Q and R are qualitatively independent. ✰

A clique in $QI(n,k)$ is equivalent to a collection of pairwise qualitatively independent k-partitions of an n-set. Thus, an r-clique in $QI(n,k)$ corresponds to a $CA(n,r,k)$. An independent set in $QI(n,k)$ is a collection of partitions in which no two are qualitatively independent.

For any positive integers k, n, it is possible to construct a $CA(n, QI(n,k), k)$. Each row of a $CA(n, QI(n,k), k)$ corresponds to a partition in $V(QI(n,k))$, which in turn corresponds to a vector in $\mathbb{Z}_k^n$. Set each row in $CA(n, QI(n,k), k)$ to be the corresponding vector. Two partitions are adjacent in $QI(n,k)$ if and only if they are qualitatively independent and the vectors corresponding to the partitions are qualitatively independent if and only if the partitions are. Thus, two rows in $CA(n, QI(n,k), k)$ are qualitatively independent if and only if the partitions corresponding to the rows are adjacent in $QI(n,k)$.

Lemma 5.2.5. *For positive integers n, k with $n \geq k^2$, we have*

$$CAN(QI(n,k), k) \leq n.$$

We conjecture that a $CA(n, QI(n,k), k)$ is an optimal covering array.

Conjecture 5.2.6. For positive integers n, k with $n \geq k^2$, we have

$$CAN(QI(n,k), k) = n.$$

5.2.3 Homomorphisms and Qualitative Independence Graphs

Recall from Section 4.1.2, that a proper colouring with n colours of a graph G is equivalent to a homomorphism from G to the complete graph K_n. We give a similar characterization for covering arrays on graphs.

Theorem 5.2.7. *For a graph G and positive integers k and n, a $CA(n, G, k)$ exists if and only if there exists a graph homomorphism $G \to QI(n, k)$.*

Proof. Assume that there exists a $CA(n, G, k)$, call this C. For a vertex $v \in V(G)$, let the vector $C_v \in \mathbb{Z}_k^n$ be the row in C corresponding to v. Consider a mapping $\phi : V(G) \to V(QI(n, k))$ which takes a vertex $v \in V(G)$ to the partition $P_v \in V(QI(n, k))$ which corresponds to the vector C_v.

The map ϕ is a homomorphism. To see this, let $v, w \in V(G)$ be adjacent vertices. Since C is a covering array, the vectors C_v and C_w are qualitatively independent, thus the corresponding partitions P_v and P_w are qualitatively independent and adjacent in $QI(n, k)$.

Conversely, assume there is a graph homomorphism $\phi : G \to QI(n, k)$. For each $v \in V(G)$, $\phi(v)$ is a k-partition and has a corresponding vector $C_v \in \mathbb{Z}_k^n$. Build a $CA(n, G, k)$ by using the vector C_v as the row corresponding to $v \in V(G)$. If the vertices $v, w \in V(G)$ are adjacent in G, then the vertices corresponding to partitions $\phi(v), \phi(w)$ are adjacent in $QI(n, k)$. This means the partitions $\phi(v)$, $\phi(w)$ are qualitatively independent, thus the corresponding vectors C_v, C_w are qualitatively independent. ✰

Example 5.2.8. For a graph G, there exists a $CA(4, G, 2)$ if and only if there is a homomorphism from G to $QI(4, 2)$. Since $QI(4, 2)$ is isomorphic to K_3, there exists a $CA(4, G, 2)$ if and only if G is 3-colourable. This gives a characterization of the graphs for which a covering array of size 4 on a 2-alphabet exists (they are the 3-colourable graphs).

Further, this shows that determining if $CAN(G, 2) = 3$ is as hard as determining whether a graph is 3-colourable. This is the approach used by Seroussi and Bshouty [63] to prove that finding $CAN(G, k)$ for a given G and k is an NP-hard problem.

Similar to the definition of chromatic number for a graph G,

$$\chi(G) = \min\{n : G \to K_n\},$$

and the characterization of fractional chromatic number (Theorem 4.1.9),

$$\chi^*(G) = \min\{n/r : G \to K_{n:r}\},$$

$CAN(G,k)$ can be characterized by a homomorphism.

Corollary 5.2.9. *For any graph G, and any positive integer k,*

$$CAN(G,k) = \min_{n\in\mathbb{N}}\{n \ : \ G \to QI(n,k)\}.$$

Knowing the chromatic number and the maximum clique size of $QI(n,k)$ will give information on which graphs have covering arrays of size n on an alphabet of size k.

Corollary 5.2.10. *Let G be a graph and n,k be positive integers. If there exists a $CA(n,G,k)$, then*

$$\chi(G) \le \chi(QI(n,k)) \text{ and } \omega(G) \le \omega(QI(n,k)).$$

With Theorem 5.2.7, Conjecture 5.2.6 can be rephrased in terms of homomorphisms.

Conjecture 5.2.11. For all integers n,k, there is no homomorphism $QI(n,k) \to QI(n-1,k)$.

It is in general difficult to prove that no homomorphism exists between two graphs. One method to prove this would be to show either $\chi(QI(n,k)) > \chi(QI(n-1,k))$ or $\omega(QI(n,k)) > \omega(QI(n-1,k))$.

With this motivation, we try to find bounds and exact values for $\chi(QI(n,k))$ and $\omega(QI(n,k))$.

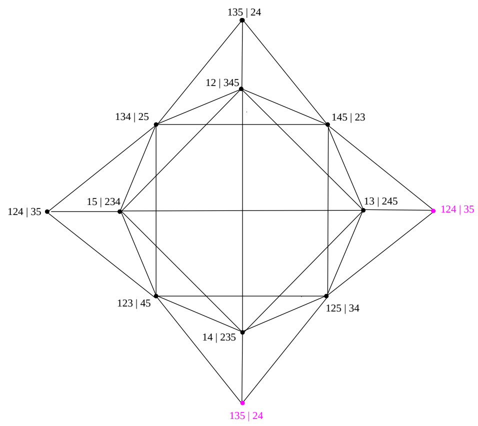


Figure 2: The graph $QI(5,2)$.

5.3 Binary Qualitative Independence Graphs

From Example 5.2.8, for a graph G, a $CA(4, G, 2)$ exists if and only if G is 3-colourable. It is not true in general that the existence of a $CA(n, G, k)$ can be characterized by the chromatic number of G. To see this we consider the graph $QI(5,2)$.

Example 5.3.1. There are 10 vertices in $\mathcal{P}_2^5$ that have each class of size at least 2:

$$1\,2 \mid 3\,4\,5,\quad 1\,3 \mid 2\,4\,5,\quad 1\,4 \mid 2\,3\,5,\quad 1\,5 \mid 2\,3\,4,\quad 1\,2\,3 \mid 4\,5,$$
$$1\,2\,4 \mid 3\,5,\quad 1\,2\,5 \mid 3\,4,\quad 1\,3\,4 \mid 2\,5,\quad 1\,3\,5 \mid 2\,4,\quad 1\,4\,5 \mid 2\,3.$$

A representation of $QI(5,2)$ is given below. The vertices $1\,2\,5 \mid 3\,4$ and $1\,3\,4 \mid 2\,5$ are repeated (in grey) to make the graph easier to read.

By inspection, $\omega(QI(5,2)) = 4$ and $\chi(QI(5,2)) = 5$. A 5-colouring of $QI(5,2)$ is given below.

colour	vertices	
1	00111	00011
2	01011	01010
3	01101	00101
4	01110	01100
5	01001	00110

From Theorem 5.2.7, $QI(5,2)$ has a covering array of size 5. This graph does not have a covering array of size 4 since otherwise there would exist a homomorphism $QI(5,2) \to QI(4,2)$, contradicting that $\chi(QI(5,2)) > 3$. Thus, $CAN(QI(5,2),2) = 5$. From Theorem 3.4.3, $CAN(K_5,2) = CAN(5,2) = 6$.

Therefore, for $QI(5,2)$, the inequality from Corollary 5.1.2 holds strictly,

$$5 = CAN(QI(5,2),2) < CAN(K_{\chi(QI(5,2))},2) = 6.$$

If for a graph G a $CA(5,G,2)$ exists, then there is a homomorphism from G to $QI(5,2)$. Since $QI(5,2)$ is not isomorphic to a complete graph, determining if a graph has a binary covering array of size 5 is not equivalent to finding a proper colouring of the graph.

Finally, it is interesting to note that the complement of $QI(5,2)$ is the Petersen graph, which is also known as the Kneser graph $K_{5:2}$.

5.3.1 Formulae for $\omega(QI(n,2))$ and $\chi(QI(n,2))$

In this section, the following formulae for maximum clique size and chromatic number of $QI(n,2)$ are established:

$$\omega(QI(n,2)) = \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1} \quad \text{and} \quad \chi(QI(n,2)) = \left\lceil \frac{1}{2}\binom{n}{\lfloor \frac{n}{2} \rfloor} \right\rceil.$$

The vertices of $QI(n,2)$ are 2-partitions of an n-set with the property that each class of the partition has at least 2 elements. For any 2-partition $P = \{P_1, P_2\}$, it is clear that $P_2 = \overline{P_1}$. So, each 2-partition can be unambiguously described by a subset of an n-set. Moreover, by choosing the smaller of P_1 and P_2, any 2-partition of an n-set can be described as an r-subset of an n-set where $r \leq n/2$.

Let $P = \{P_1, P_2\}$ and $Q = \{Q_1, Q_2\}$ be 2-partitions of an n-set. The partitions P and Q are qualitatively independent if and only if the sets P_1 and Q_1 are intersecting and incomparable (Section 3.4). The exact value of the maximum clique for $QI(n,2)$ is the maximum size of a 1-intersecting set system; this is known from Sperner's Theorem and the Erdős-Ko-Rado Theorem. In fact, Theorem 3.4.3 can be restated in terms of a maximum clique in $QI(n,2)$.

Theorem 5.3.2 ([47]). *For integer $n \geq 4$,*

$$\omega(QI(n,2)) = \max_{r \in \mathbb{N}}\{r : \exists\, CA(n, K_r, 2)\} = \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1}.$$

When n is even, the set of uniform 2-partitions of an n-set is a maximum clique. When n is odd the set of all almost-uniform 2-partitions of an n-set that have a common element in the smaller class is a maximum clique.

Next, we determine the chromatic number of the graphs $QI(n, 2)$.

Theorem 5.3.3. *For all positive integers n,*

$$\chi(QI(n,2)) = \left\lceil \frac{1}{2}\binom{n}{\lfloor \frac{n}{2} \rfloor} \right\rceil.$$

Proof. Consider the vertices in $QI(n, 2)$ not as partitions, but as subsets of an n-set of size no more than $\lfloor n/2 \rfloor$ (as described above).

From the proof of Sperner's Theorem (Theorem 3.2.2), the poset of subsets of an n-set ordered by inclusion can be decomposed into $\binom{n}{\lfloor n/2 \rfloor}$ disjoint chains and each chain contains exactly one set of size $\lfloor n/2 \rfloor$. Call these chains C_i, where $i \in \{1, \ldots, \binom{n}{\lfloor n/2 \rfloor}\}$. For any i, if the sets $A, B \in C_i$, then A and B are not qualitatively independent. In particular, any two vertices in $QI(n, k)$ which correspond to sets that are in the same chains are not adjacent in $QI(n, k)$.

It is possible to pair the $\binom{n}{\lfloor n/2 \rfloor}$ chains so that any subset of size no more than $n/2$ in one chain is disjoint from any subset of size no more than $n/2$ in the other chain. To see this, consider two cases, first when n is even and second when n is odd.

Assume n is even. For each chain C_i, let A_i be the set of size $n/2$. Match the chains C_i and C_j where $A_i = \overline{A_j}$.

Assume n is odd. For each chain C_i, let A_i be the set of size $(n - 1)/2$. The sets A_i are the vertices of the Kneser graph $K_{n:\frac{n-1}{2}}$ (see Section 4.1.5). The graph $K_{n:\frac{n-1}{2}}$ is vertex transitive so there exists a matching that is perfect or is missing just one vertex (Section 3.5 of [34]). So each set A_i (except possibly one set) is matched to another set of size $\frac{n-1}{2}$, call it A'_i. The set $A'_i \subset \overline{A_i}$. Match the chain C_i which contains A_i with the chain $C_{i'}$ that contains the set A'_i.

Any two sets in a matched pair of chains have the property that either one set contains the other or one set contains the complement of the other. In either case, the partitions are not qualitatively independent. All sets in the paired chains can be assigned the same colour in a proper colouring of $QI(n,2)$. This produces a proper $\left\lceil \frac{1}{2}\binom{n}{\lfloor n/2 \rfloor} \right\rceil$-colouring on the graph $QI(n,2)$.

To see that this is the smallest possible colouring of $QI(n,2)$, consider the vertices of $QI(n,2)$ that correspond to $\lfloor n/2 \rfloor$-sets. Two such vertices may be assigned the same colour if and only if the subsets are disjoint. It is clear that there can not be three mutually disjoint subsets of an n-set with size $\lfloor n/2 \rfloor$. So it is not possible to properly colour these vertices with fewer than $\left\lceil \frac{1}{2}\binom{n}{\lfloor n/2 \rfloor} \right\rceil$ colours. ☆

From the formulae for maximum clique size and chromatic number, for all $n \geq 4$,

$$\chi(QI(n-1,2)) < \omega(QI(n,2)) \leq \chi(QI(n,2)) < \omega(QI(n+1,2)).$$

Note that when n is even, we have $\omega(QI(n,2)) = \chi(QI(n,2))$. From this inequality we have the following result, which confirms Conjecture 5.2.6 for $k = 2$.

Corollary 5.3.4. *For all $n \geq 4$, we have $CAN(QI(n,2),2) = n$.*

Proof. Assume that there exists a $CA(n-1, QI(n,2), 2)$, then by Theorem 5.2.7 there is a homomorphism

$$QI(n,2) \rightarrow QI(n-1,2),$$

which contradicts $\chi(QI(n-1,2)) < \chi(QI(n,2))$.

From Theorem 5.2.7, there exists a $CA(n, QI(n,2), 2)$, so

$$CAN(QI(n,2),2) = n.$$

☆

With the exact value of $\chi(QI(n,2))$ we can prove that the second inequality in Corollary 5.1.2 is strict for $QI(n,2)$ for all odd n.

Corollary 5.3.5. *For n odd,*

$$CAN(QI(n,2),2) < CAN(K_{\chi(QI(n,2))},2).$$

Proof. From Corollary 5.3.4, $CAN(QI(n,2),2) = n$. From Theorem 5.3.3 and Theorem 3.4.3,

$$\begin{aligned} CAN(K_{\chi(QI(n,2))},2) &= CAN\left(\left\lceil \frac{1}{2}\binom{n}{\frac{n-1}{2}} \right\rceil, 2\right) \\ &= \min\left\{m : \binom{m-1}{\lfloor m/2 \rfloor - 1} \geq \left\lceil \frac{1}{2}\binom{n}{\frac{n-1}{2}} \right\rceil \right\} \\ &= n+1. \end{aligned}$$

☆

This gives an infinite family of graphs for which Inequality (10) holds with strict inequality. Further, we can use this to get a lower bound on $CAN(G,2)$.

Corollary 5.3.6. *For any graph G*

$$CAN(K_{\chi(G)},2) - 1 \leq CAN(G,2) \leq CAN(K_{\chi(G)},2).$$

Moreover, if $CAN(K_{\chi(G)},2)$ *is odd, then*

$$CAN(G,2) = CAN(K_{\chi(G)},2).$$

Proof. From Corollary 5.1.2, $CAN(G,2) \leq CAN(K_{\chi(G)},2)$.

Assume that $m = CAN(K_{\chi(G)},2)$ is even and that $CAN(G,2) \leq m-2$. Then there is a homomorphism from G to $QI(m-2,2)$ and

$$\chi(G) \leq \chi(QI(m-2,2)) = \frac{1}{2}\binom{m-2}{\frac{m-2}{2}}.$$

Since

$$CAN(K_{\chi(G)},2) = \min\left\{l : \binom{l-1}{\lfloor \frac{l}{2} \rfloor - 1} \geq \chi(G)\right\},$$

$CAN(K_{\chi(G)},2) \leq m-1$. This is a contradiction with $m = CAN(K_{\chi(G)},2)$.

Next, assume that $CAN(K_{\chi(G)},2)$ is odd and set $m+1 = CAN(K_{\chi(G)},2)$. If $CAN(G,2) \leq m$, then there is a homomorphism from G to $QI(m,2)$. From Theorem 5.3.3, $\chi(QI(m,2)) = \frac{1}{2}\binom{m}{m/2} = \binom{m-1}{m/2}$, and $\chi(G) \leq \binom{m-1}{m/2}$. By definition,

$$CAN(K_{\chi(G)},2) = \min\left\{l : \binom{l-1}{\lfloor \frac{l}{2} \rfloor - 1} \geq \chi(G)\right\}.$$

Since $\chi(G) \leq \binom{m-1}{m/2}$, the minimum occurs when $l \leq m$. This means that

$$CAN(K_{\chi(G)}, 2) \leq m < CAN(K_{\chi(G)}, 2),$$

and this contradiction gives that $CAN(G, 2) = CAN(K_{\chi(G)}, 2)$. ✩

If $CAN(G, 2) < CAN(K_{\chi(G)}, 2)$, then by Corollary 7.1.9 $c = CAN(K_{\chi(G)}, 2)$ is even and

$$CAN(G, 2) = CAN(K_{\chi(G)}, 2) - 1.$$

This means there is a homomorphism from G to $QI(c-1, 2)$. Since $\chi(QI(c-1, 2)) = \lceil \frac{1}{2}\binom{c-1}{\frac{c-2}{2}} \rceil$, we have that $\chi(G) \leq \lceil \frac{1}{2}\binom{c-1}{\frac{c-2}{2}} \rceil$.

If $\chi(G) \leq \frac{1}{2}\binom{c-2}{\frac{c-2}{2}}$, then there is a homomorphism from G to $K_{\frac{1}{2}\binom{c-2}{\frac{c-2}{2}}}$. For c even, $K_{\frac{1}{2}\binom{c-2}{\frac{c-2}{2}}}$ is isomorphic to the graph $UQI(c-2, 2)$. Thus, there is a homomorphism from G to $UQI(c-2, 2)$, and in particular, $CAN(G, 2) \leq c-2 = CAN(K_{\chi(G)}, 2) - 2$, contradicting Corollary 7.1.9.

Thus we have two necessary conditions on G for

$$CAN(G, 2) < CAN(K_{\chi(G)}, 2).$$

These are:

1. $CAN(K_{\chi(G)}, 2) = c$ must be even, and
2. $\frac{1}{2}\binom{c-2}{\frac{c-2}{2}} < \chi(G) \leq \left\lceil \frac{1}{2}\binom{c-1}{\frac{c}{2}-1} \right\rceil$.

5.3.2 Cores of the Binary Qualitative Independence Graphs

From Section 4.1.4, a core of a graph G is an induced subgraph, denoted $G^\bullet$, with the property that there exists a homomorphism from G to $G^\bullet$ and every endomorphism on $G^\bullet$ is an automorphism. A core of a graph is useful since it preserves any property of the graph defined by homomorphisms (for example, chromatic number, maximum clique size, and fractional chromatic number) and for any graph H, if there exists a homomorphism from H to G, then there also exists a homomorphism from H to $G^\bullet$.

In this section, we find a core for the graphs $QI(n,2)$. First we need some notation. Recall from Section 3.1.2, that a *uniform partition* is a partition in which all the classes have the same size and an *almost-uniform partition* is a partition in which the sizes of the classes differ by at most one. For n even, let $UQI(n,2)$ be the subgraph of $QI(n,2)$ induced by the vertices that correspond to uniform partitions. For n odd, let $AUQI(n,2)$ be the subgraph of $QI(n,2)$ induced by the vertices that correspond to almost-uniform partitions. These graphs are further explored in Section 6.1.

Proposition 5.3.7. *For n an even positive integer, there exists a homomorphism $QI(n,2) \rightarrow UQI(n,2)$. For n an odd positive integer, there exists a homomorphism $QI(n,2) \rightarrow AUQI(n,2)$.*

Proof. Recall from Theorem 5.3.3 that the poset of subsets of an n-set can be decomposed into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ disjoint chains.

For every $P \in V(QI(n,2))$ with $P = \{P_1, P_2\}$, assume $|P_1| \leq |P_2|$. The chain in the poset of subsets of an n-set that contains P_1 contains a unique $\lfloor \frac{n}{2} \rfloor$-set, call this set P_1'. Let $P' = \{P_1', \overline{P_1'}\}$. Define a map ϕ from $QI(n,2)$ to $UQI(n,2)$ (or $AUQI(n,2)$ if n is odd) by $\phi(P) = P'$.

The map ϕ is a homomorphism. To see this, assume partitions $P = \{P_1, P_2\}$ and $Q = \{Q_1, Q_2\}$ are qualitatively independent. Then $P_1 \cap Q_1 \neq \emptyset$, and P_1 and Q_1 are in distinct chains. Let P_1' and Q_1' be the unique sets of size $\lfloor \frac{n}{2} \rfloor$ in the chains that contain P_1 and Q_1, respectively. Then P_1' and Q_1' are distinct, and since $P \subseteq P_1'$ and $Q \subseteq Q_1'$, $P_1 \cap Q_1 \subseteq P_1' \cap Q_1'$. Thus $P_1' \cap Q_1' \neq \emptyset$. Finally, since P_1' and Q_1' are distinct $\lfloor \frac{n}{2} \rfloor$-sets, they are incomparable. Thus the partitions $P' = \{P_1', P_2'\}$ and $Q' = \{Q_1', Q_2'\}$ are qualitatively independent. ✩

From Theorem 5.2.7, for a graph G and positive integers k, n, a $CA(n, G, k)$ exists if and only if there is a homomorphism $G \rightarrow QI(n,k)$. Further, from Proposition 5.3.7, a $CA(n, G, 2)$ exists if and only if there is a homomorphism from G to $UQI(n,2)$, if n is even, or to $AUQI(n,2)$, if n is odd.

Theorem 5.3.8. *For a graph G and a positive integer n, if there exists a $CA(n, G, 2)$, then it is possible to find a covering array $CA(n, G, 2)$ in which the rows have exactly*

$\lfloor n/2 \rfloor$ 0s. Moreover, if n is even, then it is possible to find such a covering array with the rows all beginning with 0.

Proof. From Proposition 5.2.7, if there exists a $CA(n, G, 2)$, then there exists a homomorphism $\phi : G \to QI(n, 2)$.

If n is even, then there is an endomorphism $\psi : QI(n, 2) \to UQI(n, 2)$. For each vector $v \in V(G)$, replace the row of $CA(n, G, 2)$ corresponding to v by the vector corresponding to the uniform 2-partition $\psi(\phi(v))$. If this vector does not start with 0, simply relabel the 0s by 1s and the 1s by 0s.

If n is odd, then there is an endomorphism $\psi : QI(n, 2) \to AUQI(n, 2)$. For each vector $v \in V(G)$, replace the row of $CA(n, G, 2)$ corresponding to v by the vector corresponding to the almost-uniform partition $\psi(\phi(v))$. ✫

In the next two theorems, we give a core of $QI(n, 2)$.

Theorem 5.3.9. *For n even, $UQI(n, 2)$ (which is isomorphic to $K_{\frac{1}{2}\binom{n}{n/2}}$) is a core of $QI(n, 2)$.*

Proof. For n even, any two distinct uniform 2-partitions of an n-set are qualitatively independent. Thus, the graph $UQI(n, 2)$ is isomorphic to $K_{\frac{1}{2}\binom{n}{n/2}}$. From Theorem 5.3.7 there is a homomorphism from $QI(n, 2)$ to $UQI(n, 2)$. Since $K_{\frac{1}{2}\binom{n}{n/2}}$ is a core, the theorem holds. ✫

Theorem 5.3.10. *For n odd, $AUQI(n, 2)$ is a core of $QI(n, 2)$.*

Proof. From Theorem 5.3.7, there is a homomorphism from $QI(n, 2)$ to $AUQI(n, 2)$. All that is needed is to show that $AUQI(n, 2)$ is itself a core.

Let $AUQI^{\bullet}(n, 2)$ be a core of $AUQI(n, 2)$. The graph $AUQI(n, 2)$ is vertex transitive. Thus, a core $AUQI^{\bullet}(n, 2)$ is also vertex transitive (Theorem 4.1.4) and from Theorem 4.1.5,

$$|V(AUQI^{\bullet}(n, 2))| \quad \text{divides} \quad |V(AUQI(n, 2))| = \binom{n}{\frac{n-1}{2}}.$$

Since the graph $AUQI(n, k)$ is an induced subgraph of $QI(n, k)$ and there is a homomorphism from $QI(n, k)$ to $AUQI(n, k)$, we have that $\chi(AUQI(n, k)) =$

$\chi(QI(n,k))$. As $AUQI^\bullet(n,2)$ is a core, we also have that

$$\chi(AUQI^\bullet(n,k)) = \chi(QI(n,k)).$$

Thus,

$$\chi(AUQI^\bullet(n,2)) = \chi(AUQI(n,2)) = \left\lceil \frac{1}{2}\binom{n}{\frac{n-1}{2}} \right\rceil.$$

These facts give two possibilities for $|V(AUQI^\bullet(n,2))|$, either it is $\frac{1}{2}\binom{n}{\frac{n-1}{2}}$ or $\binom{n}{\frac{n-1}{2}}$.

If $|V(AUQI^\bullet(n,2))| = \frac{1}{2}\binom{n}{\frac{n-1}{2}}$ then, from the chromatic number of $AUQI^\bullet(n,2)$, the graph $AUQI^\bullet(n,2)$ would have to be the complete graph. This is not the case since

$$\omega(AUQI^\bullet(n,2)) = \omega(AUQI(n,2)) = \binom{n-1}{\frac{n-1}{2}-1} < \frac{1}{2}\binom{n}{\frac{n-1}{2}}.$$

Thus $|V(AUQI^\bullet(n,2))| = \binom{n}{(n-1)/2}$, and as it is an induced subgraph, $AUQI^\bullet(n,2) = AUQI(n,2)$. This means $AUQI(n,2)$ is a core and in particular, it is a core of $QI(n,2)$. ✫

For n odd, the graph $AUQI(n,2)$ is isomorphic to the complement of the Kneser graph $K_{n:\frac{n-1}{2}}$.

5.4 The graphs $QI(k^2,k)$

For k a positive integer, the family of graphs $QI(k^2,k)$ is a special family. First, for each k, the graph $QI(k^2,k)$ is the qualitative independence graph with the smallest number of vertices. Further, a clique in $QI(k^2,k)$ is equivalent to an orthogonal array, thus from Corollary 2.3.5, for k a prime power, we have $\omega(QI(k^2,k)) = k+1$, and from Corollary 2.3.4 we have $\omega(QI(k^2,k)) \leq k+1$ in general. We study some properties of these graphs.

Lemma 5.4.1. *For any integer $k \geq 2$, the qualitative independence graph $QI(k^2,k)$ is vertex transitive.*

Proof. The vertices of $QI(k^2,k)$ correspond to uniform k-partitions of $\{1,\ldots,k^2\}$. For any pair of uniform k-partitions P, Q, there is a permutation in $\sigma \in Sym(k^2)$ such that $\sigma(P) = Q$. ✫

Theorem 5.4.2. *For any integer k,*

$$\chi(QI(k^2,k)) \leq \binom{k+1}{2}.$$

Proof. Assign one colour to each of the $\binom{k+1}{2}$ pairs from the set $\{1,2,\ldots,k+1\}$. By the pigeon-hole principle, each partition $P \in V(QI(k^2,k))$ has at least two distinct elements from $\{1,2,\ldots,k+1\}$ in the same class $P_i \in P$. Assign P any colour that corresponds to a pair of distinct elements from $\{1,2,\ldots,k+1\}$ that occur in the same class of P.

This colouring is a proper colouring of $QI(k^2,k)$. If partitions P and Q are assigned the same colour, then for some pair $a,b \in \{1,2,\ldots,k+1\}$ (with $a \neq b$) there are i,j such that $a,b \in P_i$ and $a,b \in Q_j$. This means that P_i can not intersect all $k-1$ sets Q_l, $l \neq j$, so P and Q are not qualitatively independent. ✫

Theorem 5.4.3. *For any positive integer k,*

$$\chi^*(QI(k^2,k)) \leq k+1.$$

Proof. There is a homomorphism from $QI(k^2,k)$ to the Kneser graph $K_{k^2-1:k-1}$. Each vertex in $QI(k^2,k)$ corresponds to a k-partition of a k^2-set. For a partition P, let P_1 be the class with the element $k^2 \in P_1$. Map the partition P to the $(k-1)$-subset $P_1 \backslash \{k^2\}$.

If partitions P and Q are adjacent in $QI(k^2,k)$, then, for each class $P_i \in P$ and $Q_j \in Q$, $|P_i \cap Q_j| = 1$. In particular, if $k^2 \in P_1 \in P$ and $k^2 \in Q_1 \in Q$, then $(P_1\backslash\{k^2\}) \cap (Q_1\backslash\{k^2\}) = \emptyset$. Thus, this map defines a homomorphism from $QI(k^2,k)$ to $K_{k^2-1:k-1}$.

✫

Also by Inequality (8), Section 4.1.5, Theorem 5.4.3 implies that $\omega(QI(k^2,k)) \leq k+1$, which is another proof of Lemma 2.4.5.

Lemma 5.4.4. *For all k, the graph $QI(k^2,k)$ is $(k!)^{k-1}$-regular.*

Proof. Let P be a vertex in $QI(k^2, k)$, so P is a k-uniform k-partition.

Any k-uniform k-partition that is qualitatively independent with P must have each of the k elements in each of the k classes of P in different classes. There are $k!$ ways to place the k elements of a class of P in different classes. Since there are k classes in P, this gives $(k!)^k$ ways to arrange all the elements of all the classes. Finally, we need to divide by $k!$ so as not to over count the partitions with the same k classes in different orders. Thus there are $\frac{1}{k!}(k!)^k$ partitions qualitatively independent with P. Thus each vertex of $QI(k^2, k)$ has degree $(k!)^{k-1}$. ✰

From Lemma 5.2.4, the graphs $QI(k^2, k)$ have diameter 2. For each vertex $v \in V(QI(k^2, k))$ there are $(k!)^{k-1}$ vertices at distance 1 and

$$\frac{1}{k!}\binom{k^2}{k}\binom{k^2-k}{k}\cdots\binom{k}{k} - (k!)^{k-1} - 1$$

vertices at distance 2.

Chapter 6

Uniform Qualitative Independence Graphs

In the previous chapter, we saw that a core of the graph $QI(n, 2)$, for n even, is the subgraph induced by vertices that are uniform 2-partitions of an n-set. Motivated by this, we consider the subgraph of $QI(n, k)$ which is induced by the uniform k-partitions. This subgraph is called the *uniform qualitative independence graph*.

These graphs are regular and vertex transitive. A clique in the uniform qualitative independence graph corresponds to a *balanced covering array*; that is, a covering array with the property that each letter occurs the same number of times in each row.

We start this chapter by proving bounds on the size of a maximum clique in a uniform qualitative independence graph. These bounds are a result of the fact that the uniform qualitative independence graphs are vertex transitive. Some of these bounds are known, but the method we use to prove them is new.

Next, we consider the two ratio bounds for graphs from Section 4.2.4, which we restate here: For a d-regular graph G with smallest eigenvalue τ, we have

$$\alpha(G) \leq \frac{|V(G)|}{1 - \frac{d}{\tau}}; \tag{11}$$

further, if G is arc transitive or a single graph in an association scheme, then

$$\omega(G) \leq 1 - \frac{d}{\tau}. \tag{12}$$

To use these bounds for a graph G, it is necessary to know the largest and smallest eigenvalue of G. For certain special cases of the uniform qualitative independence graphs, it is possible to find the eigenvalues, but in general they are difficult to find. In this chapter, we find an equitable partition on the vertex set of the uniform qualitative independence graphs that reduces the calculations for the eigenvalues. With this partition, we give the eigenvalues (and their multiplicities) for several small uniform qualitative independence graphs.

Eigenvalues and their multiplicities can give more information about the graph than just these two ratio bounds. For example, Godsil and Newman [33] use the multiplicity of the smallest eigenvalue of $QI(9,3)$ and the properties of the maximum independent sets to characterize all of the maximum independent sets in the graph. This, in turn, is used to prove that the graph $QI(9,3)$ is a core. This result is restated with more detail in Section 6.4.1.

Mathon and Rosa [50] have shown that the graph $QI(9,3)$ is a graph in an association scheme on $\mathcal{U}_3^9$. We can generalize this to $\mathcal{U}_k^{ck}$ for all positive integers k and c, although it is not clear if this generalization produces an association scheme. For the graphs $UQI(12,3)$ and $UQI(15,3)$, we give the modified matrix of eigenvalues of the graphs in this generalization and conjecture that they are part of an association scheme.

6.1 Uniform Qualitative Independence Graphs

Recall from Section 3.1.2 that, for positive integers n, k, c with $n = ck$, a *uniform k-partition of an n-set* is a partition of an n-set into k classes each of size c.

Definition 6.1.1 (*Uniform Qualitative Independence Graph*). For positive integers n, k, with the property that k divides n and $n \geq k^2$, the *uniform qualitative independence graph* $UQI(n,k)$ is the graph whose vertex set is the set of all uniform k-partitions of an n-set. Vertices are adjacent if and only if the corresponding partitions are qualitatively independent.

The graphs $UQI(n,k)$ are called *partition graphs* in [33] and are denoted $P(c^k)$, where $c = n/k$.

The uniform qualitative independence graphs are vertex transitive, and from Section 3.1.2,

$$|V(UQI(n,k))| = U(n,k) = \frac{1}{k!}\binom{n}{c}\binom{n-c}{c}\cdots\binom{n-(k-1)c}{c}.$$

If k does not divide n, it is not possible to have uniform k-partitions of an n-set. In this case, *almost-uniform partitions* are considered. Again, recall from Section 3.1.2 that, for positive integers n, k, c with $n = ck + r$ where $0 \leq r < k$, an almost-uniform k-partition of an n-set is a partition of an n-set into k classes, each of size c or $c+1$. If $r = 0$ in the above definition, then an almost-uniform k-partition of an n-set is also a uniform k-partition of an n-set.

Definition 6.1.2 (*Almost-Uniform Qualitative Independence Graph*). Let n, k, c be positive integers such that $n = ck + r$ where $0 \leq r < k \leq c$. The *almost-uniform qualitative independence graph* $AUQI(n,k)$ is the graph whose vertex set is the set of all almost-uniform k-partitions of an n-set. Vertices are adjacent if and only if the corresponding partitions are qualitatively independent.

The almost-uniform qualitative independence graphs are vertex transitive. From Section 3.1.2 the number of vertices in this graph is

$$\begin{aligned}|V(AUQI(n,k))| = AU(n,k) \quad &= \quad \frac{1}{r!(k-r)!}\binom{n}{c}\binom{n-c}{c}\cdots\binom{n-(k-r-1)c}{c} \\ &\quad \binom{r(c+1)}{c+1}\binom{(r-1)(c+1)}{c+1}\cdots\binom{c+1}{c+1}.\end{aligned}$$

If k and n are positive integers, and k divides n, then $AUQI(n,k) = UQI(k,n)$; in the special case that $n = k^2$, $AUQI(k^2,k) = UQI(k^2,k) = QI(k^2,k)$.

6.2 Bounds from Vertex Transitivity

In this section, we give upper bounds on $\omega(UQI(n,k))$ and $\omega(AUQI(n,k))$ derived from the fact that the uniform and almost-uniform qualitative independence graphs are vertex transitive. An upper bound on $\omega(UQI(n,k))$ (or $\omega(AUQI(n,k))$) gives a lower bound on the size of a balanced (or an almost-balanced covering array). To see this, assume $\omega(UQI(ck,k)) \leq r$. Then, there does not exist a balanced $CA(ck, r+1, k)$, which means a balanced covering array with $r+1$ rows on a k-alphabet must have more than ck columns.

These results are not new, as versions of these bounds for transversal covers were proven by Stevens, Moura and Mendelsohn [71] (these are restated in Lemma 2.2.5). The proof from [71] uses Bollobás's Theorem, Theorem 3.2.4.

Before proving these bounds, we need some notation. Let n, k be positive integers and set $c = \lfloor \frac{n}{k} \rfloor$. For a set $A \subseteq \{1, \ldots, n\}$, define the subset $S_A \subseteq \mathcal{AU}_k^n$ by

$$S_A = \{P \in \mathcal{AU}_k^n : A \subseteq P_0, \text{ for some class } P_0 \in P \text{ with } |P_0| = c\}.$$

These sets will be seen again in Section 7.5, where they are called *trivially partially intersecting partition systems.*

Theorem 6.2.1. *For positive integers n, k, c with $n = ck + r$ where $0 \leq r < k \leq c$,*

$$\omega(AUQI(n,k)) \leq \frac{n!(k-2)!}{(k-r)(n-c+k-2)!c!}.$$

Proof. Let $A = \{1, 2, \ldots, c-(k-2)\}$. Consider the set $S_A \subseteq \mathcal{AU}_k^n$. Assume $P, Q \in S_A$, and let $P_0 \in P$ and $Q_0 \in Q$ be the classes with $A \subseteq P_0$ and $A \subseteq Q_0$ and $|P_0| = |Q_0| = c$. There are at most $c - (c-(k-2)) = k-2$ elements in P_0 that are not also in Q_0. Thus, the class P_0 can intersect at most $k-2$ classes of Q other than Q_0. Since Q has k classes, P_0 can not intersect every class in Q. This means the partitions P and Q are not qualitatively independent, and the set S_A is an independent set in $AUQI(n,k)$. Since

$$|S_A| = \binom{n-(c-(k-2))}{k-2} AU(n-c, k-1),$$

this produces a lower bound on $\alpha(AUQI(n,k))$,

$$\binom{n-(c-(k-2))}{k-2} AU(n-c, k-1) \leq \alpha(AUQI(n,k)).$$

Since the graph $AUQI(n,k)$ is vertex transitive, by Inequality (8) in Section 4.1.5

$$\begin{aligned}\omega(AUQI(n,k)) \leq \frac{|V(AUQI(n,k)|}{\alpha(AUQI(n,k))} &\leq \frac{AU(n,k)}{\binom{n-(c-(k-2))}{k-2}AU(n-c,k-1)}\\ &\leq \frac{\frac{1}{(k-r)!}\binom{n}{c}}{\frac{1}{(k-1-r)!}\binom{n-(c-(k-2))}{k-2}}\\ &= \frac{n!(k-2)!}{(k-r)(n-c+k-2)!c!}.\end{aligned}$$

☆

Next, we give several bounds for specific values of n. Starting with a bound for $\omega(QI(k^2,k))$, which is a restatement of Lemma 2.4.5.

Corollary 6.2.2. *For all positive integers k, we have $\omega(QI(k^2,k)) \leq k+1$.*

Proof. Use Theorem 6.2.1 with $n = k^2$, $c = k$ and $r = 0$ along with the fact that $QI(k^2,k) = AUQI(k^2,k)$. ☆

This next bound is for $n = k^2+1$. A stronger version of this bound is given in [71] for transversal covers and is restated as Part 3 of Lemma 2.2.5.

Corollary 6.2.3. *For all positive integers $k \geq 6$, we have $\omega(QI(k^2+1,k)) \leq k+2$.*

Proof. The graph $QI(k^2+1,k) = AUQI(k^2+1,k)$. Using Theorem 6.2.1 with $n = k^2+1$, $c = k$ and $r = 1$,

$$\begin{aligned}\omega(QI(k^2+1,k)) &\leq \frac{n!(k-2)!}{(k-r)(n-c+k-2)!c!}\\ &= \frac{(k^2+1)k}{(k-1)(k-1)}\\ &= k+2+\frac{4k-2}{k^2-2k+1}.\end{aligned}$$

For $k \geq 6$, we have $4k-2 < k^2-2k+1$ and the corollary holds. ☆

Since the rows of any covering array $CA(k^2+1, s, k)$ with $s > 1$ correspond to almost-uniform k-partitions of a (k^2+1)-set, this bound can be translated into a bound on s. In particular, Corollary 6.2.3 implies that if $k \geq 6$, then for any $CA(k^2+1, s, k)$, $s \leq k+2$. Equivalently, for $k \geq 6$, if a $CA(n, s, k)$ exists with $s \geq k+3$, then $n \geq k^2+2$. The result from [71] is: if $s \geq k+2$, then $n \geq k^2+2$.

Corollary 6.2.4. *For $k \geq 13$, we have $\omega(AUQI(k^2+2, k)) \leq k+3$.*

Proof. Using Theorem 6.2.1 with $n = k^2+2$, $c = k$ and $r = 2$,

$$\begin{aligned} \omega(AUQI(k^2+1,k)) &\leq \frac{n!(k-2)!}{(k-r)(n-c+k-2)!c!} \\ &= \frac{(k^2+1)(k^2+2)}{(k-2)(k-1)k} \\ &= k+3+\frac{10k^2-6k+2}{k^3-3k^2+2k}. \end{aligned}$$

For $k \geq 13$, we have $10k^2-6k+2 < k^3-3k^2+2k$ and the corollary holds. ✫

Corollary 6.2.4 can not be directly translated to a lower bound on the size of a covering array. The rows of a covering array $CA(n, s, k)$ correspond to a set of cardinality s of qualitatively independent k-partitions of an n-set, but when $n \geq k^2+2$ these partitions are not necessarily almost-uniform partitions. For example, such a partition could have one class of cardinality $k+2$ and all other classes with cardinality k.

However, for positive integers n, k, s such that k divides n and $s > 1$, if a $CA(n, s, k)$ is a balanced covering array, then the rows of $CA(n, s, k)$ correspond to uniform k-partitions of an n-set. An upper bound on $\omega(UQI(n, k))$ gives a lower bound on the size of a balanced covering array.

With Theorem 6.2.1, we have such a bound on general uniform qualitatively independence graphs. This bound is not new, but it is, surprisingly, equivalent to the bound in Part 4 of Lemma 2.2.5 for point-balanced transversal covers which originally appeared in [71].

Corollary 6.2.5. *For positive integers n, k, c with $n = ck$ and $k \leq c$,*

$$\omega(UQI(n,k)) \leq \frac{(ck)!(k-2)!}{k(ck-c+k-2)!c!}.$$

Proof. This is Theorem 6.2.1 with $n = ck$, and $r = 0$. ✰

This corollary can be restated in terms of covering arrays. For positive integers n, k, c, s with $n = ck$ and $s > 1$, if $CA(n, s, k)$ is a balanced covering array, then $s \leq \frac{(ck)!(k-2)!}{k(ck-c+k-2)!c!}$. Alternately, if a $CA(n, s, k)$ exists with $s > \frac{(ck)!(k-2)!}{k(ck-c+k-2)!c!}$, then the $CA(n, s, k)$ is not a balanced covering array.

There are several questions concerning these bounds. For all integers n, k, the graph $AUQI(n, k)$ is an induced subgraph of $QI(n, k)$, and, in particular,

$$\omega(AUQI(n,k)) \leq \omega(QI(n,k)).$$

The first question is: how big can $\omega(QI(n, k)) - \omega(AUQI(n, k))$ be?

From Section 5.3.2, the graph $AUQI(n, 2)$ is a core of $QI(n, 2)$; in this case $\omega(QI(n, 2)) = \omega(AUQI(n, 2))$. If this is true in general — that is, if $AUQI(n, k)$ is a core for $QI(n, k)$ — then $\omega(QI(n, k)) = \omega(AUQI(n, k))$ and an upper bound on $\omega(AUQI(n, k))$ gives a lower bound on the size of a covering array. Moreover, there would exist a balanced covering array with same size as an optimal covering array.

Question 6.2.6. For positive integers k and n, is the graph $AUQI(n, k)$ a core of $QI(n, k)$? For a positive integer r, if $n = CAN(r, k)$, does there exist a balanced $CA(n, r, k)$?

Another question is: are the bounds given here close to the actual size of the maximum cliques in $AUQI(n, k)$? This question is much easier to answer. A clique in $QI(n, k)$ is a qualitatively independent family of partitions so $\omega(QI(n, k)) = N(n, k)$ (recall from Section 2.4 that $N(n, k)$ is the maximum number of rows s in a $CA(n, s, k)$). Since $AUQI(n, k)$ is an induced subgraph of $QI(n, k)$, $\omega(AUQI(n, k)) \leq N(n, k)$. The value of $N(n, k)$ is not known in general, but from Equation (4) in Section 2.4.2,

$$\limsup_{n\to\infty} \frac{\log_2 N(n,k)}{n} = \frac{2}{k}.$$

Let $f(n,k)$ be the bound on $\omega(AUQI(n,k))$ from Lemma 6.2.1. Then

$$\limsup_{n\to\infty} \frac{\log_e f(n,k)}{n} = \log_e\left(\frac{k}{k-1}\right) + \frac{\log_e(k-1)}{k}.$$

The details of this calculation are omitted, but a similar calculation is shown in full detail in Section 7.1.3. For $k > 2$, it is clear from the asymptotic growth of $f(n,k)$ that the bound on $\omega(AUQI(n,k))$ from Theorem 6.2.1 is far from the real value of $\omega(AUQI(n,k))$ for large n.

6.3 Eigenvalues of Qualitative Independence Graphs

In this section, we give the eigenvalues for the binary qualitative independence graphs and we find the largest and smallest eigenvalues for $QI(k^2,k)$.

6.3.1 Eigenvalues for $UQI(n,2)$ and $AUQI(n,2)$

For n even, $UQI(n,2)$ is isomorphic to the complete graph on $\frac{1}{2}\binom{n}{n/2}$ vertices. By Theorem 4.2.1, the eigenvalues of the complete graph on $\frac{1}{2}\binom{n}{n/2}$ vertices are $\frac{1}{2}\binom{n}{n/2}-1$ and -1. Further, the complete graph is regular and part of an association scheme (the trivial scheme) so both ratio bounds (where d is the largest eigenvalue and τ is the smallest)

$$\alpha(UQI(n,2)) \leq \frac{|V(UQI(n,2))|}{1-\frac{d}{\tau}} \quad \text{and} \quad \omega(UQI(n,2)) \leq 1 - \frac{d}{\tau}$$

hold, and in fact, both hold with equality.

If n is odd, it is an entirely different story. The graph $AUQI(n,2)$ is isomorphic to the complement of the Kneser graph $K_{n:\frac{n-1}{2}}$ (Section 5.3.2). As stated in Example 4.2.6 in Section 4.2.2, the eigenvalues of $K_{n:\frac{n-1}{2}}$ are

$$(-1)^i\binom{\frac{n+1}{2}-i}{\frac{n-1}{2}-i} = (-1)^i\left(\frac{n+1}{2}-i\right), \qquad \text{for } i = 0,1,\ldots,\frac{n-1}{2}.$$

From Theorem 4.2.3, the eigenvalues of $AUQI(n,2)$ are

$$\binom{n}{\frac{n-1}{2}} - \frac{n+1}{2} - 1 \text{ and } -1-(-1)^i\left(\frac{n+1}{2}-i\right), \text{ for } i=1,\ldots,\frac{n-1}{2}.$$

In particular, the largest eigenvalue of $AUQI(n,2)$ is $\binom{n}{\frac{n-1}{2}} - \frac{n+1}{2} - 1$ and the smallest is $\frac{1-n}{2}$ (when $i=2$).

Since $AUQI(n,2)$ is vertex transitive, the ratio bound for independent sets holds:

$$\alpha(AUQI(n,2)) \leq \frac{|V(AUQI(n,2))|}{1-\frac{d}{\tau}} = \frac{\binom{n}{\frac{n-1}{2}}}{1-\frac{\binom{n}{\frac{n-1}{2}}-\binom{\frac{n+1}{2}}{}-1}{-\frac{n-1}{2}}} = \frac{n-1}{2-4\binom{n}{\frac{n-1}{2}}^{-1}}.$$

It is not hard to see that $\alpha(AUQI(n,2)) = 2$ for all n. So, this bound is tight only for $n=5$, and for large n this bound is not good at all.

Next, consider the ratio bound for maximum cliques, Inequality (12). From Theorem 5.3.2, $\omega(AUQI(n,2)) = \frac{1}{2}\binom{n}{\frac{n-1}{2}}$. Let d be the largest eigenvalue of $AUQI(n,2)$ and τ the smallest, then $1-\frac{d}{\tau} = \frac{2}{n-1}\left(\binom{n}{\frac{n-1}{2}}-2\right)$. For $n>5$, the ratio bound for maximum cliques does not hold. This means that, for n odd and $n>5$, the graph $AUQI(n,2)$ is not a single graph in an association scheme nor is it arc transitive (Theorem 4.2.12).

In conclusion, it seems that the ratio bounds are more appropriate for the uniform qualitative independence graphs. Motivated by this, we will focus on these graphs.

6.3.2 Eigenvalues for $QI(k^2,k)$

In this section, we consider the qualitative independence graphs $QI(k^2,k)$. In this case, $QI(k^2,k) = UQI(k^2,k)$. Cliques in $QI(k^2,k)$ correspond to orthogonal arrays. The size of the largest clique in $QI(k^2,k)$ is at most $k+1$ (Corollary 6.2.2, or, equivalently, Lemma 2.4.5), and for k a prime power $\omega(QI(k^2,k)) = k+1$ (Corollary 2.3.5 and Lemma 2.4.4). The equivalence between orthogonal arrays and transversal designs and MOLS, from Theorem 2.3.3, makes this case particularly interesting.

For all positive integers k, we find the largest and the smallest eigenvalues for the graph $QI(k^2,k)$. These can be used with the ratio bounds to find upper bounds on $\alpha(QI(k^2,k))$ and $\omega(QI(k^2,k))$.

For the rest of this section, k will be a positive integer and $\mathbf{1}$ will denote the all ones vector of length $U(k^2, k)$. Following the notation from Section 6.2, for any positive integer n, and $a, b \in \{1, \dots, n\}$, $S_{\{a,b\}} \subseteq \mathcal{U}_k^n$ denotes the set of all k-partitions of an n-set that have a class that contains both the elements a and b. For any distinct $a, b \in \{1, 2, \dots, k^2\}$, the set $S_{\{a,b\}}$ is an independent set in $QI(k^2, k)$. A proof of this is contained in the proof of Theorem 6.2.1.

Lemma 6.3.1. *For positive integer k, the largest eigenvalue for $QI(k^2, k)$ is $(k!)^{k-1}$ with multiplicity one. The corresponding eigenvector is $\mathbf{1}$.*

Proof. From Lemma 5.4.4 and Lemma 5.2.4, $QI(k^2, k)$ is $(k!)^{k-1}$-regular and connected. The result follows from Theorem 4.2.2. ✩

Theorem 6.3.2. *For any k, $\frac{-(k!)^{k-1}}{k}$ is the smallest eigenvalue for $QI(k^2, k)$. Further, for any distinct $a, b \in \{1, 2, \dots, k^2\}$, if v is the characteristic vector for $S_{\{a,b\}}$, then $v - \frac{1}{k+1}\mathbf{1}$ is an eigenvector for $\frac{-(k!)^{k-1}}{k}$.*

Proof. Let A be the adjacency matrix for $QI(k^2, k)$ and let $x = v - \frac{1}{k+1}\mathbf{1}$. To show that x is an eigenvector corresponding to the eigenvalue $\frac{-(k!)^{k-1}}{k}$, all that is needed is to show that $Ax = \frac{-(k!)^{k-1}}{k}x$.

For $i \in \{1, 2, \dots, U(k^2, k)\}$, denote row i of A by A_i. Each row A_i corresponds to a vertex in $QI(k^2, k)$, which is a k-uniform k-partition; call this partition P^i.

Consider two cases, first when $P^i \in S_{\{a,b\}}$ and second when $P^i \notin S_{\{a,b\}}$. Since the degree of P^i is $(k!)^{k-1}$ (Lemma 5.4.4), if $P^i \in S_{\{a,b\}}$, then P^i is adjacent to $(k!)^{k-1}$ vertices in $QI(k^2, k)$, none of which are in $S_{\{a,b\}}$. Thus

$$A_i \cdot x = (k!)^{k-1}\left(\frac{-1}{k+1}\right) = \frac{-(k!)^{k-1}}{k}\left(1 - \frac{1}{k+1}\right).$$

If $P^i \notin S_{\{a,b\}}$ then P^i is adjacent to $(k!)^{k-1}$ vertices in $QI(k^2, k)$. By a counting argument, similar to, but slightly more complicated than, the counting argument for the degree of each vertex (see the proof of Lemma 5.4.4), $(k-1)!(k!)^{k-2}$ of these are in $S_{\{a,b\}}$ and $(k-1)^2(k-2)!(k!)^{k-2}$ are not in $S_{\{a,b\}}$. Thus

$$\begin{aligned} A_i \cdot x &= (k-1)^2(k-2)!(k!)^{k-2}\left(\frac{-1}{k+1}\right) + (k-1)!(k!)^{k-2}\left(1-\frac{1}{k+1}\right) \\ &= \frac{-(k!)^{k-1}}{k}\left(\frac{-1}{k+1}\right). \end{aligned}$$

Thus,

$$Ax = A\left(v - \frac{1}{k+1}\mathbf{1}\right) = \frac{-(k!)^{k-1}}{k}\left(v-\frac{1}{k+1}\mathbf{1}\right) = \frac{-(k!)^{k-1}}{k}x.$$

Finally, we need to show that $\frac{-(k!)^{k-1}}{k}$ is the smallest eigenvalue for $QI(k^2,k)$. From above, a partition $P \in S_{\{a,b\}}$ is adjacent to no partitions in $S_{\{a,b\}}$ and $(k!)^{k-1}$ partitions not in $S_{\{a,b\}}$; a partition $P \notin S_{\{a,b\}}$ is adjacent to $(k-1)!(k!)^{k-2}$ partitions in $S_{\{a,b\}}$ and $(k-1)^2(k-2)!(k!)^{k-2}$ partitions not in $S_{\{a,b\}}$. This means the partition $\{S_{\{a,b\}}, V(QI(k^2,k))\backslash S_{\{a,b\}}\}$ is an equitable partition (see Section 4.2.2). From Theorem 4.2.11,

$$|S_{\{a,b\}}| = \frac{|V(QI(k^2,k))|}{1-\frac{(k!)^{k-1}}{\tau}}$$

where τ is the least eigenvalue. Since the cardinality of $S_{\{a,b\}}$ is $\binom{k^2-2}{k-2}U(k^2-k,k-1)$ and $|V(QI(k^2,k))| = U(k^2,k)$, the least eigenvalue $\tau = \frac{-(k!)^{k-1}}{k}$. ✰

It is possible to see that the partition $\{S_{\{a,b\}}, V(QI(k^2,k))\backslash S_{\{a,b\}}\}$ is an equitable partition in another manner. Recall from Section 4.2.2 that the orbits of a group action on the vertices of a graph form an equitable partition. For any vertex $P \in V(QI(k^2,k))$ and any permutation $\sigma \in Sym(k^2)$ let $\sigma(P)$ be the partition with $\sigma(a) \in (\sigma(P))_i$ if and only if $a \in P_i$. In this manner, any subgroup of $Sym(k^2)$ induces a subgroup of $Aut(QI(k^2,k))$. Let $H' = \{\sigma \in Sym(k^2) : \sigma(\{a,b\}) = \{a,b\}\}$. Then H' is a subgroup of $Sym(k^2)$ and H' induces a subgroup H of $Aut(QI(k^2,k))$. The group action of H on the vertices of $QI(k^2,k)$ has two orbits: $S_{\{a,b\}}$ and $V(QI(k^2,k))\backslash S_{\{a,b\}}$.

From Lemma 6.3.2 and Lemma 6.3.1, we know the largest and the smallest eigenvalues of $QI(k^2,k)$, so now we can use the ratio bound on the size of the largest independent set in $QI(k^2,k)$. This result will be seen again in Section 7.5.

Lemma 6.3.3. *For all k,*

$$\alpha(QI(k^2,k)) = \binom{k^2-2}{k-2} U(k^2-k, k-1). \tag{13}$$

Proof. From the ratio bound for independent sets (Inequality (11)), with the largest and smallest eigenvalues for $QI(k^2,k)$ (Theorem 6.3.1 and Theorem 6.3.2), we have

$$\alpha(QI(k^2,k)) \leq \frac{U(k^2,k)}{1 - \frac{(k!)^{k-1}}{\frac{-(k!)^{k-1}}{k}}} = \binom{k^2-2}{k-2} U(k^2-k, k-1).$$

For any distinct $i, j \in \{1, \ldots, k^2\}$, the set $S_{\{i,j\}} \subset \mathcal{U}_k^n$ is an independent set of size $\binom{k^2-2}{k-2} U(k^2-k, k-1)$. ✰

With the exact value of $\alpha(QI(k^2,k))$ and Corollary 4.1.10, we also have the exact value of the fractional chromatic number of $QI(k^2,k)$.

Corollary 6.3.4. *For any positive integer k,*

$$\chi^*(QI(k^2,k)) = k+1.$$

Before we can apply the ratio bound for maximum cliques (Inequality (12)), we need to prove that $QI(k^2,k)$ is arc transitive.

Theorem 6.3.5. *For all positive integers k, the graph $QI(k^2,k)$ is arc transitive.*

Proof. To prove that $QI(k^2,k)$ is arc transitive, we build an automorphism that takes an arbitrary arc to any other arc. Let u and v be arcs in $QI(k^2,k)$. Then $u = (P,Q)$ and $v = (R,S)$, where $P, Q, R, S \in \mathcal{U}_k^{k^2}$ and both P and Q are qualitatively independent, and R and S are qualitatively independent. Let P_i and Q_j, $i, j \in \{1, 2, \ldots, k\}$, be the classes in the partitions P and Q, and similarly, R_i and S_j the classes of R and S. Then for any $i, j \in \{1, 2, \ldots, k\}$, $|P_i \cap Q_j| = 1$ and $|R_i \cap S_j| = 1$.

Define a permutation $\phi \in Sym(k^2)$ by $\phi(P_i \cap Q_j) = R_i \cap S_j$ for all $i, j \in \{1, 2, \ldots, k\}$. The permutation ϕ induces an automorphism on $QI(k^2,k)$ with the property that $\phi(P) = R$ and $\phi(Q) = S$. ✰

This proof can be extended to the graph $QI(k^2+1,k)$ but not to any other qualitative independence graph.

Theorem 6.3.6. *For all positive integers k, the graph $QI(k^2+1,k)$ is arc transitive.*

Since the graph $QI(k^2+1,k)$ is arc transitive, the ratio bound for cliques can be used to bound $\omega(QI(k^2+1,k))$. This makes it particularly interesting to try to find the largest and smallest eigenvalues of $QI(k^2+1,k)$.

Question 6.3.7. What are the eigenvalues of the graph $QI(k^2+1,k)$?

Since the graph $QI(k^2,k)$ is arc transitive, the ratio bound for cliques, Inequality (12), holds. The ratio bound for cliques gives the now very well-known bound

$$\omega(QI(k^2,k)) \leq 1 - \frac{(k!)^{k-1}}{\frac{-(k!)^{k-1}}{k}} = k+1.$$

We have now found this bound in three different ways. The first proof was Lemma 2.4.5. This lemma used the fact that a clique of size r in $QI(k^2,k)$ corresponds to an orthogonal array $OA(n,r,k,2)$ which in turn corresponds to a set of $(r-2)$-$MOLS$ of order k. Since there can be at most $k-1$ MOLS of order k, we have the bound $r \leq k+1$. This method can not be generalized to $AUQI(n,k)$, as cliques in this graph do not correspond to sets of MOLS.

The second method used to find this bound was in Corollary 6.2.2. This proof used bounds from vertex transitivity of the graph $QI(k^2,k)$. This method can be used to bound the number of rows in a balanced covering array (Corollary 6.2.5). The comments on the asymptotic growth of this bound indicate that this bound does not seem to be very good.

The third method (Theorem 4.2.12) uses the ratio bound for cliques and the eigenvalues of the graph. This method has two problems: first, finding the eigenvalues of graphs can be difficult, and second, it is not clear if the ratio bound for cliques holds for all uniform qualitative independence graphs. Indeed, Theorem 4.2.12 holds for arc-transitive graphs and for graphs that are a single graph in an association scheme. As it is clear that the first two methods cannot be extended, we turn our attention to this third method. In the next section, we find the eigenvalues for several qualitative independence graphs and also consider whether or not they are graphs in association schemes.

6.4 Equitable Partitions

Throughout this section, we assume that k, c and n are positive integers with $n = kc$.

For $k > 2$, the eigenvalues of the graphs $QI(n, k)$ and $UQI(n, k)$ are not known in general. One notable exception is the graph $QI(9, 3)$. For this graph, all eigenvalues and their multiplicities are known. These are found using an equitable partition on the vertices of $QI(9, 3)$. This example is particularly interesting because the eigenvalues and their multiplicities can be used to prove that $QI(9, 3)$ is a core.

In this section, we give two equitable partitions of the vertices of $UQI(n, k)$ that can be used to find some of the eigenvalues of $UQI(n, k)$. The first is a simple partition motivated by the comments following the proof of Theorem 6.3.2. The second is a natural extension of the partition used to find the eigenvalues of $QI(9, 3)$. This second equitable partition reduces the work to find eigenvalues but it does not solve the problem completely. Computation is still needed, and for large graphs the computation takes too long. Hence only the eigenvalues for the graphs $QI(9, 3)$, $UQI(12, 3)$, $UQI(15, 3)$, $UQI(18, 3)$ and $QI(16, 4)$ are given.

6.4.1 Eigenvalues for $QI(9, 3)$

One of the qualitative independence graphs, $QI(9, 3)$, has previously appeared in the literature. Mathon and Rosa [50] give an association scheme which has $QI(9, 3)$ as one of the graphs, and in this paper they also state all the eigenvalues and their multiplicities for $QI(9, 3)$. Mathon and Rosa do not focus on the graph $QI(9, 3)$, rather they focus on a different graph in the scheme that is strongly regular.

For two partitions P and Q the *meet of P and Q* is defined by

$$P \wedge Q = |\{(i, j) \ : \ P_i \cap Q_j \neq \emptyset\}|.$$

If $P, Q \in \mathcal{U}_3^9$, the value of $P \wedge Q$ is one of 3, 5, 6, 7 or 9. In addition, $P \wedge Q = 3$ if and only if $P = Q$; and $P \wedge Q = 9$ if and only if P and Q are qualitatively independent.

Define an association scheme with graphs G_i for $i = 1, \ldots, 4$ as follows. The vertex set of each graph is the set of all uniform 3-partitions of an 9-set, $\mathcal{U}_3^9$. The graphs G_i for $i = 1, 2, 3, 4$ have an edge between vertices if and only if the meet of the corresponding partitions is 5, 6, 7 or 9 respectively.

The modified matrix of eigenvalues for this scheme, Table 3, was given by Mathon and Rosa in [50]. In this table, the last four columns contain all of the eigenvalues of the graphs G_i for $i = 1, \ldots, 4$. The first column contains the multiplicities of the eigenvalues (see Section 4.2.3).

$$\left(\begin{array}{r|rrrr} 1 & 27 & 162 & 54 & 36 \\ 27 & 11 & -6 & 6 & -12 \\ 48 & 6 & -6 & -9 & 8 \\ 120 & -3 & -6 & 6 & 2 \\ 84 & -3 & 12 & -6 & -4 \end{array}\right)$$

Table 3: The modified matrix of eigenvalues for the association scheme on $\mathcal{U}_3^9$

These eigenvalues can be found using equitable partitions. This is similar to how the eigenvalues of the Kneser graphs are found (Example 4.2.6, Section 4.2.2). We will define an equitable partition π on $\mathcal{U}_3^9$ (which is the vertex set of the graphs G_i for $i = 1, \ldots, 4$). Fix a vertex $P \in \mathcal{U}_3^9$. Partition the other vertices of $\mathcal{U}_3^9$ into 4 classes, C_j, $j = 1, \ldots, 4$, such that all vertices in a class have the same meet with P.

In Theorem 6.4.8, it will be shown that this vertex partition is also formed by the orbits of a group acting on the partitions of $\mathcal{U}_3^9$, so for all $i = 1, \ldots, 4$, this vertex partition is equitable on G_i. It is possible to build the adjacency matrix of the quotient graph G_i/π; indeed, for $j, k \in \{1, \ldots, 4\}$, the (j,k)-entry of matrix $A(G_i/\pi)$ is the number of vertices in the class C_k which are adjacent to a single (but arbitrary) vertex in class C_j in graph G_i. The fact that these numbers can be found and do not depend on the arbitrary vertex in class C_j also proves that the partition is equitable.

The adjacency matrices of the quotient graphs G_i/π, for $i = 1, \ldots, 4$, are listed below:

$$A(G_1/\pi) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 27 & 8 & 2 & 3 & 0 \\ 0 & 12 & 15 & 18 & 18 \\ 0 & 6 & 6 & 6 & 0 \\ 0 & 0 & 4 & 0 & 9 \end{pmatrix}, \quad A(G_2/\pi) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 12 & 15 & 18 & 18 \\ 162 & 90 & 96 & 90 & 90 \\ 0 & 36 & 30 & 30 & 36 \\ 0 & 24 & 20 & 24 & 18 \end{pmatrix},$$

$$A(G_3/\pi) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 6 & 6 & 6 & 0 \\ 0 & 36 & 30 & 30 & 36 \\ 54 & 12 & 10 & 9 & 12 \\ 0 & 0 & 8 & 8 & 6 \end{pmatrix}, \quad A(G_4/\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 9 \\ 0 & 24 & 20 & 24 & 18 \\ 0 & 0 & 8 & 8 & 6 \\ 36 & 12 & 4 & 4 & 2 \end{pmatrix}.$$

This last matrix is the adjacency matrix for $QI(9,3)/\pi$. Table 3 is the modified matrix of eigenvalues for this scheme, so the first column contains the multiplicities of the eigenvalues and the last four columns contain the eigenvalues of these four matrices. How the multiplicities of the eigenvalues are found is discussed in Section 6.4.3.

The eigenvalues for $QI(9,3)$ are $(36,-12,8,2,-4)$ with corresponding multiplicities $(1,27,48,120,84)$. The largest eigenvalue, 36, is the degree of the vertices in $QI(9,3)$. The smallest eigenvalue is $\tau = -12 = -\frac{(3!)^{3-1}}{3}$, which is expected from Lemma 6.3.2.

We have more information than just the eigenvalues. But first, we need some notation. Let v be the characteristic vector for the set $S_{\{a,b\}}$, for distinct $a,b \in \{1,2,\ldots,k^2\}$. From Theorem 6.3.2, $v - \frac{1}{k+1}\mathbf{1}$ is an eigenvector for $QI(k^2,k)$ corresponding to the smallest eigenvalue, $\frac{-(k!)^{k-1}}{k}$. Let $V_k \subset \mathbb{R}^{U(k^2,k)}$ be the vector space spanned by the set of $\binom{k^2}{2}$ characteristic vectors of $S_{\{a,b\}}$, for all distinct $a,b \in \{1,2,\ldots,k^2\}$.

Godsil and Newman [33] show that the dimension of the vector space V_3 is 27. Using the fact that this is exactly the dimension of the eigenspace corresponding to the smallest eigenvalue of $QI(9,3)$, and properties of the maximum independent sets in $QI(9,3)$, they prove that all independent sets in $QI(9,3)$ are of the form $S_{\{i,j\}}$ for distinct $i,j \in \{1,2,\ldots,9\}$. With this result, they are also able to prove that the graph $QI(9,3)$ is a core. This proof is similar to the proof of Lemma 4.1.12, which shows that the Kneser graphs are cores.

Lemma 6.4.1 ([33]). *The graph $QI(9,3)$ is a core.*

Moreover, in [58], Newman shows that the dimension of the vector space V_k is $\binom{k^2}{2}-\binom{k^2}{1}$ for all k. If it was known that the dimension of the eigenspace corresponding to the smallest eigenvector was $\binom{k^2}{2}-\binom{k^2}{1}$, then we might be able to show that all the independent sets in $QI(k^2,k)$ are sets $S_{\{i,j\}}$ for some distinct $i,j \in \{1,2,\dots,k^2\}$. Further, it may be possible to use this to prove that the graphs $QI(k^2,k)$ are cores in general.

It is not clear whether or not this can be done directly. One problem is that for $k > 3$ it is not clear that it is possible to construct an association scheme where $QI(k^2,k)$ is a graph (this is discussed in Section 6.5). Another problem is that we would need to generalize the extra properties of the independent sets. But this motivates trying to find the eigenvalues and the multiplicities of the qualitative independence graphs. It also motivates searching for association schemes (or possibly asymmetric association schemes) that have $QI(k^2,k)$ as one of the graphs. We end this section with several conjectures on extending Lemma 6.4.1 to all $QI(k^2,k)$.

Conjecture 6.4.2. For all positive integers k, the eigenspace corresponding to the smallest eigenvector of $QI(k^2,k)$, $\frac{-(k!)^{k-1}}{k}$, has dimension $\binom{k^2}{2}-\binom{k^2}{1}$.

Conjecture 6.4.3. For all positive integers k, all maximum independent sets in $QI(k^2,k)$ correspond to sets $S_{\{a,b\}}$ for distinct $a,b \in \{1,\dots,k^2\}$.

Conjecture 7.5.2 in Section 7.5 is a generalization of this conjecture to *intersecting partition systems.*

Conjecture 6.4.4. For any k, the graph $QI(k^2,k)$ is a core.

6.4.2 A Simple Equitable Partition

From the proof of Theorem 6.3.2, the partition $\pi = \{S_{\{1,2\}}, V(QI(k^2,k))\backslash S_{\{1,2\}}\}$ is an equitable partition. This partition can not be used to find all eigenvalues for $QI(k^2,k)$, but it can be used to find their largest and smallest eigenvalues.

The adjacency matrix for the quotient graph $QI(k^2,k)/\pi$ is

$$\begin{pmatrix} 0 & k!^{k-1} \\ \frac{k!^{k-1}}{k} & k!^{k-1} - \frac{k!^{k-1}}{k} \end{pmatrix}$$

which has eigenvalues $k!^{k-1}$ and $-\frac{k!^{k-1}}{k}$. These are the largest and smallest eigenvalues of $QI(k^2,k)$.

It would be interesting to find a similar equitable partition on the vertices of the graph $UQI(ck,k)$, where c is a positive integer with $k < c$. From the proof of Theorem 6.2.1, for a set $A \subset \{1,\ldots,ck\}$ with $|A| = c-(k-2)$, S_A is an independent set in $QI(ck,k)$. But, the partition $\{S_A, V(UQI(ck,k))\backslash S_A\}$ is not equitable. So instead, we consider the set $S_{\{1,2\}}$, which is not an independent set in $UQI(ck,k)$ for $k < c$. But, from the comments following the proof of Lemma 6.3.2, the vertex partition

$$\{S_{\{1,2\}}, V(QI(k^2,k))\backslash S_{\{1,2\}}\}$$

is the orbit partition formed by the subgroup $H < Aut(QI(k^2,k))$ induced by the group $H' = \{\sigma \in Sym(k^2) : \sigma(\{1,2\}) = \{1,2\}\}$. Similarly, the partition

$$\{S_{\{1,2\}}, V(UQI(ck,k))\backslash S_{\{1,2\}}\}$$

is the orbit partition formed by the subgroup $H < Aut(QI(ck,k))$ induced by the group $H' = \{\sigma \in Sym(ck) : \sigma(\{1,2\}) = \{1,2\}\}$. From the comments in Section 4.2.2, this means $\{S_{\{1,2\}}, V(UQI(ck,k))\backslash S_{\{1,2\}}\}$ is an equitable partition.

For a given vertex $P \in S_{\{1,2\}}$ in $UQI(ck,k)$, it is much harder to count the number of vertices in $S_{\{1,2\}}$ adjacent to P and the number of vertices not in $S_{\{1,2\}}$ adjacent to P. So, let a be the number of partitions with 1 and 2 in the same class adjacent to a fixed partition also with 1 and 2 in the same class. Let b be the number of partitions with 1 and 2 in the same class and adjacent to a fixed partition with 1 and 2 in different classes. Let d be the degree of the vertices in $UQI(ck,k)$.

Then the adjacency matrix for this quotient graph is

$$\begin{pmatrix} a & d-a \\ b & d-b \end{pmatrix}.$$

The eigenvalues for this quotient graph are $a - b$ and d. It is no surprise that d is an eigenvalue since the graph is d-regular. Since the partition is equitable, $a - b$ is an eigenvalue of $UQI(ck, k)$.

Question 6.4.5. For integers c, k with $k \leq c$, what is the exact value of a and b for $UQI(ck, k)$?

For $QI(k^2, k)$ the value of $a - b$ is $-\frac{k!^{k-1}}{k}$. From Lemma 6.3.2, the eigenvalue $a - b$ is the smallest eigenvalue for $QI(k^2, k)$. Is this true for all $UQI(ck, k)$?

Question 6.4.6. For integers c, k with $k \leq c$, is the eigenvalue $a - b$ from above the smallest eigenvalue for $UQI(ck, k)$?

One obvious problem with this equitable partition is that it can only be used to find at most two eigenvalues of the graph $UQI(ck, k)$. In the next section, we give an equitable partition that can be used to find all eigenvalues of $UQI(ck, k)$.

6.4.3 A Better Equitable Partition

Recall from Theorem 4.2.5, that if G is a vertex-transitive graph and π is the orbit partition for some subgroup of $Aut(G)$ with the property that π has a singleton class $\{u\}$, then every eigenvalue of G is an eigenvalue of G/π.

A partition with these properties will be used to simplify the calculations of the eigenvalues of $UQI(n, k)$.

For $P \in \mathcal{U}_k^n$, define a subgroup

$$\text{fix}'(P) = \{\sigma \in Sym(n) \ : \ \sigma(P) = P\},$$

where $\sigma(P)$ is defined by: $\sigma(a) \in (\sigma(P))_i$ if and only if $a \in P_i$. The group $\text{fix}'(P)$ induces a subgroup of $Aut(UQI(n, k))$, which we call $\text{fix}(P)$. The orbits of the group action of $\text{fix}(P)$ on the vertices of $UQI(n, k)$ contain the partition P as a singleton class. Thus the orbit partition of $\text{fix}(P)$ forms an equitable partition of the vertex set of $UQI(n, k)$ with a singleton class.

Corollary 6.4.7. *Fix an arbitrary partition $P \in V(UQI(n,k))$, let π be the partition of $V(UQI(n,k))$ induced by the group* fix(P). *Then every eigenvalue of $UQI(n,k)$ is an eigenvalue of the quotient graph $UQI(n,k)/\pi$.*

It is easier to calculate the eigenvalues of the quotient graph $UQI(n,k)/\pi$ than $UQI(n,k)$. Since the group fix(P) is very large and difficult to work with, we give a characterization for when two partitions in $V(UQI(n,k))$ are in the same orbit under the group action of fix(P). For the graph $QI(9,3)$, this gives the same partition of the vertices as the one defined using the meet of two partitions in Section 6.4.1.

For partitions $P, Q \in V(QI(n,k))$ define the *meet table of P and Q* to be the $k \times k$ array with the i,j entry $|P_i \cap Q_j|$. Denote the meet table of P and Q by $M_{P,Q}$. Partitions P and Q are qualitatively independent if and only if the meet table of P and Q has all entries non-zero.

Two meet tables are *isomorphic* if some permutation of the rows and columns of one array produces the other array. Different orderings on the classes in the partitions P and Q could produce different meet tables, but the tables would be isomorphic. If $P = Q$, then the meet table $M_{P,Q}$ is isomorphic to the array with diagonal entries $c = n/k$ and all other entries are zero.

Theorem 6.4.8. *Let $P, Q, R \in V(QI(n,k))$. The meet table for P and Q is isomorphic to the meet table for P and R if and only if there exists $g \in$* fix(P) *such that $g(Q) = R$.*

Proof. For a partition P, let P_i for $i \in \{0,\dots,k-1\}$ denote the classes of P.

Assume the meet tables $M_{P,Q}$ and $M_{P,R}$ are isomorphic. Then for some permutations $\sigma, \phi \in Sym(k)$

$$[M_{P,Q}]_{i,j} = [M_{P,R}]_{\sigma(i),\phi(j)}, \text{ for } i,j \in \{0,1,\dots,k-1\}.$$

For all $i,j \in \{1,\dots,k\}$, we have $|P_i \cap Q_j| = |P_{\sigma(i)} \cap R_{\phi(j)}|$. For fixed i,j, set $m = |P_i \cap Q_j|$. Denote $P_i \cap Q_j = \{a_1, a_2, \dots, a_m\}$ and $P_{\sigma(i)} \cap R_{\phi(j)} = \{b_1, b_2, \dots, b_m\}$. Define $g_{i,j}$ to be the mapping that assigns a_l to b_l for $l = 1,\dots,m$. For distinct pairs (i_0, j_0) and (i_1, j_1) the mappings g_{i_0,j_0} and g_{i_1,j_1} have disjoint domains and map these sets to disjoint sets. In particular, for $j_0 \neq j_1$, $g_{i,j_0}(P)$ and $g_{i,j_1}(P)$ are disjoint.

Define $g_i = \Pi_{j=0}^{k-1} g_{i,j}$, then $g_i(P_i) = P_{\sigma(i)}$. Define $g = \Pi_{i=0}^{k-1} g_i$, so $g(P_i) = P_{\sigma(i)}$ for all $i \in \{0, \ldots, k-1\}$. Similarly, for all $j \in \{0, \ldots, k-1\}$, $g(Q_j) = R_{\phi(j)}$. Thus, the permutation g is in fix(P) and $g(Q) = R$.

Next, assume that there exists a $g \in$ fix(P) such that $g(Q) = R$ and show that $M_{P,Q}$ is isomorphic to $M_{P,R}$. Define a permutation on the rows $i = 0, \ldots, k-1$ of $M_{P,Q}$ by $\sigma(i) = i'$ if and only if $g(P_i) = P_{i'}$. Similarly, define a permutation ϕ on the columns $i = 0, \ldots, k-1$ of $M_{P,Q}$ by $\phi(j) = j'$ if and only if $g(Q_j) = R_{j'}$. Thus,

$$[M_{P,Q}]_{\sigma(i),\phi(j)} = [M_{P,R}]_{i,j}, \text{ for } i, j \in \{0, 1, \ldots, k-1\}$$

and the meet tables are isomorphic. ✰

With this characterization of an equitable partition, we use a computer program to build the quotient graph of several of the graphs $UQI(ck, k)$. Then, using `Maple` we can find the eigenvalues of these graphs. First, we fix a partition $P \in V(UQI(ck, k))$. Let π be the partition of the vertices in $UQI(ck, k)$ formed by the orbits of fix(P). Next, we build the list of the classes in π. To do this, we create a list of all non-isomorphic meet tables. This is done by going through all the partitions in $\mathcal{U}_k^n$ and, for each one, building the meet table with P. For each meet table we construct, we check if it is isomorphic to a meet table already in the list. If not, we add the meet table to the list. The classes of π (the non-isomorphic meet tables) are stored as a single partition that has the particular meet table with P; we call this partition the *representative of the orbit*. Next, the adjacency matrix for $UQI(ck, k)/\pi$ is built. Again, we go through the list of all partitions in $\mathcal{U}_k^{ck}$, and for each orbit we count the number of partitions which are qualitatively independent with the representative of the orbit.

Since this process requires going through the entire set of partitions $\mathcal{U}_k^n$ twice, it can take a very long time to run. Currently, the largest graph our program can complete in a reasonable time (2 days) is $QI(16, 4)$, which has more than 2.6 million vertices. Fortunately, the quotient graphs are small: the graph $QI(16, 4)/\pi$ has only 43 vertices.

6.4.4 Multiplicities of the Eigenvalues

Once the adjacency matrix for $UQI(ck,k)/\pi$ is built, we also have enough information to find the multiplicities of the eigenvalues. First we need some notation. For a graph G, let $\phi(G,x)$ denote the characteristic polynomial of the adjacency matrix $A(G)$, and let $\phi'(G,x)$ denote the derivative of this polynomial. Let $ev(G)$ denote the set of distinct eigenvalues of G, and for $\lambda \in ev(G)$, let m_λ be the multiplicity of λ.

Corollary 6.4.9 (Section 5.3, [31]). *Let $\pi = \{C_1, C_2, \ldots, C_r\}$ be an equitable partition of the vertices of a graph G with the property that C_1 is a singleton class. Then*

$$\frac{\phi'(G,x)}{\phi(G,x)} = \frac{|V(G)|\phi((G/\pi)\backslash C_1, x)}{\phi(G/\pi, x)}.$$

It is a straightforward application of the multiplication rule for derivatives to see that

$$\frac{\phi'(G,x)}{\phi(G,x)} = \sum_{\lambda \in ev(G)} \frac{m_\lambda}{x-\lambda}.$$

With the adjacency matrix for $UQI(n,k)/\pi$, using `Maple`, it is simple to find the partial fraction expansion of $\frac{|V(G)|\phi((G/\pi)\backslash C_1,x)}{\phi(G/\pi,x)}$ where $G = UQI(ck,k)$, and $C_1 = \{P\}$. Then the numerators in the partial fraction expansion are the multiplicities of the eigenvalues for $UQI(ck,k)$.

6.4.5 Eigenvalues for Several Small Uniform Qualitative Independence Graphs

The equitable partition described in Section 6.4.3 can be used to reduce the calculation needed to find the eigenvalues of $UQI(ck,k)$ and their multiplicities. As stated in the comments following the proof of Theorem 6.4.8, the adjacency matrix for $UQI(ck,k)/\pi$ can be built using a computer program, but this can be very time consuming. For large c and k, this method takes too much time. To date, we have been able to find the eigenvalues for the graphs $QI(9,3)$, $UQI(12,3)$, $UQI(15,3)$, $UQI(18,3)$ and $QI(16,4)$. These are given, along with their multiplicities, in Table 4.

Graph	Eigenvalues / Corresponding multiplicities (in the same order as the eigenvalues)
$UQI(9,3)$	(-4, 2, 8, -12, 36)
	(84, 120, 48, 27, 1)
$UQI(12,3)$	(0, 8, -12, 18, -27, 48, 108, -252, 1728)
	(275, 2673, 462, 616, 1408, 132, 154, 54, 1)
$UQI(15,3)$	(4, 8, -10, -22, 29, 34, -76, 218, -226, 284, 1628, -5060, 62000)
	(1638, 21450, 910, 25025, 32032, 22113, 11583, 1925, 7007, 2002, 350, 90,1)
$UQI(18,3)$	(8, 15, 18, -60, 60, -102, -120, 120, 368, 648, -655, -2115, 2370, -2115, 2370, 2460, -4140, 24900, -89550, 1876500, $954 \pm 18\sqrt{10209}$)
	(787644, 678912, 136136, 87516, 331500, 259896, 102102, 219912, 99144, 11934, 88128, 22848, 4641, 5508, 2244, 663, 135, 1, 9991)
$UQI(16,4)$	(-72, $-56 \pm 8\sqrt{193}$, $-96 \pm 96\sqrt{37}$, $24 \pm 24\sqrt{97}$, -96, 96, -288, 8, -144, 24, 192, 32, 1728, -64, -16, 432, 48, 1296,-48, -576, 128, -3456, 576, 13824, -1152, 144)
	(266240, 137280, 7280, 76440, 69888, 91520, 24960, 262080, 73920, 24024, 65520, 150150, 440, 51480, 753324, 20020, 420420, 1260, 23100, 10752, 60060, 104, 4070, 1, 1260, 32032)

Table 4: Eigenvalues and their multiplicities for small, uniform qualitative independence graphs

The largest and smallest eigenvalues for each of the graphs in Table 4 are particularly interesting. Since the graphs $UQI(ck,k)$ are vertex transitive, the ratio bound for independent sets (Inequality (11)) gives an upper bound on the size of the maximum independent set for each of the graphs in Table 4. Moreover, from the proof of Theorem 6.2.1, the set S_A with $|A| = c - k + 2$ is an independent set of $UQI(ck,k)$, and the cardinality of S_A is a lower bound for $\alpha(UQI(ck,k))$.

Corollary 6.4.10.

1. $\alpha(QI(9,3)) = 70$
2. $315 \leq \alpha(UQI(12,3)) \leq 735$
3. $1386 \leq \alpha(UQI(15,3)) \leq 9516$
4. $6006 \leq \alpha(UQI(18,3)) \leq 130215$
5. $\alpha(QI(16,4)) = 525525$

Question 6.4.11. What is the size of a largest independent set in each of $UQI(12,3)$, $UQI(15,3)$ and $UQI(18,3)$?

Considering Conjecture 6.4.2, it is also interesting to note the multiplicity of the smallest eigenvalue for each of the graphs in Table 4. The smallest eigenvalue for $UQI(16,4)$ is -3456 and it has multiplicity 104. This confirms Conjecture 6.4.2 for $k = 4$. In fact, for each of the graphs $QI(3c,3)$, $c = 3,4,5,6$, the dimension of the eigenspace corresponding to the smallest eigenvalue is $\binom{ck}{2} - \binom{ck}{1}$. This suggests that Conjecture 6.4.2 can be extended from $QI(k^2,k)$ to $UQI(ck,k)$.

Conjecture 6.4.12. For all positive integers c,k, the eigenspace corresponding to the smallest eigenvalue of $UQI(ck,k)$ has dimension $\binom{ck}{2} - \binom{ck}{1}$.

6.5 Other Schemes

The association scheme from Section 6.4.1 that has $QI(9,3)$ as one of its graphs can also be defined by meet tables. There are only five possible values for the meet of two partitions in $\mathcal{U}_3^9$; for each value, there is (up to isomorphism) exactly one meet table. In this association scheme there are four graphs G_i, one for each non-isomorphic meet table, except the table corresponding to meet 3 (two partitions have meet 3 if and only if they are identical). Two partitions $P,Q \in \mathcal{U}_3^9$ are adjacent in the graph G_i if the meet table $M_{P,Q}$ is isomorphic to the meet table corresponding to G_i.

This can be generalized to $\mathcal{U}_k^n$ for positive integers k and n, where k divides n. Assume that there are m pairwise non-isomorphic meet tables for the partitions in $\mathcal{U}_k^n$. For each meet table from this list, define the graph G_i, for $i = 0,\dots,m-1$ as follows: the vertex set of G_i is the set of all uniform partitions $\mathcal{U}_k^n$, and two partitions $P,Q \in \mathcal{U}_k^n$ are adjacent in G_i if the meet table $M_{P,Q}$ is isomorphic to the meet table corresponding to G_i.

The big question here is: Does this, in general, define an association scheme for $\mathcal{U}_k^n$?

The first problem is that for $P,Q \in \mathcal{U}_k^n$ it is possible that the meet tables $M_{P,Q}$

and $M_{Q,P}$ are not isomorphic. For example, consider the partitions from $UQI(18,3)$

$$\begin{aligned} P &= 1\ 2\ 3\ 4\ 5\ 6 \mid 7\ 8\ 9\ 10\ 11\ 12 \mid 13\ 14\ 15\ 16\ 17\ 18 \\ Q &= 1\ 2\ 3\ 4\ 7\ 13 \mid 5\ 6\ 8\ 9\ 14\ 15 \mid 10\ 11\ 12\ 16\ 17\ 18\ . \end{aligned}$$

The meet arrays for P and Q and for Q and P are

$$M_{P,Q} = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad M_{Q,P} = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 3 & 3 \end{bmatrix}.$$

These two tables are transposes of each other, but they are not isomorphic. This means that the graphs G_i, as defined above, may actually be directed graphs. So this method does not in general define an association scheme. But, it may define an *asymmetric association scheme* (see Section 4.2.3 for definition).

This example of meet tables $M_{P,Q}$ and $M_{Q,P}$ which are not isomorphic is the smallest example for $UQI(3c,3)$. For $c = 4,5$, for all partitions $P, Q \in \mathcal{U}_3^{3c}$ the meet tables $M_{P,Q}$ and $M_{Q,P}$ are isomorphic. The two cases $\mathcal{U}_3^{12}$ and $\mathcal{U}_3^{15}$ are considered in the next sections.

6.5.1 Partitions in $\mathcal{U}_3^{12}$

For $\mathcal{U}_3^{12}$ there are 9 non-isomorphic meet tables. Let $P = 1\ 2\ 3\ 4 \mid 5\ 6\ 7\ 8 \mid 9\ 10\ 11\ 12$ be the fixed partition. Each of the following nine partitions is a representative for a class of non-isomorphic meet tables. Only the last representative is qualitatively independent with P.

1 2 3 4 | 5 6 7 8 | 9 10 11 12	1 2 3 4 | 5 6 7 9 | 8 10 11 12
1 2 3 4 | 5 6 9 10 | 7 8 11 12	1 2 3 5 | 4 6 7 9 | 8 10 11 12
1 2 3 5 | 4 6 9 10 | 7 8 11 12	1 2 3 5 | 4 9 10 11 | 6 7 8 12
1 2 5 6 | 3 4 9 10 | 7 8 11 12	1 2 5 9 | 3 4 6 10 | 7 8 11 12
1 2 5 9 | 3 6 7 10 | 4 8 11 12	

For each of the nine non-isomorphic meet tables, define a graph G_i with vertex set $\mathcal{U}_3^{12}$. Two partitions will have an edge in graph G_i if and only if their meet table is isomorphic to the meet table corresponding to the graph G_i. Using the equitable partition described in Section 6.4.3, it is possible to calculate the eigenvalues of these graphs. The table below is the modified matrix of eigenvalues of these graphs. As only the last representative is qualitatively independent with P, the final column of this table is the set of eigenvalues of the graph $UQI(12, 3)$.

$$\left(\begin{array}{c|cccccccc}
275 & 12 & 21 & -48 & -24 & -16 & 18 & 36 & 0 \\
2673 & -4 & 1 & -8 & 24 & 0 & -2 & -20 & 8 \\
462 & -12 & 9 & 36 & -72 & 8 & 6 & 36 & -12 \\
616 & 6 & -9 & -18 & -36 & 20 & 18 & 0 & 18 \\
1408 & 3 & -6 & 6 & 3 & -7 & -9 & 36 & -27 \\
132 & 6 & -9 & 72 & -36 & -40 & 48 & -90 & 48 \\
154 & 18 & 9 & 36 & -72 & 8 & -54 & -54 & 108 \\
54 & 26 & 21 & 92 & 144 & 40 & 18 & -90 & -252 \\
1 & 48 & 54 & 576 & 1728 & 128 & 216 & 1296 & 1728
\end{array}\right)$$

Table 5: Modified matrix of eigenvalues for the graphs G_i (with vertex set $\mathcal{U}_3^{12}$)

If the set of graphs $\{G_i : i = 1, \ldots, 9\}$ describes an association scheme, then the ratio bound for cliques, Inequality (12), would hold for the graph $UQI(12, 3)$. If this bound held, then $\omega(UQI(12, 3)) \leq 7$. This particular bound is interesting because the exact value of $N(12, 3)$ is not known; it is known that $7 \leq N(12, 3)$ and that $12 \leq CAN(8, 3) \leq 13$. If it is true that $\omega(UQI(12, 3)) \leq 7$, it still may be true that a $CA(12, 8, 3)$ exists, but it would not be balanced. So, if a $CA(12, 8, 3)$ exists and the above is an association scheme, then the graph $UQI(12, 3)$ would not be a core of $QI(12, 3)$. This would mean the answer to Question 6.2.6 is negative.

Conjecture 6.5.1. The set of graphs, $\{G_i : i = 2, \ldots, 9\}$, defined above form an association scheme on $\mathcal{U}_3^{12}$.

Conjecture 6.5.2. A $CA(12, 8, 3)$ does not exist, so $CAN(8, 3) = 13$.

6.5.2 Partitions in $\mathcal{U}_3^{15}$

For $\mathcal{U}_3^{15}$ there are 13 non-isomorphic meet tables. Use $P = 1\ 2\ 3\ 4\ 5 \mid 6\ 7\ 8\ 9\ 10 \mid 11\ 12\ 13\ 14\ 15$ as the fixed partition. Below is a list of representatives of each non-isomorphic meet table with respect to P. In this case, the last three representatives are qualitatively independent with the fixed partition.

1 2 3 4 5 | 6 7 8 9 10 | 11 12 13 14 15
1 2 3 4 5 | 6 7 8 11 12 | 9 10 13 14 15
1 2 3 4 6 | 5 7 8 11 12 | 9 10 13 14 15
1 2 3 4 6 | 5 11 12 13 14 | 7 8 9 10 15
1 2 3 6 7 | 4 5 11 12 13 | 8 9 10 14 15
1 2 3 6 11 | 4 7 8 9 12 | 5 10 13 14 15
1 2 6 7 11 | 3 4 8 12 13 | 5 9 10 14 15
1 2 3 4 5 | 6 7 8 9 11 | 10 12 13 14 15
1 2 3 4 6 | 5 7 8 9 11 | 10 12 13 14 15
1 2 3 4 6 | 5 7 11 12 13 | 8 9 10 14 15
1 2 3 6 7 | 4 5 8 11 12 | 9 10 13 14 15
1 2 3 6 11 | 4 5 7 8 12 | 9 10 13 14 15
1 2 3 6 11 | 4 7 8 12 13 | 5 9 10 14 15

Similar to the case for $\mathcal{U}_3^{12}$, for each of the 13 non-isomorphic meet tables, it is possible to a build graph G_i, for $i = 1, \ldots, 13$, with vertex set $\mathcal{U}_3^{15}$ and vertices adjacent in G_i if and only if their meet table is isomorphic to the meet table corresponding to G_i. The table below is the modified matrix of eigenvalues of these graphs.

$$\left(\begin{array}{c|cccccccccccc}
2002 & 15 & -18 & 126 & 144 & -276 & -50 & 234 & 80 & -540 & 176 & 162 & -45 \\
32032 & 0 & -3 & -24 & 9 & 54 & -5 & 54 & -25 & -90 & 56 & -108 & 81 \\
90 & 47 & 132 & 450 & 1440 & 1240 & 110 & 600 & 320 & 720 & 160 & -1980 & -3240 \\
910 & 27 & 102 & 0 & 240 & -60 & 10 & 150 & 20 & -480 & -640 & -180 & 810 \\
7007 & 15 & -18 & 36 & -36 & 96 & 10 & 54 & -100 & 360 & -64 & -108 & -54 \\
25025 & -9 & 6 & 12 & 36 & -72 & 10 & -54 & 20 & 72 & 32 & -108 & 54 \\
350 & 36 & 66 & 252 & 186 & 228 & 55 & -984 & -340 & -1184 & 512 & 792 & 324 \\
1638 & 20 & 102 & -84 & -306 & -116 & -24 & 24 & 20 & 306 & 256 & 72 & -324 \\
1 & 75 & 300 & 1500 & 9000 & 6000 & 250 & 9000 & 2000 & 36000 & 8000 & 27000 & 27000 \\
11583 & 5 & -8 & 16 & -16 & 64 & -30 & -156 & 40 & 160 & -64 & -48 & 36 \\
22113 & 5 & -8 & -44 & 64 & -16 & 10 & -26 & 20 & -40 & -24 & 172 & -114 \\
1925 & 21 & -24 & 60 & -432 & 96 & 58 & 54 & 164 & -216 & 56 & -108 & 270 \\
21450 & -9 & 6 & 30 & -72 & 36 & 2 & 54 & -16 & -36 & -64 & 162 & -90
\end{array}\right)$$

Table 6: Modified matrix of eigenvalues for the graphs G_i (with vertex set $\mathcal{U}_3^{15}$)

There are three representatives that are qualitatively independent with the partition P, this means there are three graphs, namely G_{11}, G_{12} and G_{13}, in which partitions that are adjacent are qualitatively independent. Thus the graph $QI(15,3)$ is not a single graph in this scheme. So we end with one last conjecture.

Conjecture 6.5.3. The graphs G_i, for $i = 2, \ldots, 13$, described above give an association scheme for $\mathcal{U}_3^{15}$.

Chapter 7

Partition Systems

In Chapter 3, we considered extremal problems for set systems. These problems included Sperner's Theorem and the Erdős-Ko-Rado Theorem. There have been attempts to extend these results to extremal problems in which the elements in the system are families of sets, called *clouds*, rather than sets. Such problems are called *higher order extremal problems*. Ahlswede, Cai and Zhang [3] give a good overview of such problems. Most problems considered in [3] require that the clouds be pairwise disjoint, that is, no set can occur in two distinct clouds. With this restriction, the direct generalization of the Erdős-Ko-Rado Theorem for disjoint clouds proved to be false [1]. Erdős and Székely [25] survey higher order Erdős-Ko-Rado Theorems where clouds are substituted by set systems with additional structure and the disjointness requirement for pairs of set systems is dropped. They consider, among other cases, the particular case in which each structure is a set partition. It is this type of problem that is connected to covering arrays.

Sperner's Theorem and the Erdős-Ko-Rado Theorem completely determine the maximum cardinality of a set of qualitatively independent sets. Since the rows of a binary covering array correspond to subsets of an n-set, this determines the minimum size of a binary covering array (Theorem 3.4.2). The rows of a general covering array, $CA(n, r, k)$, correspond to k-partitions of an n-set, so it seems likely that results on extremal partition systems could help determine the size of general covering arrays. In this chapter, we give theorems for partition systems that are extensions of Sperner's

Theorem and the Erdős-Ko-Rado Theorem to partition systems.

Where appropriate, the notation in this chapter will follow the notation used in Chapter 3. In particular, the collection of all c-sets of an n-set is denoted by $\binom{[n]}{c}$, the set of all k-partitions of an n-set is denoted by $\mathcal{P}_k^n$ with $S(n,k) = |\mathcal{P}_k^n|$ and the the set of all uniform k-partitions of an n-set is denoted $\mathcal{U}_k^n$ with $U(n,k) = |\mathcal{U}_k^n|$.

The results in this chapter are contained in the papers [52, 53].

7.1 Sperner Partition Systems

In this section, theorems similar to Sperner's Theorem (Theorem 3.2.2) for partition systems are proven.

Recall from Section 3.2 that sets A and B are *incomparable* if they have the property that $A \not\subseteq B$ and $B \not\subseteq A$. From Definition 3.2.1, a Sperner set system is a set system $\mathcal{A}$ with the property that any distinct $A, B \in \mathcal{A}$ are incomparable.

We will extend this property to partitions; this definition does not coincide with the extension of incomparability to sequences given by Gargano, Körner and Vaccaro [28, 29, 30].

Definition 7.1.1 (*Sperner Property for Partitions*). Let n and k be positive integers. Two k-partitions of an n-set, $P, Q \in \mathcal{P}_k^n$, with $P = \{P_1, \ldots, P_k\}$ and $Q = \{Q_1, \ldots, Q_k\}$ have the *Sperner property* if

$$P_i \not\subseteq Q_j \text{ and } Q_i \not\subseteq P_j \text{ for all } i, j \in \{1, \ldots, k\}.$$

Definition 7.1.2 (*Sperner Partition System*). Let n and k be positive integers. A partition system $\mathcal{P} \subseteq \mathcal{P}_k^n$ is a *Sperner partition system* if all distinct $P, Q \in \mathcal{P}$ have the Sperner property.

Let $SP(n,k)$ denote the maximum cardinality of a Sperner partition system in $\mathcal{P}_k^n$.

If $\mathcal{P}$ is a partition system, then $\mathcal{P}$ is a Sperner partition system if and only if all the partitions in $\mathcal{P}$ are disjoint (no two partitions have a common class) and the union of all the partitions in $\mathcal{P}$ is a Sperner set system. In this sense, Sperner partition systems can be considered as resolvable Sperner set systems.

Further, any set of qualitatively independent partitions is a Sperner partition system. Moreover, for $k = 2$, the definition of the Sperner property for partitions is equivalent to the definition of qualitative independence.

7.1.1 Sperner's Theorem for Partition Systems in $\mathcal{P}_k^{ck}$

In this section, we show that for positive integers n, k, c with $n = ck$, the largest Sperner k-partition system of an n-set is a uniform partition system with cardinality $\binom{n-1}{c-1}$.

When $n = ck$, a 1-factor of the complete uniform hypergraph $K_n^{(c)}$ is equivalent to a uniform k-partition of an n-set, and a 1-factorization of $K_n^{(c)}$ corresponds to a Sperner partition system. From Theorem 3.1.7, if c divides n, the hypergraph $K_n^{(c)}$ has a 1-factorization with $\binom{n-1}{c-1}$ factors.

Lemma 7.1.3. *Let n, k, c be positive integers with $n = ck$, then there exists a Sperner partition system in $\mathcal{P}_k^n$ of cardinality $\binom{n-1}{c-1}$.*

The proof that this is the largest Sperner partition system uses a result by Kleitman and Milner on Sperner set systems. For a set system $\mathcal{A}$, define the *volume* of $\mathcal{A}$ to be $t(\mathcal{A}) = \sum_{A\in\mathcal{A}} |A|$.

Theorem 7.1.4 (Kleitman and Milner [46]). *Let $\mathcal{A}$ be a Sperner set system on an n-set with $|\mathcal{A}| \geq \binom{n}{c}$ and $c \leq \frac{n}{2}$, then*

$$\frac{t(\mathcal{A})}{|\mathcal{A}|} \geq c.$$

This inequality is strict in all cases except when $\mathcal{A} = \binom{[n]}{c}$.

Theorem 7.1.5. *Let n, k, c be positive integers with $n = kc$. Then $SP(n, k) = \frac{1}{k}\binom{n}{c}$. Moreover, a Sperner partition system has cardinality $\frac{1}{k}\binom{n}{c}$ only if it is a c-uniform partition system.*

Proof. If $k = 1$ then $\mathcal{P}_k^n$ has only one partition, namely $\{\{1, \dots, n\}\}$. So $SP(n, k) = 1$.

Let $\mathcal{P} \subseteq \mathcal{P}_k^n$ be a Sperner partition system. Let $\mathcal{A}$ be the Sperner set system formed by taking all classes from all the partitions in $\mathcal{P}$. Thus $|\mathcal{A}| = k|\mathcal{P}|$.

We show that $|\mathcal{A}| \leq \binom{n}{c}$. Let p_i, $i \in \{1, \dots, n\}$ be the number of sets in $\mathcal{A}$ with size i.

By the LYM Inequality (Inequality (5), Section 3.2), we have

$$\sum_{i=1}^{n} \frac{p_i}{\binom{n}{i}} \leq 1.$$

Following the notation from [27], define the function $f(i) = \binom{n}{i}^{-1}$. With this, we get

$$\sum_{i=1}^{n} \frac{p_i}{|\mathcal{A}|} f(i) \leq \frac{1}{|\mathcal{A}|}. \tag{14}$$

In [27] it is shown that the function $f(i)$ can be extended to a convex function on the real numbers by

$$f(i+u) = (1-u)f(i) + uf(i+1) \quad \text{for} \quad 0 \leq u \leq 1.$$

Since the set system $\mathcal{A}$ is formed from a partition system with $n = ck$,

$$\sum_{i=1}^{n} ip_i = \sum_{A \in \mathcal{A}} |A| = n|\mathcal{P}| = ck|\mathcal{P}| = c|\mathcal{A}|. \tag{15}$$

Using Equation (15); the fact that f is a convex function and that $\sum_{i=1}^{n} \frac{p_i}{|\mathcal{A}|} = 1$; and Inequality (14),

$$f(c) = f\left(\sum_{i=1}^{n} i \frac{p_i}{|\mathcal{A}|}\right) \leq \sum_{i=1}^{n} f(i) \frac{p_i}{|\mathcal{A}|} \leq \frac{1}{|\mathcal{A}|}.$$

Thus, $|\mathcal{A}| \leq \binom{n}{c}$ and

$$|\mathcal{P}| \leq \frac{1}{k}\binom{n}{c}.$$

Next, we need to show that a Sperner partition system meets this bound, then it is c-uniform. If $k = 1$ this is clearly true since the only partition in $\mathcal{P}_1^n$ is the

n-uniform partition $\{\{1,\ldots,n\}\}$. Assume that for $k \geq 2$, $|\mathcal{P}| = \frac{1}{k}\binom{n}{c}$. Let $\mathcal{A}$ be the Sperner set system formed from all the classes in $\mathcal{P}$, then $|\mathcal{A}| = \binom{n}{c}$ and $c \leq \frac{n}{2}$.

Since $\frac{t(\mathcal{A})}{|\mathcal{A}|} = \frac{\Sigma_{A\in\mathcal{A}}|A|}{|\mathcal{A}|} = c$, from Theorem 7.1.4 it follows that $\mathcal{A} = \binom{[n]}{c}$ and $\mathcal{P}$ is a uniform partition system.

Finally, a Sperner partition system $\mathcal{P}$ with $|\mathcal{P}| = \frac{1}{k}\binom{n}{c}$ exists for all n, k, c with $n = ck$ from Lemma 7.1.3. ☆

Theorem 7.1.5 is a natural extension of Sperner's Theorem for sets (Theorem 3.2.2). Sperner's Theorem for set systems says that the Sperner set system with maximum cardinality on an n-set is the collection of all $\lfloor \frac{n}{2} \rfloor$-sets. Theorem 7.1.5 says that for integers n, k such that k divides n, the Sperner k-partition system on an n-set with the largest cardinality is the collection of all $(\frac{n}{k})$-sets arranged in resolution classes.

A collection of qualitatively independent partitions is a Sperner partition system, but the converse of this is not true in general. In fact, for $k > 2$, from Theorem 7.1.5, the cardinality of the largest Sperner k-partition system can be much larger than the cardinality of a collection of qualitatively independent k-partitions. For example, for a positive integer k, there can be at most $k+1$ qualitatively independent k-partitions of a k^2-set (Inequality (1), Section 2.1 and Theorem 2.3.3), while $SP(k^2, k) = \binom{k^2-1}{k-1}$.

7.1.2 A Bound on the Cardinality of Sperner Partition Systems in $\mathcal{P}_k^n$

For integers k and n, if k does not divide n, then we have a bound on the cardinality of a Sperner partition system in $\mathcal{P}_k^n$.

Theorem 7.1.6. *Let n, k, c be positive integers with $n = ck + r$, where $0 \leq r < k$. Then,*

$$SP(n,k) \leq \frac{\binom{n}{c}}{(k-r) + \frac{r(c+1)}{n-c}}.$$

Proof. Let $\mathcal{P} \subseteq \mathcal{P}_k^n$ be a Sperner partition system. Let $\mathcal{A}$ be the Sperner set system formed by taking all classes from all the partitions in $\mathcal{P}$. Thus $|\mathcal{A}| = k|\mathcal{P}|$.

Let p_i, $i \in \{1,\ldots,n\}$ be the number of sets in $\mathcal{A}$ with size i.

Again, using the function $f(i) = \binom{n}{i}^{-1}$ from [27] and the LYM Inequality (Inequality (5) in Section 3.2)

$$\sum_{i=1}^{n} \frac{p_i}{|\mathcal{A}|} f(i) \leq \frac{1}{|\mathcal{A}|}.$$

Since the function $f(i)$ can be extended to a convex function on the real numbers,

$$f\left(\frac{n}{k}\right) = f\left(\sum_{A\in\mathcal{A}} \frac{|A|}{|\mathcal{A}|}\right) = f\left(\sum_{i=1}^{n} i\frac{p_i}{|\mathcal{A}|}\right) \leq \sum_{i=1}^{n} f(i)\frac{p_i}{|\mathcal{A}|} \leq \frac{1}{|\mathcal{A}|}.$$

By the definition of $f(i)$,

$$f\left(\frac{n}{k}\right) = f\left(\frac{ck+r}{k}\right) = f\left(c+\frac{r}{k}\right) = \left(1-\frac{r}{k}\right)\binom{n}{c}^{-1} + \frac{r}{k}\binom{n}{c+1}^{-1}.$$

Thus,

$$|\mathcal{A}| \leq \frac{1}{(1-\frac{r}{k})\binom{n}{c}^{-1} + \frac{r}{k}\binom{n}{c+1}^{-1}} = \binom{n}{c}\frac{k}{(k-r)+\frac{r(c+1)}{n-c}}$$

and $|\mathcal{P}| \leq \frac{\binom{n}{c}}{(k-r)+\frac{r(c+1)}{n-c}}$. ✩

It would be better to know the exact cardinality and structure of the largest Sperner partition system. We conjecture that the largest Sperner partition system is an almost-uniform partition system (see Section 3.1.2 for definition of almost-uniform partition systems).

Conjecture 7.1.7. Let n, k be positive integers. The largest Sperner partition system in $\mathcal{P}_k^n$ is an almost-uniform partition system.

Similar to the case where k divides n, the maximum cardinality of a system of qualitatively independent k-partitions of an n-set is much smaller than $SP(n,k)$. This is best seen by considering the asymptotic growth of $SP(n,k)$ and comparing it to the asymptotic growth of the maximum cardinality of a system of qualitatively independent k-partitions of an n-set, $N(n,k)$.

7.1.3 Asymptotic Growth of Maximal Sperner Partition Systems

In this section, we give the asymptotic growth of Sperner partition systems. This result is similar to the result for the asymptotic growth of qualitatively independent partition systems cited in Section 2.4.2.

Lemma 7.1.8. *Let n, k, c be positive integers with $n = ck + r$ and $0 \leq r < k$. Then $SP(n, k) \leq SP(n + 1, k)$.*

Proof. Let $\mathcal{P} \subset \mathcal{P}_k^n$ be a Sperner partition system in $\mathcal{P}_k^n$ with $|\mathcal{P}| = SP(n, k)$. For each partition in $\mathcal{P}$, add the element $n + 1$ to the class in the partition that contains the element 1. This is a Sperner partition system in $\mathcal{P}_k^{n+1}$ and the result follows. ✰

The following bounds on $SP(n, k)$ follow from Theorem 7.1.5 and Lemma 7.1.8.

Corollary 7.1.9. *Let n, k, c be positive integers with $k \geq 2$ and $n = ck + r$, where $0 \leq r < k$. Then*

$$\frac{1}{k}\binom{ck}{c} \leq SP(n, k) \leq \frac{1}{k}\binom{(c+1)k}{c+1}.$$

The above upper bound on $SP(n, k)$ is weaker than the upper bound in Theorem 7.1.6.

The bounds in Corollary 7.1.9 will be used to produce a result on the asymptotic growth of $SP(n, k)$. Following the result of Gargano, Körner and Vaccaro [29] (see Section 2.4.2), we consider the growth of $\lim_{n\to\infty} \frac{\log SP(n,k)}{n}$.

First we need a technical result.

Lemma 7.1.10 ([35]). *For all positive integers n,*

$$\log_e n! = n \log_e n - n + \frac{\log_e n}{2} + O(1).$$

Theorem 7.1.11. *For positive integers n, k, c with $k \geq 2$ and $n = kc + r$ where $0 \leq r < k$,*

$$\limsup_{n\to\infty} \frac{\log SP(n, k)}{n} = \log \frac{k}{k-1} + \frac{\log(k-1)}{k}.$$

Proof. Since different logarithm bases differ by a constant factor, to prove the theorem it is sufficient to prove that for a positive integer k with $k \geq 2$,

$$\lim_{n\to\infty} \frac{\log_e SP(n,k)}{n} = \log_e k - \frac{k-1}{k}\log_e(k-1).$$

Further, by Corollary 7.1.9, it is enough to show that

$$\lim_{n\to\infty} \frac{\log_e \frac{1}{k}\binom{ck}{c}}{n} = \log_e \frac{k}{k-1} + \frac{\log_e(k-1)}{k}, \tag{16}$$

and

$$\lim_{n\to\infty} \frac{\log_e \frac{1}{k}\binom{(c+1)k}{c+1}}{n} = \log_e \frac{k}{k-1} + \frac{\log_e(k-1)}{k}, \tag{17}$$

where $c = c(n) = \lfloor \frac{n}{k} \rfloor$ and $r = r(n)$ such that $n = ck + r$ and $0 \leq r(n) < k$.

First expand $\log_e \binom{ck}{c}$ using Lemma 7.1.10:

$$\begin{aligned}
\log_e \binom{ck}{c} &= \log_e (ck)! - \log_e c! - \log_e (ck-c)! \\
&= ck\log_e(ck) - ck + \frac{\log_e(ck)}{2} - c\log_e c + c - \frac{\log_e c}{2} \\
&\quad -(ck-c)\log_e(ck-c) + ck - c - \frac{\log_e(ck-c)}{2} + O(1) \\
&= ck\log_e k + \frac{\log_e k}{2} - ck\log_e(k-1) + c\log_e(k-1) - \frac{\log_e(k-1)}{2} \\
&\quad -\frac{\log_e c}{2} + O(1).
\end{aligned}$$

From this and the fact that $n = ck + r$, we obtain Equation (16):

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log_e \frac{1}{k}\binom{ck}{c}}{n} &= \lim_{n\to\infty} \Bigg(\frac{ck\log_e k}{n} + \frac{\log_e k}{2n} - \frac{ck\log_e(k-1)}{n} \\
&\quad + \frac{c\log_e(k-1)}{n} - \frac{\log_e(k-1)}{2n} - \frac{\log_e c}{2n} + \frac{O(1)}{n} - \frac{\log_e k}{n} \Bigg) \\
&= \lim_{n\to\infty} \Bigg(\frac{(n-r)\log_e k}{n} + \frac{\log_e k}{2n} - \frac{(n-r)\log_e(k-1)}{n} \\
&\quad + \frac{(n-r)\log_e(k-1)}{kn} - \frac{\log_e(k-1)}{2n} - \frac{\log_e \frac{n-r}{k}}{2n} + \frac{O(1)}{n} - \frac{\log_e k}{n} \Bigg) \\
&= \log_e k - \log_e(k-1) + \frac{1}{k}\log_e(k-1)
\end{aligned}$$

Similarly, to obtain Equation (17), we first use Lemma 7.1.10:

$$\begin{aligned}\log_e \binom{(c+1)k}{c+1} &= (c+1)k\log_e k + \frac{\log_e k}{2} - (c+1)k\log_e(k-1) \\ &\quad +(c+1)\log_e(k-1) - \frac{\log_e(k-1)}{2} - \frac{\log_e(c+1)}{2} + O(1).\end{aligned}$$

Dividing by n, and taking the limit in the above equation, we obtain Equation (17):

$$\begin{aligned}\lim_{n\to\infty} \frac{\log_e\left(\frac{1}{k}\binom{(c+1)k}{c+1}\right)}{n} &= \lim_{n\to\infty}\left(\frac{(c+1)k\log_e k}{n} + \frac{\log_e k}{2n} - \frac{(c+1)k\log_e(k-1)}{n}\right. \\ &\quad +\frac{(c+1)\log_e(k-1)}{n} - \frac{\log_e(k-1)}{2n} - \frac{\log_e(c+1)}{2n} \\ &\quad \left.+\frac{O(1)}{n} - \frac{\log_e k}{n}\right) \\ &= \lim_{n\to\infty}\left(\frac{(n-r+k)\log_e k}{n} + \frac{\log_e k}{2n} - \frac{(n-r+k)\log_e(k-1)}{n}\right. \\ &\quad +\frac{(n-r+k)\log_e(k-1)}{kn} - \frac{\log_e(k-1)}{2n} - \frac{\log_e \frac{n-r+k}{k}}{2n} \\ &\quad \left.+\frac{O(1)}{n} - \frac{\log_e k}{n}\right) \\ &= \log_e k - \log_e(k-1) + \frac{1}{k}\log_e(k-1).\end{aligned}$$

☆

Recall from Section 2.4.2 that for all $k \geq 2$,

$$\limsup_{n\to\infty} \frac{\log_2 N(n,k)}{n} = \frac{2}{k},$$

and from Theorem 7.1.11 above

$$\begin{aligned}\limsup_{n\to\infty} \frac{\log_2 SP(n,k)}{n} &= \limsup_{n\to\infty} \frac{1}{\log_e 2}\frac{\log_e SP(n,k)}{n} \\ &= \frac{1}{\log_e 2}\left(\log_e \frac{k}{k-1} + \frac{\log_e(k-1)}{k}\right).\end{aligned}$$

It is no surprise that for $k = 2$, the asymptotic growth of $N(n, k)$ is the same as the asymptotic growth for $SP(n, k)$. For $k > 2$, asymptotic growth of $SP(n, k)$ is larger than that of $N(k, n)$. These results on Sperner partition systems are not enough to give a bound on the maximal collection of qualitatively independent partitions. In the next section we extend the Erdős-Ko-Rado Theorem to partition systems.

7.2 Intersecting Partitions

Following Section 3.4, it would be interesting to find a version of the Erdős-Ko-Rado Theorem for partitions. It is not obvious how to extend the definition of intersection from sets to partitions. Two sets intersect if they have an element in common, and since partitions are a collection of sets, a natural extension of intersection is that two partitions intersect if they have a common class. Further, two partitions *t-intersect* if they have at least t classes in common. This is the type of intersection considered in this section and Section 7.3, while in Section 7.5, a different type of intersection, which we call *partial intersection*, is considered.

Definition 7.2.1 (*t-Intersecting Partition System*). A partition system $\mathcal{P} \subseteq \mathcal{P}_k^n$ is *t-intersecting* if $|P \cap Q| \geq t$, for all $P, Q \in \mathcal{P}$.

Recall from Section 3.3.1, for positive integers t, k and n, a k-uniform trivially t-intersecting set system of an n-set is a set system formed by all k-sets of an n-set that contain a given t-set. Similarly, a partition system $\mathcal{P} \subseteq \mathcal{P}_k^n$ is a *trivially t-intersecting partition system* if $\mathcal{P}$ is equal, up to a permutation on $\{1, \dots, n\}$, to

$$\mathcal{Q}(n, k, t) = \{P \in \mathcal{P}_k^n : \{\{1\}, \{2\}, \dots, \{t\}\} \subseteq P\}.$$

The cardinality of a trivially t-intersecting partition system in $\mathcal{P}_k^n$ is $S(n - t, k - t) = |\mathcal{P}_{k-t}^{n-t}|$ (recall that $S(n, k)$ is the Stirling number of the second type, defined in Section 3.1.2).

For positive integers n, k, c with $n = ck$, $\mathcal{P} \subseteq \mathcal{U}_k^n$ is a *trivially t-intersecting* *uniform partition system* if $\mathcal{P}$ is equal, up to a permutation on $[1, n]$, to

$$\begin{aligned}\mathcal{P}(n,k,t) &= \{P \in \mathcal{U}_k^n : \\ &\qquad \{[1,c],[c+1,2c],\ldots,[(t-1)c+1,tc]\} \subseteq P\}\,.\end{aligned}$$

The cardinality of a trivially t-intersecting uniform partition system in $\mathcal{U}_k^{ck}$ is

$$U((k-t)c, k-t) = |\mathcal{U}_{k-t}^{(k-t)c}|. \tag{18}$$

Erdős and Székely observe that the following Erdős-Ko-Rado type theorem for t-intersecting partition systems holds.

Theorem 7.2.2 ([25]). *Let $k \geq t \geq 1$ be positive integers. There exists a function $n_0(k,t)$ such that if $n \geq n_0(k,t)$, and $\mathcal{P} \subseteq \mathcal{P}_k^n$ is a t-intersecting partition system, then $|\mathcal{P}| \leq S(n-t, k-t)$. This bound is attained by a trivially t-intersecting partition system.*

We prove analogous theorems for uniform partition systems that guarantee the uniqueness, up to isomorphism, of the maximal system. Our first theorem completely settles the case $t = 1$.

Theorem 7.2.3. *Let $n \geq k \geq 1$ be positive integers such that k divides n. Let $\mathcal{P} \subseteq \mathcal{U}_k^n$ be a 1-intersecting uniform partition system. Let $c = n/k$ be the size of a class in each partition. Then, $|\mathcal{P}| \leq U(n-c, k-1)$. Moreover, this bound holds with equality if and only if $\mathcal{P}$ is a trivially 1-intersecting uniform partition system.*

Our second theorem deals with general t and determines the cardinality and structure of maximal t-intersecting uniform partition systems when n is sufficiently large. In this theorem, n can be sufficiently large with respect to k and t. Alternately, if $c \geq t+2$, then n can be sufficiently large with respect to c and t.

Theorem 7.2.4. *Let $k \geq t \geq 1$. There exist functions $n_0(k,t)$ and $n_1(c,t)$ such that if ($n \geq n_0(k,t)$) or ($c \geq t+2$ and $n \geq n_1(c,t)$) and $\mathcal{P} \subseteq \mathcal{U}_k^n$ is a t-intersecting uniform partition system and $c = n/k$ is the size of a class in each partition, then*

1. $|\mathcal{P}| \leq U(n - tc, k - t)$*;*

2. this bound is tight if and only if $\mathcal{P}$ *is a trivially* t*-intersecting uniform partition system.*

In Section 7.2.1, we give a straightforward lemma from which we can easily prove Theorem 7.2.3 for all cases except $c = 2$ and Theorem 7.2.4. Indeed, the proof of Theorem 7.2.3 for $c = 2$ is the only more involved case. Since this proof applies to all c, it is presented in this generality in Section 7.2.2. In the proofs of Lemmas 7.2.5–7.2.8 in the following sections, we apply a version of the *kernel method* introduced by Hajnal and Rothschild [38].

7.2.1 Erdős-Ko-Rado Theorem for Partitions for $c \neq 2$

A *blocking set* $\mathcal{B} \subset \binom{[n]}{c}$ for a uniform partition system $\mathcal{P} \subseteq \mathcal{U}_k^n$ is a collection of c-sets, where $c = n/k$, such that $|\mathcal{B} \cap P| \geq 1$, for all $P \in \mathcal{P}$. For an intersecting partition system the set of classes in any partition in the system forms a blocking set. In particular, if $\mathcal{P} \subseteq \mathcal{U}_k^n$, then $\mathcal{P}$ has a blocking set with k sets.

Let $\mathcal{P} \subseteq \mathcal{U}_k^n$, $c = n/k$ and let A be a c-set of $[1, n]$; define $\mathcal{P}_A = \{P \in \mathcal{P} : A \in P\}$.

Lemma 7.2.5. *Let* $n \geq k \geq t \geq 1$ *be positive integers, and let* $\mathcal{P} \subseteq \mathcal{U}_k^n$ *be a* t*-intersecting partition system. Let* $c = n/k$ *be the size of a class in each partition. Assume that there does not exist a* c*-set that occurs as a class in every partition in* $\mathcal{P}$*. Then,*

$$|\mathcal{P}| \leq k\binom{k-2}{t} U(n - (t+1)c, k - (t+1)).$$

Proof. Let A be a class from a partition in $\mathcal{P}$. Since no single class occurs in every partition in $\mathcal{P}$, there is a partition $Q \in \mathcal{P}$ that does not contain A. Every partition in $\mathcal{P}_A$ must t-intersect Q. There are at most $k - 2$ classes in Q that do not contain an element in A. Each partition in $\mathcal{P}_A$ must contain at least t of these $k - 2$ classes. Thus, for any class A, $|\mathcal{P}_A| \leq \binom{k-2}{t} U(n - (t+1)c, k - (t+1))$.

Let $R \in \mathcal{P}$. Then, R is a blocking set of $\mathcal{P}$, and $\mathcal{P} = \cup_{A \in R} \mathcal{P}_A$. Thus, since

$|R| = k$, we get

$$|\mathcal{P}| \leq k\binom{k-2}{t}U(n-(t+1)c, k-(t+1)).$$

☆

Proof of Theorem 7.2.3 for $c \neq 2$**.** Let $n \geq k \geq 1$ and $c = n/k \neq 2$. Let $\mathcal{P} \subseteq \mathcal{U}_k^n$ be a maximal 1-intersecting uniform partition system that is not trivially 1-intersecting. The theorem is clearly true when $c = 1$ and when $k = 1$. By Lemma 7.2.5

$$|\mathcal{P}| \leq k(k-2)U(n-2c, k-2).$$

For $k \geq 2$ and $c \geq 3$,

$$\binom{kc-c}{c} \quad \geq \quad \binom{3k-3}{3} > k(k-1)(k-2).$$

Thus, we have $|\mathcal{P}| < \frac{1}{k-1}\binom{n-c}{c}U(n-2c, k-2) = U(n-c, k-1)$ and any 1-intersecting uniform partition system that is not trivially 1-intersecting has cardinality strictly less than $U(n-c, k-1)$.

Finally, from Equation (18), a trivially 1-intersecting uniform partition system has cardinality $U(n-c, k-1)$ and the theorem holds for $c \neq 2$. ☆

Proof of Theorem 7.2.4. Let $n \geq k \geq t \geq 1$. Let $\mathcal{P} \subseteq \mathcal{U}_k^n$ be a maximal t-intersecting uniform partition system that is not trivially t-intersecting. Let $c = n/k$ be the size of a class in each partition. From Equation (18), a trivially t-intersecting uniform partition system has cardinality $U(n-tc, k-t)$, thus, it is enough to show that for n sufficiently large $|\mathcal{P}| < U(n-tc, k-t)$.

For $c = 1$, there is only one partition and $|\mathcal{P}| = 1$, so we assume $c \geq 2$. If $t = k$ or $k-1$, then two partitions are t-intersecting if and only if they are identical. So we may also assume that $t \leq k-2$.

Let $\mathcal{A}$ be the set of all c-sets that occur in every partition in $\mathcal{P}$. Let $s = |\mathcal{A}|$ and since $\mathcal{P}$ is not trivially t-intersecting, we have $0 \leq s < t$. Consider the system $\mathcal{P}' = \{P \backslash \mathcal{A} : P \in \mathcal{P}\}$. The system $\mathcal{P}'$ is a t'-intersecting partition system contained in $\mathcal{U}_{k'}^{n'}$, with $k' = k - s$, $t' = t - s$ and $n' = n - sc = c(k - s)$, and $|\mathcal{P}| = |\mathcal{P}'|$. Furthermore, there exists no c-set in every partition in $\mathcal{P}'$, so by Lemma 7.2.5,

$$\begin{aligned} |\mathcal{P}'| &\leq k'\binom{k'-2}{t'}U(n'-(t'+1)c, k'-(t'+1)) \\ &= (k-s)\binom{k-s-2}{t-s}U(n-(t+1)c, k-(t+1)) \\ &\leq k\binom{k-2}{t}U(n-(t+1)c, k-(t+1)). \end{aligned}$$

Fix the value of t and k. Then for some n sufficiently large, relative to t and k we have

$$k\binom{k-2}{t} < \frac{1}{k-t}\binom{n-tc}{c}. \tag{19}$$

Thus, there exists a function $n_0(k,t)$ such that for $n \geq n_0(k,t)$ Inequality (19) holds.

Since

$$\frac{1}{k-t}\binom{n-tc}{c} = \binom{kc-tc-1}{c-1},$$

if c and t are fixed with $1 \leq t < c-1$, for k sufficiently large Inequality (19) holds. Thus, if $1 \leq t < c-1$ then there exists a function $n_1(c,t)$ such that for $n \geq n_1(c,t)$ Inequality (19) holds.

Therefore,

$$\begin{aligned} |\mathcal{P}'| &< \frac{1}{k-t}\binom{n-tc}{c}U(n-(t+1)c, k-(t+1)) \\ &= U(n-tc, k-t). \end{aligned}$$

☆

7.2.2 General Erdős-Ko-Rado Theorem for Partitions

It only remains to prove the case $c = 2$ of Theorem 7.2.3, but we give the proof for general c.

Let $\mathcal{P} \subseteq \mathcal{U}_k^n$, with $n = ck$, and let A be a c-set of an n-set. We denote $\mathcal{P}_A = \{P \in \mathcal{P} : A \in P\}$. Further, if for some positive integer ℓ, $\mathcal{A} = \{A_1, A_2, \ldots, A_\ell\}$ is a disjoint collection of c-sets of an n-set, then $\mathcal{P}_\mathcal{A} = \{P \in \mathcal{P} : A_i \in P \text{ for all } i = 1, \ldots, \ell\}$.

Lemma 7.2.6. *Let $n \geq k \geq 1$, and let $\mathcal{P} \subseteq \mathcal{U}_k^n$ be a 1-intersecting partition system that is not trivially 1-intersecting. Let $c = n/k$ be the size of a class in each partition. Let ℓ be the size of the smallest blocking set for $\mathcal{P}$. Then, for any $1 \leq i < \ell$, any given set of i classes of a partition can occur together in at most*

$$(k-i)(k-(i+1))\cdots(k-(\ell-1))U(c(k-\ell), k-\ell)$$

partitions in $\mathcal{P}$.

Proof. First, since the set of classes of any partition from $\mathcal{P}$ is a blocking set, we have $\ell \leq k$.

Use induction on $\ell - i$. If $i = \ell - 1$, consider a set of $(\ell - 1)$ disjoint c-sets $\mathcal{A} = \{A_1, A_2, \ldots, A_{\ell-1}\}$. Since $|\mathcal{A}| < \ell$, the set $\mathcal{A}$ is not a blocking set for $\mathcal{P}$. So, there exists a partition $Q \in \mathcal{P}$ that does not contain any of the $A_j \in \mathcal{A}$. Since the c-sets A_i are disjoint, $|\cup_{i=1}^{\ell-1} A_i| = c(\ell-1)$. This means there are at least $\ell - 1$ classes in Q that contain some element of $A_1 \cup A_2 \cup \cdots \cup A_{\ell-1}$. So, there are at most $k - (\ell - 1)$ classes in Q that could appear in a partition in $\mathcal{P}_\mathcal{A}$. Each partition in $\mathcal{P}_\mathcal{A}$ must contain at least one of these $k - (\ell - 1)$ classes. Thus,

$$\begin{aligned} |\mathcal{P}_\mathcal{A}| &\leq (k-(\ell-1))U(n-(\ell-1)c-c, k-(\ell-1)-1) \\ &= (k-\ell+1)U(n-\ell c, k-\ell). \end{aligned}$$

This completes the case $\ell = i + 1$.

Now, for $\ell \geq i + 1$, we assume that any set of i disjoint c-sets can occur together in at most

$$(k-i)(k-(i+1))\cdots(k-\ell+1)U(n-\ell c, k-\ell)$$

partitions in $\mathcal{P}$. Consider any set of $(i-1)$ disjoint c-sets $\mathcal{A} = \{A_1, A_2, \ldots, A_{i-1}\}$. Since $i - 1 < \ell$, there exists a partition $Q \in \mathcal{P}$ that does not contain any of the

$A_j \in \mathcal{A}$. There are at most $k-(i-1)$ classes in Q that could appear in a partition in $\mathcal{P}_{\mathcal{A}}$. By the induction hypothesis, each of these $k-(i-1)$ classes can occur together with all $A_j \in \mathcal{A}$ in at most $(k-i)(k-i+1)\cdots(k-\ell+1)U(n-\ell c, k-\ell)$ partitions. Thus,

$$|\mathcal{P}_{\mathcal{A}}| \leq (k-(i-1))(k-i)\cdots(k-\ell+1)U(n-\ell c, k-\ell).$$

✫

A slightly stronger version of the previous lemma is needed for $i = 1$.

Lemma 7.2.7. *Let $n \geq k \geq 1$, and let $\mathcal{P} \subseteq \mathcal{U}_k^n$ be a 1-intersecting system that is not trivially 1-intersecting. Let $c = n/k$ be the size of a class in each partition. If the size of a smallest blocking set for $\mathcal{P}$ is $\ell < k$, then any class can occur in at most*

$$(k-2)\left(\prod_{i=2}^{\ell-1}(k-i)\right)U(n-\ell c, k-\ell)$$

partitions in $\mathcal{P}$.

Proof. Let A be a class in a partition in $\mathcal{P}$. Since the system is not trivially 1-intersecting, there exists a partition $Q \in \mathcal{P}$ which does not contain A. Any partition in $\mathcal{P}_A$ must intersect Q. The elements from A must be in at least two separate classes in Q, thus there are at most $k-2$ classes in Q which could be in this intersection.

If $\ell = 2$ then each of these $k-2$ classes can occur in at most $(k-2)U(n-2c, k-2)$ partitions in $\mathcal{P}_A$. If $\ell > 2$ from Lemma 7.2.6, for the case $i = 2$, we have that any pair of classes can occur in at most $(k-2)\cdots(k-(\ell-1))U(n-\ell c, k-\ell)$ partitions. So each of these $k-2$ classes can occur in at most $(k-2)\cdots(k-(\ell-1))U(n-\ell c, k-\ell)$ partitions in $\mathcal{P}_A$. Thus, for all ℓ

$$|\mathcal{P}_A| \leq (k-2)\left(\prod_{i=2}^{\ell-1}(k-i)\right)U(n-\ell c, k-\ell).$$

✫

Lemma 7.2.8. *Let $n \geq k \geq 4$, and let $\mathcal{P} \subseteq \mathcal{U}_k^n$ be a 1-intersecting partition system that is not trivially t-intersecting. Let ℓ be the size of the smallest blocking set for $\mathcal{P}$.*

If $\ell = k-1$ or k, then any set of $i < k-2$ classes of a partition can occur in at most $(k-i)(k-i-1)(k-i-2)\cdots 3$ partitions in $\mathcal{P}$.

Proof. Since $\ell \geq k-1$, for any set $\mathcal{A}$ of $k-2$ pairwise disjoint classes, there exists a partition $Q \in \mathcal{P}$ that does not contain any of the classes in $\mathcal{A}$. Any partition in $\mathcal{P}_{\mathcal{A}}$ must intersect Q and there are at most 2 classes in Q which could be in this intersection. Let Q_0 and Q_1 be these two classes. For $P \in \mathcal{P}_{\mathcal{A}}$, if $Q_0 \in P$ then the first $k-2$ classes of P form $\mathcal{A}$, and another class is Q_0. Since Q_1 is disjoint from all the sets in $\mathcal{A}$ and from Q_0, the final class of P must be Q_1. Thus there is only one partition in $\mathcal{P}_{\mathcal{A}}$, and any set of $k-2$ classes can occur in at most one partition in $\mathcal{P}$.

We will use induction on $k-i$. If $i = k-3$, consider a set $\mathcal{A}$ of $k-3$ classes. Since $|\mathcal{A}| < \ell - 1$, there is a partition $Q \in \mathcal{P}$ that does not contain any of the classes in $\mathcal{A}$. There are at most $k-(k-3) = 3$ classes in Q that could appear in a partition in $\mathcal{P}_{\mathcal{A}}$. Since no set of $(k-2)$ c-sets can occur in more than one partition, $|\mathcal{P}_{\mathcal{A}}| \leq 3$.

Now, if $i \leq k-3$ we assume that any set of i classes of a partition can occur together in at most $(k-i)(k-i-1)(k-i-2)\cdots 3$ partitions in $\mathcal{P}$. Consider any set $\mathcal{A}$ of $i-1$ classes. There exists a partition $Q \in \mathcal{P}$ which does not contain any of the classes in $\mathcal{A}$. There are at most $k-(i-1)$ classes in Q which could occur in a partition in $\mathcal{P}_{\mathcal{A}}$. Thus, $|\mathcal{P}_{\mathcal{A}}| \leq (k-(i-1))(k-i)(k-i-1)(k-i-2)\cdots 3$. ✫

Before giving the proof of Theorem 7.2.3, for general c, we need two technical lemmas.

Lemma 7.2.9. *For $k-2 \geq \ell \geq 2$,*

$$\ell(k-2)\prod_{i=2}^{\ell-1}(k-i) < \prod_{i=1}^{\ell-1}(2(k-i)-1).$$

Proof. We prove this by induction on ℓ. If $\ell = 2$,

$$2(k-2) < 2k-3.$$

For $\ell \geq 2$, assume

$$\ell(k-2)\prod_{i=2}^{\ell-1}(k-i) < \prod_{i=1}^{\ell-1}(2k-2i-1), \tag{20}$$

then

$$\begin{aligned}
&(\ell+1)(k-2)\prod_{i=2}^{\ell}(k-i) = \\
&\quad = \tfrac{(\ell+1)}{\ell}(k-\ell)\ell(k-2)\prod_{i=2}^{\ell-1}(k-i) \\
&\quad < \tfrac{(\ell+1)}{\ell}(k-\ell)\prod_{i=1}^{\ell-1}(2k-2i-1) && \text{(by Inequality (20))} \\
&\quad \leq (2k-2\ell-1)\prod_{i=1}^{\ell-1}(2k-2i-1) && \text{(since } 2 \leq \ell \leq k-2\text{)} \\
&\quad = \prod_{i=1}^{\ell}(2k-2i-1).
\end{aligned}$$

☆

Lemma 7.2.10. *For integers $k > j \geq 2$, $c \geq 1$ and $n = ck$,*

$$U(c(k-1), k-1) = \left(\prod_{i=1}^{j-1}\binom{ck-ic-1}{c-1}\right)U(c(k-j), k-j). \tag{21}$$

Proof. We prove this by induction on j.

If $j = 2$, rewriting the size of a trivially 1-intersecting system, we get

$$\begin{aligned}
U(c(k-1), k-1) &= \frac{1}{k-1}\binom{ck-c}{c}U(c(k-2), k-2) \\
&= \binom{ck-c-1}{c-1}U(c(k-2), k-2).
\end{aligned}$$

For $k-1 > j \geq 2$, assume Equation (21) holds, then

$$\begin{aligned}
&U(c(k-1), k-1) \\
&= \left(\prod_{i=1}^{j-1}\binom{ck-ic-1}{c-1}\right)\frac{1}{k-j}\binom{ck-jc}{c}U(c(k-j)-c, k-j-1) \\
&= \left(\prod_{i=1}^{j-1}\binom{ck-ic-1}{c-1}\right)\binom{ck-jc-1}{c-1}U(c(k-j)-c, k-j-1) \\
&= \left(\prod_{i=1}^{j}\binom{ck-ic-1}{c-1}\right)U(c(k-(j+1)), k-(j+1)).
\end{aligned}$$

☆

Proof of Theorem 7.2.3 for all c**.** Let $\mathcal{P} \subseteq \mathcal{U}_k^n$ be a maximal 1-intersecting partition system that is not trivially 1-intersecting. It is enough to show that $|\mathcal{P}| < U(n-c, k-1)$. If $k = 2$ then every maximal 1-intersecting partition system is trivially 1-intersecting.

Consider the case when $k = 3$. Assume that $P, Q, R \in \mathcal{U}_3^n$ with $P = \{P_1, P_2, P_3\}$, $Q = \{Q_1, Q_2, Q_3\}$ and $R = \{R_1, R_2, R_3\}$. Further assume P, Q and R are distinct intersecting partitions that do not have a common class. Since P, and Q are distinct, $|P \cap Q| = 1$, so we may assume $P_1 = Q_1$. Since R intersects P and Q but the partitions do not contain a common class, we may assume $R_1 = P_2$ and $R_2 = Q_2$. Thus $P_2 \cap Q_2 = \emptyset$, but this means $P_2 = Q_3$ and $Q_2 = P_3$ contradicting that P and Q are distinct. Thus R can not intersect P and Q and for $k = 3$ it is not possible to have a non-trivial intersecting partition system.

We can assume that $k \geq 4$. For the same reason, we know $c \geq 2$. Let ℓ be the size of a smallest blocking set for $\mathcal{P}$. Since $\mathcal{P}$ is not trivially 1-intersecting, we know that $\ell > 1$.

Case 1. $2 \leq \ell \leq k-2$.

There exists a blocking set $\mathcal{B}$ for $\mathcal{P}$ with $|\mathcal{B}| = \ell$, and from Lemma 7.2.7 each class in $\mathcal{B}$ can be in at most $(k-2)\left(\prod_{i=2}^{\ell-1}(k-i)\right) U(c(k-\ell), k-\ell)$ partitions in $\mathcal{P}$. Thus,

$$|\mathcal{P}| \leq \ell(k-2)\left(\prod_{i=2}^{\ell-1}(k-i)\right) U(c(k-\ell), k-\ell). \tag{22}$$

From Lemma 7.2.9, and the fact that $2(k-i)-2 < \binom{c(k-i)-1}{c-1}$ for all $c \geq 2$, we get

$$\ell(k-2)\prod_{i=2}^{\ell-1}(k-i) < \prod_{i=1}^{\ell-1}(2(k-i)-1) \leq \prod_{i=1}^{\ell-1}\binom{c(k-i)-1}{c-1}. \tag{23}$$

Therefore,

$$\begin{aligned}
|\mathcal{P}| &\leq \ell(k-2)\left(\textstyle\prod_{i=2}^{\ell-1}(k-i)\right) U(c(k-\ell), k-\ell) && \text{(by Inequality (22))}\\
&< \left(\textstyle\prod_{i=1}^{\ell-1}\binom{c(k-i)-1}{c-1}\right) U(c(k-\ell), k-\ell) && \text{(by Inequality (23))}\\
&= U(c(k-1), k-1) && \text{(by Lemma 7.2.10 with } j = \ell)\\
&= U(n-c, k-1).
\end{aligned}$$

Case 2. $k \geq \ell \geq k-1$.

By Lemma 7.2.8, any single class can occur in at most $(k-1)(k-2)\cdots 3$ partitions in $\mathcal{P}$. Since there exists a blocking set of cardinality k,

$$|\mathcal{P}| \leq k(k-1)(k-2)(k-3)\cdots 3 = \prod_{i=1}^{k-2}(k-i+1). \tag{24}$$

We have

$$k-i+1 < 2k-2i-1, \quad \text{for all } i \leq k-3 \tag{25}$$

$$\text{and} \qquad 2(k-i)-1 \leq \binom{c(k-i)-1}{c-1} \quad \text{for all } c \geq 2. \tag{26}$$

Therefore,

$$\begin{aligned}
|\mathcal{P}| &\leq \prod_{i=1}^{k-2}(k-i+1) && \text{(by Inequality (24))}\\
&= 3\prod_{i=1}^{k-3}(k-i+1)\\
&< 3\prod_{i=1}^{k-3}(2k-2i-1) && \text{(by Inequality (25) and } k \geq 4\text{)}\\
&= \prod_{i=1}^{k-2}(2k-2i-1)\\
&\leq \prod_{i=1}^{k-2}\binom{c(k-i)-1}{c-1} && \text{(by Inequality (26) and } c \geq 2\text{)}\\
&= U(c(k-1),k-1) && \text{(by Lemma 7.2.10 with } j=k-1 \text{ and } U(c,1)=1\text{)}\\
&= U(n-c,k-1).
\end{aligned}$$

☆

7.3 Intersecting Packing Systems

These theorems can be seen as results on maximal families of 1-regular c-uniform hypergraphs on n vertices that intersect in at least t edges. Alternatively, these hypergraphs can be thought of as perfect matchings on $K_n^{(c)}$, the complete c-uniform hypergraph on n vertices. Thus, we can generalize our results for the case when c does not divide n, by considering maximal matchings in place of perfect ones.

Define an (n,c)*-packing* to be a set of disjoint c-sets of an n-set. Let $\mathcal{PC}_{n,c}$ denote the set of all maximum (n,c)-packings, that is, all (n,c)-packings with $\lfloor \frac{n}{c} \rfloor$ c-sets. Set $P(n,c) = |\mathcal{PC}_{n,c}|$, then for $k = \lfloor \frac{n}{c} \rfloor$,

$$P(n,c) = \frac{1}{k!}\binom{n}{c}\binom{n-c}{c}\cdots\binom{n-(k-1)c}{c}.$$

An (n,c)-packing system is a collection of (n,c)-packings.

An (n,c)-packing system $\mathcal{P} \subseteq \mathcal{PC}_{n,c}$ is *t-intersecting* if $|P \cap Q| \geq t$ for all $P, Q \in \mathcal{P}$. It is straightforward to define a trivially intersecting t-intersecting (n,c)-packing system.

Generalizations of Theorems 7.2.3 and 7.2.4 are stated next without proof. The proofs for these are very similar to the ones used for the original theorems. Indeed, the only change to the original proofs is that Lemma 7.2.5 needs to have $k-1$ in place of $k-2$ in the upper bound on $|\mathcal{P}|$.

Lemma 7.3.1. *Let n, c, k and t be positive integers such that $n \geq c$ and $t \leq k = \lfloor n/c \rfloor$. Let $\mathcal{P} \subseteq \mathcal{P}_{n,c}$ be a t-intersecting (n,c)-packing system. Assume that there does not exist a c-set that occurs in every (n,c)-packing in $\mathcal{P}$. Then,*

$$|\mathcal{P}| \leq k\binom{k-1}{t}P(n-(t+1)c, c).$$

Theorem 7.3.2. *Let n, c, k and t be positive integers such that $n \geq c$ and $k = \lfloor \frac{n}{c} \rfloor$. Let $\mathcal{P} \subseteq \mathcal{P}_{n,c}$ be a 1-intersecting (n,c)-packing system. Then $|\mathcal{P}| \leq P(n-c, c)$. Moreover, this bound is tight if and only if $\mathcal{P}$ is a trivially 1-intersecting (n,c)-packing system.*

Theorem 7.3.3. *Let n, c, k and t be positive integers such that $n \geq c$ and $t \leq k = \lfloor \frac{n}{c} \rfloor$. Let $\mathcal{P} \subseteq \mathcal{P}_{n,c}$ be a t-intersecting (n,c)-packing system. Then there exist functions $n_0(k,t)$ and $n_1(c,t)$ such that if ($n \geq n_0(k,t)$) or ($c \geq t+2$ and $n \geq n_1(c,t)$) then,*

1. *$|\mathcal{P}| \leq P(n-tc, c)$;*

2. *moreover, this bound is tight if and only if $\mathcal{P}$ is a trivially t-intersecting (n, c)-packing system.*

7.4 Towards a Complete Theorem for t-Intersecting Partition Systems

Ahlswede and Khachatrian [4] have extended the Erdős-Ko-Rado Theorem for set systems by determining the cardinality and structure of all maximal t-intersecting set systems $P \subseteq \binom{[n]}{k}$ for all possible $n \leq (k-t+1)(t+1)$ (see Theorem 3.3.5). This remarkable result went beyond proving a conjecture by Frankl [26] that stated a specific list of candidates for maximal set systems. Next, we state conjectures for uniform t-intersecting partition systems, which parallel the conjecture of Frankl and the theorem of Ahlswede and Khachatrian, respectively.

For $0 \leq i \leq \lfloor (k-t)/2 \rfloor$, define the partition system

$$\begin{aligned} \mathcal{P}_i(n,k,t) \quad = \quad & \{P \in \mathcal{U}_k^n : |P \cap \{[1,c],[c+1,2c], \\ & \ldots, [(t+2i-1)c+1, (t+2i)c]\}| \geq t+i\} \,. \end{aligned}$$

Note that $\mathcal{P}_0(n,k,t) = \mathcal{P}(n,k,t)$. Theorem 7.2.4 says that for n sufficiently large $\mathcal{P}_0(n,k,t)$ is the unique (up to permutations on $[1,n]$) largest t-intersecting uniform partition system in $\mathcal{U}_k^n$. We conjecture that for any n the unique (up to permutations on $[1,n]$) largest t-intersecting uniform partition system in $\mathcal{U}_k^n$ is one of $\mathcal{P}_i(n,k,t)$ for $i \in \{0, \ldots \lfloor (k-t)/2 \rfloor\}$.

Conjecture 7.4.1. Let n, k and t be positive integers with $n \geq k \geq t \geq 1$, and let $\mathcal{P} \subseteq \mathcal{U}_k^n$ be a t-intersecting partition system. Then

$$|\mathcal{P}| \leq \max_{0 \leq i \leq \frac{k-t}{2}} |\mathcal{P}_i(n,k,t)|.$$

Conjecture 7.4.2. Let c, t and i be positive integers. There exists a function $n_0(c, t, i)$, such that, for an integer k with $k \geq t$ and $n_0(c, t, i+1) < ck < n_0(c, t, i)$, if $\mathcal{P} \subseteq \mathcal{U}_k^{ck}$ is a t-intersecting partition system, then $|\mathcal{P}| \leq |\mathcal{P}_i(ck, k, t)|$. Moreover, this bound is tight if and only if $\mathcal{P}$ is equal (up to permutations on $[1, ck]$) to $\mathcal{P}_i(ck, k, t)$.

One could hope to be able to use the ideas in [4] to prove these conjectures; however, key techniques such as *left compression*, which are used in their proofs, do not seem to have an extension to partition systems.

We conclude with an infinite sequence of parameters (n, k, t) for which $|\mathcal{P}_1(n, k, t)| > |\mathcal{P}(n, k, t)|$. This is not a counter example to Conjecture 7.4.2 since, in this example, we require that $t = k-3$, thus $n = c(t+3)$. In fact, this example gives a lower bound for the function $n_0(c, t, 0)$, specifically $n_0(c, t, 0) \geq c(t+3)$.

Proposition 7.4.3. *For positive integers c, k, t with $k \geq 3$ and $t = k-3$, let $n = ck$. For a function $k_0(c)$, if $k > k_0(c)$, then $|\mathcal{P}_1(n, k, t)| > |\mathcal{P}(n, k, t)|$.*

Proof. From Equation (18),

$$\begin{aligned} |\mathcal{P}(n, k, k-3)| &= U(n-(k-3)c, k-(k-3)) \\ &= \frac{1}{3!}\binom{3c}{c}\binom{2c}{c} \\ &= \binom{3c-1}{c}\binom{2c-1}{c-1}. \end{aligned}$$

Since

$$\begin{aligned} \mathcal{P}_1(n, k, t) &= \{P \in \mathcal{U}_k^n : |P \cap \{[1, c], [c+1, 2c], \\ &\qquad \ldots, [(t+1)c+1, (t+2)c]\}| \geq t+1\}, \end{aligned}$$

for $t = k-3$,

$$\begin{aligned} |\mathcal{P}_1(n, k, t)| &= \binom{t+2}{t+1} U(n-(t+1)c, k-(t+1)) \\ &\quad - \binom{t+2}{t+2} U(n-(t+2)c, k-(t+2)) + 1 \\ &= (t+2)\binom{2c-1}{c-1} - t - 1. \end{aligned}$$

For k (and t, since $t = k-3$) sufficiently large

$$\binom{3c-1}{c-1}\binom{2c-1}{c-1} < (t+2)\binom{2c-1}{c-1} - t - 1.$$

☆

7.5 Partially Intersecting Partitions

Erdős and Székely [25] define another type of intersecting partitions, which we call here *partially t-intersecting.* Two partitions $P, Q \in \mathcal{P}_k^n$ are said to be partially t-intersecting if there exist classes $P_i \in P$ and $Q_j \in Q$ such that $|P_i \cap Q_j| \geq t$. A partition system $\mathcal{P} \subseteq \mathcal{P}_k^n$ that is pairwise partially t-intersecting is called a *partially t-intersecting partition system.*

A partition system is called a *trivially partially t-intersecting partition system* if it is equal, up to permutations on $[1, n]$, to

$$\mathcal{R}(n,k,t) \;=\; \{P \in \mathcal{P}_k^n : [1,t] \subseteq A, \text{ for some } A \in P\}.$$

Conjecture 7.5.1. (Czabarka's Conjecture, see [25]) Let $n \leq 2k-1$ and $\mathcal{P} \subseteq \mathcal{P}_k^n$ be a partially 2-intersecting partition system. Then, $|\mathcal{P}| \leq S(n-1,k)$.

This bound is attained by the system

$$\{P \in \mathcal{P}_k^n : [1,2] \subseteq P_i, \text{ for some } P_i \in P\}.$$

We pose a similar conjecture for uniform partition systems.

Conjecture 7.5.2. Let k, c, t be positive integers with $t \leq c$ and $n = ck$. Let $\mathcal{P} \subseteq \mathcal{U}_k^n$ be a partially t-intersecting uniform partition system. Then, $|\mathcal{P}| \leq \binom{n-t}{c-t} U(n-c, k-1)$. Moreover, this bound is tight if and only if $\mathcal{P}$ is equal, up to permutations of $[1, n]$, to

$$\{P \in \mathcal{U}_k^n : [1,t] \subseteq P_i, \text{ for some } P_i \in P\}.$$

Note that Theorem 7.2.3 confirms Conjecture 7.5.2 for $t = c$.

For some values of t, k, n, a partially t-intersecting k-partition system of an n-set corresponds to an independent set in the graph $UQI(n, k)$.

Proposition 7.5.3. *For positive integers c, k, t with $t \geq c - k + 2$, a partially t-intersecting c-uniform k-partition system is an independent set in the graph $UQI(ck, k)$.*

Proof. Assume $\mathcal{P}$ is a partially t-intersecting c-uniform k-partition system in $\mathcal{P}_k^{ck}$ with $t \geq c - k + 2$. Note that $\mathcal{P} \subseteq V(UQI(ck, k))$.

To show $\mathcal{P}$ is an independent set in $UQI(ck, k)$ we will show that for any $P, Q \in V(UQI(ck, k))$, P and Q are not qualitatively independent. Since P and Q are partially t-intersecting there are classes $P_1 \in P$ and $Q_1 \in Q$ such that $|P_1 \cap Q_1| \geq t$. Since P is a c-uniform partition and $t \geq c-k+2$ there are at most $c-(c-k+2) = k-2$ elements in P_1 which are not in Q_1. Thus the class P_1 can intersect at most $k - 2$ classes in Q, other than Q_1. Since Q is a k-partition, P_1 can not intersect every class in Q. Hence, P and Q are not qualitatively independent. ✰

Proposition 7.5.4. *Let c, k, r be positive integers with $n = ck + r$ and $0 \leq r < k$. For $t \geq c - k + 3$, a partially t-intersecting almost-uniform k-partition system on an n-set is an independent set in $AUQI(n, k)$.*

Proof. Assume $\mathcal{P}$ is a partially t-intersecting almost-uniform k-partition system in $\mathcal{P}_k^n$ with $t \geq c - k + 3$ and $n = ck + r$.

To show $\mathcal{P}$ is an independent set in $AUQI(n, k)$ we will show that for any $P, Q \in V(AUQI(n, k))$, P and Q are not qualitatively independent. Since P and Q are partially t-intersecting there are classes $P_1 \in P$ and $Q_1 \in Q$ such that $|P_1 \cap Q_1| \geq t$. Since P is an almost uniform partition $|P_1| \leq c + 1$. Since, $t \geq c - k + 3$ there are at most $(c+1) - (c-k+3) = k-2$ elements in P_1 which are not in Q_1. Thus the class P_1 can intersect at most $k-2$ classes in Q, other than Q_1. Since Q is a k-partition, P_1 can not intersect every class in Q. Hence, P and Q are not qualitatively independent. ✰

We can prove the bound in Conjecture 7.5.2 for specific partition systems; to do this we need the following theorem.

Theorem 7.5.5. *For positive integers n, k, c and t with $n = ck$ and $t \leq c$, if there exists a resolvable t-$(n, c, 1)$ design, then the size of the largest partially t-intersecting uniform k-partition system from $\mathcal{U}_k^n$ is bounded above by*

$$\frac{1}{(k-1)!}\binom{n-t}{c-t}\binom{n-c}{c}\cdots\binom{c}{c} = \binom{n-t}{c-t}U(n-c, k-1).$$

Further, this bound is attained by a trivially partially t-intersecting k-partition system.

Proof. Define a graph $PIP(n, k)$ whose vertex set is the set of all uniform k-partitions of an n-set and two vertices are adjacent if and only if they are partially t-intersecting. The cardinality of the vertex set is

$$U(n, k) = \frac{1}{k!}\binom{n}{c}\binom{n-c}{c}\cdots\binom{c}{c}.$$

Further, the graph $PIP(n, k)$ is vertex transitive and a clique in $PIP(n, k)$ corresponds to a partially t-intersecting partition system.

An independent set in $PIP(n, k)$ is a set $\mathcal{Q}$ of partitions such that no two distinct partitions in $\mathcal{Q}$ are partially t-intersecting. Consider a resolvable t-$(n, c, 1)$ design. Each resolution class in the resolvable t-$(n, c, 1)$ design corresponds to a c-uniform k-partition of an n-set. Further, for each t-set there is exactly one resolution class in the resolvable t-$(n, c, 1)$ design which contains the t-set in one of its blocks. Thus, a resolvable t-$(n, c, 1)$ design corresponds to an independent set in $PIP(n, k)$.

The number of blocks in a resolvable t-$(n, c, 1)$ design is $\frac{n!(c-t)!}{c!(n-t)!}$, and the number of resolution classes is $\frac{(n-1)!(c-t)!}{(c-1)!(n-t)!}$. The maximum independent set in $PIP(n, k)$ has size at least $\frac{(n-1)!(c-t)!}{(c-1)!(n-t)!}$.

Since $PIP(n, k)$ is vertex transitive, with Inequality (8) from Section 4.1.5 we have

$$\begin{aligned}
\omega(PIP(n, k)) &\leq \frac{|V(PIP(n, k))|}{\alpha(PIP(n, k))} \\
&\leq \frac{\frac{1}{k!}\binom{n}{c}\binom{n-c}{c}\cdots\binom{c}{c}}{\frac{(n-1)!(c-t)!}{(c-1)!(n-t)!}} \\
&= \frac{1}{(k-1)!}\binom{n-t}{c-t}\binom{n-c}{c}\cdots\binom{c}{c}.
\end{aligned}$$

☆

Corollary 7.5.6. *For $n = 3k$ and k odd, the largest partially 2-intersecting uniform k-partition system from $\mathcal{U}_k^n$ has cardinality*

$$(3k - 2)U(3k - 3, k).$$

This bound is attained by a trivially partially 2-intersecting k-partition system.

Proof. If k is odd, then $3k \equiv 3 \pmod 6$ and a resolvable $(3k, 3, 1)$-BIBD (equivalently, a resolvable 2-$(3k, 3, 1)$ design) exists [20]. The result follows from Theorem 7.5.5 ✰

Corollary 7.5.7. *For $n \equiv 4 \pmod{12}$ and $k = n/4$, the largest partially 2-intersecting uniform k-partition system from $\mathcal{U}_k^n$ has cardinality*

$$\binom{4k-2}{2} U(n - 4, k).$$

This bound is attained by a trivially partially 2-intersecting k-partition system.

Proof. If $n \equiv 4 \pmod{12}$, then a resolvable $(n, 4, 1)$-BIBD (equivalently, a 2-$(n, 4, 1)$ design) exists [20]. The result follows from Theorem 7.5.5 ✰

Corollary 7.5.8. *For k a prime power and $n = k^2$, the largest partially 2-intersecting uniform k-partition system from $\mathcal{U}_k^n$ has cardinality*

$$\binom{k^2-2}{k-2} U(k^2 - k, k - 1).$$

This bound is attained by a trivially partially 2-intersecting k-partition system.

Proof. For all k a prime power there exists a resolvable $(k^2, k, 1)$-BIBD (Section 8.3 [73]). ✰

We have already seen a stronger version of this result for all k in another form. For $n = k^2$, by Proposition 7.5.3, a partially 2-intersecting uniform k-partition system of an n-set is an independent set in $QI(k^2, k)$ and from Lemma 6.3.3, $\alpha(QI(k^2, k)) = \binom{k^2-2}{k-2} U(k^2 - k, k - 1)$. Thus the bound from Corollary 7.5.8 holds for all values of k.

Theorem 7.5.9. *For all integers k and $n = k^2$, the largest partially 2-intersecting uniform k-partition system from $\mathcal{U}_k^n$ has cardinality*

$$\binom{k^2-2}{k-2} U(k^2-k, k-1).$$

This bound is attained by a trivially partially 2-intersecting k-partition system.

It is interesting to note that Lemma 6.3.3 was proven with the ratio bound and the eigenvalues of $QI(k^2, k)$. It may be possible to prove Conjecture 7.5.2 using the eigenvalues of the graph $PIP(n, k)$ and the ratio bound.

Question 7.5.10. Is it possible to find the eigenvalues of $PIP(n, k)$? Does the ratio bound (Theorem 4.2.12) give a good bound on $\alpha(PIP(n, k))$?

Corollary 7.5.11. *For v a positive integer with $v \equiv 4, 8 \pmod{12}$, except possibly for $v \in \{220, 236, 292, 364, 460, 596, 676, 724, 1076, 1100, 1252, 1316, 1820, 2236, 2308, 2324, 2380, 2540, 2740, 2812, 3620, 3820, 6356\}$, the largest partially 3-intersecting uniform $v/4$-partition system has cardinality*

$$(v-3)U(v-4, v/4-1).$$

This bound is attained by a trivially partially 3-intersecting $v/4$-partition system.

Proof. A resolvable 3-$(v, 4, 1)$ design exists for all such v [20]. ✫

There is another approach that can be used to bound partially t-intersecting uniform k-partition systems. Recall from Definition 3.1.5 that a relaxation of a design is a *packing*. A t-(n, k, λ) packing is a set system with the property that every t-set occurs in at most λ sets. Thus, the argument used in Theorem 7.5.5 follows exactly for t-$(n, k, 1)$ packings except with a different (possibly weaker) bound. We state this theorem without proof.

Theorem 7.5.12. *For positive integers k and c with $n = ck$, if there exists a resolvable t-$(n, c, 1)$ packing with B blocks, then the cardinality of the largest partially t-intersecting uniform k-partition system is bounded by*

$$\frac{\frac{1}{(k-1)!}\binom{n}{c}\binom{n-c}{c}\cdots\binom{c}{c}}{B}$$

Less is known about resolvable t-$(n, c, 1)$ packings, but we do have a result for $c = 3$.

Corollary 7.5.13. *Let $v = 3k$ where k is even and $k > 4$. Then the largest partially 2-intersecting uniform k-partition system has cardinality no more than*

$$(3k-1)U(3k-3, k).$$

Proof. From [20], an optimal resolvable 2-$(v, 3, 1)$ packing exists for all $v \equiv 0 \pmod 6$ except $v = 6, 12$. This packing has $\lfloor \frac{3k}{3} \lfloor \frac{3k-1}{2} \rfloor \rfloor$ blocks, which is the maximum possible. ✫

Chapter 8

Higher Order Problems

In the previous chapter, we consider ways to extend Sperner's Theorem and the Erdős-Ko-Rado Theorem to partition systems. In this chapter, these results are reformatted as part of a more general scheme of extremal results on partition systems.

The scheme we use here is based on the framework used by Ahlswede, Cai and Zhang [2, 3] for disjoint *clouds*. Recall from the introduction to Chapter 7 that a cloud is a collection of sets and that a c-uniform cloud is a collection of c-sets. Ahlswede, Cai and Zhang consider bounds on the cardinality of the largest system of pairwise disjoint clouds for which certain kinds of binary relations hold. They consider the following four binary relations between the sets in the clouds: comparable, incomparable, disjoint, and intersecting. Further, the binary relations can hold for clouds in four different ways called *types*. For example, for two clouds, a binary relation holds with type $(\forall, \forall)$ if it holds between *all* sets from the first cloud and *all* sets from the second cloud. The three other types that they consider are $(\exists, \forall)$, $(\forall, \exists)$ and $(\exists, \exists)$; each of these types are defined in the following section.

Ahlswede, Cai and Zhang consider all four types of all four binary relations for both disjoint c-uniform clouds and disjoint clouds. For many of these 32 different problems they give either an exact solution (with the system that attains the maximum cardinality) or an asymptotic solution. These problems include Sperner's Theorem and the Erdős-Ko-Rado Theorem. Other results for specific problems in this scheme are also given in [1] and [5].

We will consider the same 32 problems for extremal set-partition systems — that is, both uniform k-partitions and k-partitions and all types of the four binary relations. Since we require that the families of sets be partitions, rather than clouds, we require more structure than a cloud. Moreover, we are looking for systems from $\mathcal{P}_k^n$, not systems of subsets of the n-set, so we do not require that the partitions be disjoint. Also, we require that partitions in our systems be k-partitions, whereas in [3] two clouds in a system could have a different number of sets. Most of the problems in this chapter are new; it is not clear yet which may have applications.

Where appropriate, we will use the notation for partition systems and uniform partition systems given in Chapter 3.

8.1 Types of Relations

In this section, we detail the different problems that are considered in this chapter.

Let n be a positive integer and A and B be subsets of an n-set. The four binary relations that we consider are the following:

1. comparable ($A \subseteq B$ or $B \subseteq A$),
2. incomparable ($A \not\subseteq B$ and $B \not\subseteq A$),
3. disjoint ($A \cap B = \emptyset$),
4. intersecting ($A \cap B \neq \emptyset$).

Unlike previous work on clouds [1, 2, 3, 5], we do not require that the partitions in the system be pairwise disjoint. Hence, the definition of comparable sets includes sets that are equal and the definition of incomparable sets requires that sets not be equal.

For each binary relation, there are four *types* of problems. The first type is *type* $(\forall, \forall)$. The partition system $\mathcal{P}$ is of type $(\forall, \forall)$ for a binary relation if for any distinct $P, Q \in \mathcal{P}$ *for all* classes $P_i \in P$ and *for all* classes $Q_j \in Q$, P_i and Q_j satisfy the binary relation. A partition system $\mathcal{P}$ is of type $(\exists, \forall)$ if for any distinct $P, Q \in \mathcal{P}$

there exists a class $P_i \in P$ such that *for all* classes $Q_j \in Q$ the binary relation holds for P_i and Q_j. This is a weaker condition than type $(\forall, \forall)$. For a partition system $\mathcal{P}$ to be of type $(\forall, \exists)$, for any distinct $P, Q \in \mathcal{P}$ it is required that *for all* classes $P_i \in P$ *there exists* a class $Q_j \in Q$ such that the relation holds for P_i and Q_j. This is a weaker condition than both type $(\exists, \forall)$ and $(\forall, \forall)$. The final type of problem is *type* $(\exists, \exists)$ and it is the weakest condition. All that is required for a partition system $\mathcal{P}$ is that for every distinct $P, Q \in \mathcal{P}$ *there exists* a class $P_i \in P$ and that *there exists* a class $Q_j \in Q$ such that the relation holds for P_i and Q_j.

Note that each of the relations, comparable, incomparable, disjoint and intersecting, are symmetric, but not all the types of relations are symmetric. In particular, only the types $(\forall, \forall)$ and $(\exists, \exists)$ are symmetric for the relations we consider here.

Following the notation in [3], $\mathcal{A}_n(X, Y, k)$ will denote a k-partition system of an n-set of type (X, Y) for the relation A, and $A_n(X, Y, k)$ will denote the cardinality of the largest such partition system. The parameter A can take the following values: N for incomparable, C for comparable, D for disjoint, and I for intersecting. The values of X and Y can each be either $\forall$ or $\exists$. Thus, $D_n(\forall, \exists, k)$ denotes the cardinality of the largest partition system $\mathcal{P} \subseteq \mathcal{P}_k^n$ with the property that for any two distinct partitions $P, Q \in \mathcal{P}$, for all $P_i \in P$ there exists a $Q_j \in Q$ such that $P_i \cap Q_j = \emptyset$.

Similarly, we will denote a *uniform* k-partition system of an n-set of type (X, Y) for the relation A by $\mathcal{A}_n^*(X, Y, k)$ and the size of the the largest such partition system by $A_n^*(X, Y, k)$.

If partitions P and Q satisfy a binary relation of type $(\exists, \forall)$, then Q and P satisfy the same binary relation of type $(\forall, \exists)$. To see this, let P_0 be the class in P such that the relation holds between P_0 and all classes $Q_i \in Q$. Then for every $Q_i \in Q$ the relation holds for some class in P, namely P_0. Thus, for all binary symmetric relations A,

$$A_n(\forall, \forall, k) \leq A_n(\exists, \forall, k) \leq A_n(\forall, \exists, k) \leq A_n(\exists, \exists, k), \tag{27}$$

and

$$A_n^*(\forall, \forall, k) \leq A_n^*(\exists, \forall, k) \leq A_n^*(\forall, \exists, k) \leq A_n^*(\exists, \exists, k). \tag{28}$$

Further, for every type (X, Y) and every binary relation A, if k divides n, then $A_n^*(X, Y, k) \leq A_n(X, Y, k)$.

8.2 Uniform Partition Systems

First, we consider uniform k-partition systems. Throughout this section, n, c, k will denote positive integers with $n = ck$.

The requirement that the partitions in the system be uniform k-partitions makes this problem more tractable than the general case. In particular, many of the higher order extremal problems considered here, with uniform partitions, can be translated to finding maximum cliques and independent sets in a vertex-transitive graph.

8.2.1 Uniform Partition System Graphs

In the same way that we used the vertex-transitive graph $UQI(n, k)$ to find bounds on the cardinality of a set of qualitatively independent uniform partitions (Section 6.2) and the graph $PIP(n, k)$ to bound the cardinality of a set of partially intersecting partitions (Section 7.5), we can build vertex-transitive graphs that can be used to find bounds on the cardinality of uniform partition systems that satisfy a given type of binary relation.

We define the graph $G_{\mathcal{A}_n^*(X,X,k)}$ to be the graph whose vertex set is $\mathcal{U}_k^n$ and two partitions are adjacent if and only if they satisfy the binary relation A with type (X, X) (where $A \in \{N, C, D, I\}$ and the value of X can be either $\forall$ or $\exists$). Since for the two types $(\forall, \forall)$ and $(\exists, \exists)$ each of the binary relations are symmetric, the graph $G_{\mathcal{A}_n^*(X,X,k)}$ is well-defined.

In the graph $G_{\mathcal{I}_n^*(\forall,\forall,k)}$, two partitions $P, Q \in \mathcal{U}_k^n$ are adjacent if and only if for all $P_i \in P$ and for all $Q_j \in Q$, $P_i \cap Q_j \neq \emptyset$. This requirement is exactly that P and Q be qualitatively independent, so $G_{\mathcal{I}_n^*(\forall,\forall,k)}$ is exactly $UQI(n, k)$. The size of the maximum clique in the graph is $I_n^*(\forall, \forall, k)$. Moreover, two partitions are not adjacent in $G_{\mathcal{I}_n^*(\forall,\forall,k)}$ if and only if they do not satisfy this relation. The only way two partitions P, Q could not satisfy this relation is if there exists a class $P_i \in P$ and a class $Q_j \in Q$

such that $P_i \cap Q_j = \emptyset$. This means P and Q are not adjacent in $G_{\mathcal{I}_n^*(\forall,\forall,k)}$ if and only if they satisfy the relation disjoint of type $(\exists,\exists)$. Thus, $D_n^*(\exists,\exists,k)$ is the size of the maximum independent set in the graph $G_{\mathcal{I}_n^*(\forall,\forall,k)}$. Moreover, $G_{\mathcal{I}_n^*(\forall,\forall,k)}$ and $G_{\mathcal{D}_n^*(\exists,\exists,k)}$ are graph complements of each other.

For each of these symmetric relations there is a *converse relation*. Below, each relation is matched with its converse:

$$\begin{array}{ccc}
N_n^*(\forall,\forall,k) & \longleftrightarrow & C_n^*(\exists,\exists,k) \\
C_n^*(\forall,\forall,k) & \longleftrightarrow & N_n^*(\exists,\exists,k) \\
I_n^*(\forall,\forall,k) & \longleftrightarrow & D_n^*(\exists,\exists,k) \\
D_n^*(\forall,\forall,k) & \longleftrightarrow & I_n^*(\exists,\exists,k)
\end{array}$$

All the graphs $G_{\mathcal{A}_n^*(X,X,k)}$, for $A \in \{N, C, D, I\}$ and X either $\forall$ or $\exists$, are vertex transitive. From Corollary 4.1.10, if G is a vertex-transitive graph, $\omega(G) \leq \frac{|V(G)|}{\alpha(G)}$. Thus, for uniform k-partition systems we have more bounds on the size of $A_n^*(X,X,k)$.

Proposition 8.2.1. *Let n, k be positive integers such that k divides n. Then*

1. $N_n^*(\forall,\forall,k) \leq \frac{U(n,k)}{C_n^*(\exists,\exists,k)}$,
2. $C_n^*(\forall,\forall,k) \leq \frac{U(n,k)}{N_n^*(\exists,\exists,k)}$,
3. $I_n^*(\forall,\forall,k) \leq \frac{U(n,k)}{D_n^*(\exists,\exists,k)}$,
4. $D_n^*(\forall,\forall,k) \leq \frac{U(n,k)}{I_n^*(\exists,\exists,k)}$.

We could define graphs $G_{\mathcal{A}_n^*(X,Y,k)}$ for all types of relations, that is, for $A \in \{N, C, D, I\}$ and the values of X and Y each either $\forall$ or $\exists$. But, if $X \neq Y$, then these graphs may be directed and we would not have bounds equivalent to the bounds in Proposition 8.2.1.

8.2.2 Summary of Results

The following chart organizes the results for all the different types of problems for uniform partition systems. The entries either give the exact value of $A_n^*(X,Y,k)$ or a bound on $A_n^*(X,Y,k)$.

$A^*\backslash(X,Y)$	$(\forall,\forall)$	$(\exists,\forall)$	$(\forall,\exists)$	$(\exists,\exists)$
Incomparable	$\binom{n-1}{c-1}$	$U(n,k)$	$U(n,k)$	$U(n,k)$
Comparable	1	1	1	$U(n-c,k-1)$
Disjoint	1	1	$U(n,k)$ if $c<k$ $\geq U(n-c,k-1)$ if $c \geq k$	$U(n,k)$ if $c<k$ $\geq \binom{n-c+k-2}{k-2} U(n-c,k-1)$ if $c \geq k$
Intersecting	1 if $c<k$ $\leq \frac{\binom{n-1}{c-1}}{\binom{n-(c-k+2)}{k-2}}$ if $c \geq k$	1 if $c<k$ $\leq \binom{n-1}{c-1}$ if $c \geq k$	$U(n,k)$	$U(n,k)$

We will look at the different types of problems paired with their converse problem: incomparable with comparable, and intersecting with disjoint.

8.2.3 Incomparable and Comparable

For uniform partitions, we can solve all types of problems where the binary relation is either incomparable or comparable. This is due to the fact that any two sets of the same cardinality will be comparable if and only if they are identical.

Theorem 8.2.2. *For all positive integers c, k, n with $n = ck$,*

1. $N_n^*(\exists,\forall,k) = N_n^*(\forall,\exists,k) = N_n^*(\exists,\exists,k) = U(n,k)$,
2. $C_n^*(\forall,\exists,k) = C_n^*(\exists,\forall,k) = C_n^*(\forall,\forall,k) = 1$,
3. $N_n^*(\forall,\forall,k) = \binom{n-1}{c-1}$,
4. $C_n^*(\exists,\exists,k) = U(n-c,k-1)$.

Proof of case (1): Let $P, Q \in \mathcal{U}_k^n$. Since P and Q are uniform, any classes $P_i \in P$ and $Q_j \in Q$ will be incomparable if and only if they are not equal. This means any two distinct partitions are incomparable of type $(\exists,\forall)$. Thus $N_n^*(\exists,\forall,k) = U(n,k)$ and by Inequality (28) the other equations hold.

Proof of case (2): Two partitions from $\mathcal{U}_k^n$ will be comparable with type $(\forall, \exists)$ if and only if they are equal. Thus $C_n^*(\forall, \exists, k) = 1$ and the other equations follow from Inequality (28).

Proof of cases (3) and (4): In any $\mathcal{N}_n^*(\forall, \forall, k)$, each c-set of the n-set can be a class in at most one partition in the system. Since each partition in the system has exactly k c-sets, this gives the bound

$$N_n^*(\forall, \forall, k) \leq \frac{1}{k}\binom{n}{c}.$$

Moreover, a 1-factorization of the complete hypergraph $K_n^{(c)}$ corresponds to a partition system that is incomparable of type $(\forall, \forall)$. From Theorem 3.1.7, since $n = ck$, there is a 1-factorization of the complete hypergraph $K_n^{(c)}$, so $N_n^*(\forall, \forall, k) = \frac{1}{k}\binom{n}{c}$.

From the last equation and Proposition 8.2.1, we have

$$C_n^*(\exists, \exists, k) \leq \frac{U(n,k)}{\frac{1}{k}\binom{n}{c}} = U(n-c, k-1).$$

It is possible to construct a $C_n^*(\exists, \exists, k)$ of cardinality $U(n-c, k-1)$. Simply fix any c-set of the n-set, then the system of all partitions from $\mathcal{U}_k^n$ that have the given c-set as a class (from Section 7.2 this partition system is called a trivially 1-intersecting uniform partition system) is a comparable system of type $(\exists, \exists)$ with cardinality $U(n-c, k-1)$. Thus, $C_n^*(\exists, \exists, k) = U(n-c, k-1)$. ✰

The result that $C_n^*(\exists, \exists, k) = U(n-c, k-1)$ is not new; it is a weaker version of Theorem 7.2.3. Theorem 7.2.3 further states that the only system that meets this bound is a trivially 1-intersecting uniform partition system. It is interesting that it is so easy to get the bound in Theorem 7.2.3 using the graph $G_{\mathcal{C}_n^*(\exists,\exists,k)}$.

Godsil and Newman [33] (see Section 6.4.1) use graph properties of $QI(9,3)$ to show that every maximum independent set in $QI(9,3)$ is a trivially partially 2-intersecting uniform partition system. Would it be possible to completely prove Theorem 7.2.3 using properties of the graph $G_{\mathcal{C}_n^*(\exists,\exists,k)}$? In particular, could we prove that every maximum clique in $G_{\mathcal{C}_n^*(\exists,\exists,k)}$ (or equivalently, every maximum independent set in $G_{\mathcal{N}_n^*(\forall,\forall,k)}$) is a trivially 1-intersecting uniform partition system using graph properties of either $G_{\mathcal{C}_n^*(\exists,\exists,k)}$ or $G_{\mathcal{N}_n^*(\forall,\forall,k)}$?

8.2.4 Disjoint and Intersecting

The binary relations disjoint and intersecting are more complicated than comparable and incomparable. We start by stating the trivial cases without proof.

Theorem 8.2.3. *For positive integers c, k, n with $n = ck$,*

1. $D_n^*(\exists, \forall, k) = 1$;
2. *if* $n < k^2$, *then* $D_n^*(\forall, \exists, k) = U(n, k)$;
3. $I_n^*(\forall, \exists, k) = U(n, k)$;
4. *if* $n < k^2$, *then* $I_n^*(\exists, \forall, k) = 1$;
5. $D_n^*(\forall, \forall, k) = 1$ *and* $I_n^*(\exists, \exists, k) = U(n, k)$;
6. *if* $n < k^2$, *then* $D_n^*(\exists, \exists, k) = U(n, k)$ *and* $I_n^*(\forall, \forall, k) = 1$.

Bounds for $D_n^*(\forall, \exists, k)$ for $k^2 \leq n$

A trivially 1-intersecting uniform partition system is a $\mathcal{D}_n^*(\forall, \exists, k)$ with cardinality $U(n - c, k - 1)$. Thus

$$D_n^*(\forall, \exists, k) \geq U(n - c, k - 1).$$

Conjecture 8.2.4. For positive integers c, k, n with $n = ck$ and $k \leq c$,

$$D_n^*(\forall, \exists, k) = U(n - c, k - 1).$$

Bounds for $I_n^*(\exists, \forall, k)$ for $k^2 \leq n$

In any $\mathcal{I}_n^*(\exists, \forall, k)$ the partitions in the system must be disjoint; as such, no class can be repeated. Thus,

$$I_n^*(\exists, \forall, k) \leq \binom{n-1}{c-1}.$$

Conjecture 8.2.5. For positive integers c, k, n with $n = ck$ and $k \leq c$,

$$I_n^*(\exists, \forall, k) = \binom{n-1}{c-1}.$$

Ahlswede, Cai and Zhang [3] conjecture that the largest system of c-uniform clouds which is intersecting with type $(\exists, \forall)$ has cardinality $\binom{n-1}{c-1}$. In [1] this conjecture is shown to be false; they give a system of clouds that is larger than this bound for all $c \geq 8$. The counter example given in [1] requires that each cloud in the system contain intersecting sets.

Bounds for $D_n^*(\exists, \exists, k)$ and $I_n^*(\forall, \forall, k)$ where $k \leq c$

The next pair to consider is $I_n^*(\forall, \forall, k)$ and $D_n^*(\exists, \exists, k)$. Two partitions are intersecting of type $(\forall, \forall)$ if and only if they are qualitatively independent. Thus, the graph $G_{\mathcal{I}_n^*(\forall,\forall,k)}$ is exactly the graph $UQI(n, k)$. The value of $I_n^*(\forall, \forall, k)$ is the size of the maximum clique in the uniform qualitative independence graph. This is not an easy question; perhaps, trying to find the value of $D_n^*(\exists, \exists, k)$ might be easier or at least provide a new approach to tackle this question.

From the proof of Theorem 6.2.1, with $n = ck$, we have the following two bounds:

$$\begin{aligned} D_n^*(\exists, \exists, k) &\geq \binom{n-c+k-2}{k-2} U(n-c, k-1), \\ I_n^*(\forall, \forall, k) &\leq \frac{\frac{1}{k!}\binom{n}{c}}{\frac{1}{(k-1)!}\binom{n-(c-(k-2))}{k-2}} = \frac{\binom{n-1}{c-1}}{\binom{n-(c-k+2)}{k-2}}. \end{aligned}$$

For $k = 2$, from Theorem 3.4.2, $I_n^*(\forall, \forall, 2) = \binom{n-1}{\lfloor n/2 \rfloor - 1}$ and $D_n^*(\exists, \exists, 2) = 1$.

In Section 2.4.1, for k a prime power, the block-size recursive construction can be used to construct a uniform covering array $CA((i+1)k^2 - ik, k^i(k+1), k)$. So, for k a prime power,

$$\frac{\log_2 k}{k(k-1)} \leq \limsup_{n \to \infty} \frac{\log_2(I_n^*(\forall, \forall, k))}{n}.$$

It is not clear if this is the exact asymptotic growth of $I_n^*(\forall, \forall, k)$. From Equation (4), Section 2.4.2, we also have the following upper bound from a cardinality of sets of qualitatively independent partitions (not necessarily uniform)

$$\limsup_{n \to \infty} \frac{\log_2(I_n^*(\forall, \forall, k))}{n} \leq \frac{2}{k}.$$

The exact value of $\limsup_{n\to\infty} \frac{\log_2(I_n^*(\forall,\forall,k))}{n}$ is an open question. If it is strictly less then $\frac{2}{k}$ then the size of a maximum clique in $UQI(n,k)$ would be strictly smaller than the size of a maximum clique in $QI(n,k)$. In particular $UQI(n,k)$ would not be a core of $QI(n,k)$. This gives us a new way to consider Question 6.2.6. In particular, a negative answer to the question below would yield a negative answer to Question 6.2.6.

Question 8.2.6. Is it true that

$$\limsup_{n\to\infty} \frac{\log_2(I_n^*(\forall,\forall,k))}{n} = \frac{2}{k} \quad ?$$

8.3 Non-Uniform Partitions

Throughout this section, n,k and c will be positive integers with $n = ck + r$ for some integer r with $0 \le r < k$. We will consider all partition systems from $\mathcal{P}_n^k$ and not just the uniform ones. We summarize our results in the table below. The entry "??" denotes that we have no bounds and "asymptotics" indicates that we know the asymptotic growth of $\limsup_{n\to\infty} \frac{\log_2(A_n(X,Y,k))}{n}$.

$A \backslash (X,Y)$	$(\forall,\forall)$	$(\exists,\forall)$	$(\forall,\exists)$	$(\exists,\exists)$
Incomparable	$\binom{n-1}{c-1}$ if $n = ck$ asymptotics	??	$S(n,k)$	$S(n,k)$
Comparable	1	1	??	$\geq S(n-1,k-1)$
Disjoint	1	1	$\geq S(n-1,k-1)$	??
Intersecting	$\frac{1}{2}\binom{n}{\lfloor n/2 \rfloor}$ for $k=2$ asymptotics	??	$S(n,k)$	$S(n,k)$

We do not have bounds analogous to the ones in Proposition 8.2.1 for the non-uniform case. For the symmetric relations, $(\forall,\forall)$ and $(\exists,\exists)$, graphs $G_{\mathcal{A}_n(X,X,k)}$ can be defined as before, but since they are not vertex transitive, we do not get bounds from the converse relations.

8.3.1 Incomparable

It is trivial to see that $N_n(\forall, \exists, k) = S(n, k)$ and $N_n(\exists, \exists, k) = S(n, k)$.

Two partitions are incomparable of type $(\forall, \forall)$ if and only if they have the Sperner property for partitions (Section 7.1). Thus, Theorem 7.1.5 can be restated as follows.

Theorem 8.3.1. *If k divides n, then $N_n(\forall, \forall, k) = \binom{n-1}{c-1}$.*

From Theorem 7.1.6 we have a bound for general $N_n(\forall, \forall, k)$.

Theorem 8.3.2. *If $n = ck + r$ with $0 \leq r < k$, then $N_n(\forall, \forall, k) \leq \frac{1}{(k-r)+\frac{r(c+1)}{n-c}}\binom{n}{c}$.*

Finally, from Theorem 7.1.11 the asymptotic growth of $N_n(\forall, \forall, k)$ is:

$$\limsup_{n\to\infty} \frac{\log N_n(\forall, \forall, k)}{n} = \log\left(\frac{k}{k-1}\right) + \frac{\log(k-1)}{k}.$$

8.3.2 Comparable

For the relation comparable we have only the trivial results:

Theorem 8.3.3. *For positive integers k, n,*

1. $C_n(\forall, \forall, k) = 1$,
2. $C_n(\exists, \forall, k) = 1$,
3. $C_n(\exists, \exists, k) \geq S(n-1, k-1)$.

8.3.3 Disjoint

We start by stating the trivial results for disjoint partition systems without proof.

Theorem 8.3.4. *For positive integers k, n,*

1. $D_n(\forall, \forall, k) = 1$,
2. $D_n(\exists, \forall, k) = 1$,
3. *if $n < k^2$ then* $D_n(\exists, \exists, k) = S(n, k)$.

The system of all partitions that contain the set $\{1\}$ as a class is a disjoint system of type $(\forall, \exists)$. Thus,

$$D_n(\forall, \exists, k) \geq S(n-1, k-1).$$

Finally, for $k^2 \leq n$ the system of all partitions from $\mathcal{P}_k^n$ with at least one class of size strictly less than k is a system of disjoint type $(\exists, \exists)$. It is not clear how to count the number of partitions in such a system.

8.3.4 Intersecting

It is straightforward that $I_n(\forall, \exists, k) = S(n, k)$ and $I_n(\exists, \exists, k) = S(n, k)$.

Recall from Section 2.4, that for positive integers n and k, $N(n, k)$ is the largest r for which a $CA(n, r, k)$ exists. So, by definition, $I_n(\forall, \forall, k) = N(n, k)$. Finding an exact solution for $I_n(\forall, \forall, k)$ is very hard, since it is equivalent to finding the maximum set of qualitatively independent partitions, which is also equivalent to determining the minimum size of covering arrays.

We know the exact result for $k = 2$, from Theorem 3.4.2, $I_n(\forall, \forall, 2) = \binom{n-1}{\lfloor n/2 \rfloor - 1}$. Poljak and Tuza [61] prove (using Bollabás's Theorem) that $N(n, k) \leq \frac{1}{2}\binom{\lfloor 2n/k \rfloor}{\lfloor n/k \rfloor}$ (see Section 3.2.1). Thus,

$$I_n(\forall, \forall, k) \leq \frac{1}{2}\binom{\lfloor 2n/k \rfloor}{\lfloor n/k \rfloor}.$$

For general k, Gargano, Körner and Vaccaro give an asymptotic result (Equation (4), Section 2.4.2)

$$\limsup_{n \to \infty} \frac{log_2(I_n(\forall, \forall, k))}{n} = \frac{2}{k}.$$

Finally, there are many more constructions (see Section 2.4.1) and results from heuristic searches that give bounds on $N(n, k)$, and hence $I_n(\forall, \forall, k)$, for specific values of n and k [21, 49, 65, 70, 71].

Chapter 9

Conclusion

In this thesis, we introduce the qualitative independence graphs and the uniform and almost-uniform qualitative independence graphs. We also introduce extremal partition theory, in particular we prove versions of Sperner's Theorem and the Erdős-Ko-Rado Theorem for partitions. These are all interrelated and have applications to both covering arrays and covering arrays on graphs.

The qualitative independence graphs are particularly useful for studying covering arrays. For a graph G and integers n, k, a $CA(n, G, k)$ exists if and only if there is a homomorphism from G to $QI(n, k)$. In addition, an r-clique in the qualitative independence graph $QI(n, k)$ corresponds to a covering array $CA(n, r, k)$ and an independent set in $QI(n, k)$ is related to a partially intersecting partition system in $\mathcal{P}_k^n$.

We give several results on binary covering arrays on graphs. In particular, we prove that a core of $QI(n, 2)$ is $UQI(n, 2)$ if n is even, and $AUQI(n, 2)$ if n is odd. This raises the question: does there exist a homomorphism from $QI(n, k)$ to $UQI(n, k)$ or to $AUQI(n, k)$ for all integers n and k? If there is such a homomorphism, bounds on the size of the maximum clique in $UQI(n, k)$ and $AUQI(n, k)$ would provide many new bounds on the size of covering arrays. Moreover, we would know that there exists an optimal covering array whose rows correspond to uniform or almost-uniform partitions rather than to general partitions. Since we know that the asymptotic growth of the maximum cardinality of a set of qualitatively independent k-partitions is $\frac{2}{k}$ (see Section 3.2.1), it is interesting to consider the same asymptotic growth for a set of qualitatively independent uniform k-partitions. In particular, if the limit $\limsup_{n\to\infty} \frac{\log_2(I_n^*(\forall,\forall,k))}{n}$ is strictly smaller than $\frac{2}{k}$, then we would know that there is no homomorphism from $QI(n,k)$ to $UQI(n,k)$.

The uniform qualitative independence graphs $UQI(ck,k)$, where c,k are positive integers, are particularly interesting. We exhibit an equitable partition on the vertices of $UQI(ck,k)$ that can be used to find the eigenvalues of these graphs; we give the spectra for many small uniform qualitative independence graphs. Mathon and Rosa [50] give an association scheme that has $QI(9,3)$ as one of its graphs. In Section 6.5, we describe sets of graphs that are an extension of this scheme. It seems likely that these sets of graphs form either an association scheme or some generalization of an association scheme (perhaps an asymmetric association scheme). This is very interesting, since association schemes are difficult to construct and can give more information about their graphs. These results open a new and exciting direction for covering array research.

Sperner's Theorem for set systems is essential in our proof that a core of $QI(n,2)$ is $UQI(n,2)$ if n is even, and $AUQI(n,2)$ if n is odd. Motivated by this, we give an extension of Sperner's Theorem to partition systems. We prove that, for integers n and k where k divides n, the largest Sperner partition system in $\mathcal{P}_k^n$ is a uniform partition system. We also conjecture, for all values of n and k, that the largest Sperner partition system in $\mathcal{P}_k^n$ is an almost-uniform partition system. Unlike the case for $k = 2$, this extension of Sperner's Theorem to partition systems is not enough to prove the existence of a homomorphism from $QI(n,k)$ to $UQI(n,k)$. It would be interesting to know if the largest qualitatively independent partition system is a uniform (or an almost-uniform) partition system. This would be true if there is a homomorphism from $QI(n,k)$ to $UQI(n,k)$ (or to $AUQI(n,k)$).

There are two ways to extend the Erdős-Ko-Rado Theorem to partition systems. For the first extension, we define two partitions to be intersecting if they have a class in common. We prove that the largest uniform partition system with this type of intersection is a trivially intersecting system. For the second extension, we consider a different type of intersection called partial intersection. This type of intersection is related to independent sets in qualitative independence graphs. We conjecture that the largest partially t-intersecting partition system is a trivially partially t-intersecting partition system. We can prove this conjecture for partially t-intersecting partition systems from $\mathcal{U}_k^n$ for many specific values of t, n and k. For example, a partially 2-intersecting partition system in $\mathcal{U}_k^{k^2}$ is equivalent to an independent set in the graph $QI(k^2, k)$. Using the eigenvalues of $QI(k^2, k)$, we prove that a trivially partially 2-intersecting partition system is the largest partially 2-intersecting partition system in $\mathcal{U}_k^{k^2}$.

There are many open problems related to the extension of Sperner's Theorem and the Erdős-Ko-Rado Theorem to partition systems. First, it would be nice to find a tighter connection between Sperner's Theorem for partition systems, the Erdős-Ko-Rado Theorem for partition systems and covering arrays (see Question 9.1.14). Second, it would be interesting to know the exact bound for n in the Erdős-Ko-Rado Theorem for partition systems (see Theorem 7.2.4) and to find an extension of the complete Erdős-Ko-Rado Theorem for partition systems (see Conjecture 7.4.2). Wilson's proof [76] of the exact bound for n in the Erdős-Ko-Rado Theorem used the ratio bound, Lemma 4.2.10, with the eigenvalues of the Johnson scheme (Example 4.2.9, Section 4.2.3). Is it possible to prove a complete Erdős-Ko-Rado Theorem for intersecting partition systems (or partially intersecting partitions) in a similar manner? Finally, it would be better to have a version of the Erdős-Ko-Rado Theorem for all partially intersecting partition systems, rather than the collection of specific cases (see Corollaries 7.5.6, 7.5.7, and 7.5.8).

These extensions of Sperner's Theorem and the Erdős-Ko-Rado Theorem to partition systems are a starting point for the study of extremal partition theory. Throughout the thesis, we give motivation for why these problems are interesting and in Chapter 8, we present a framework in which to consider a variety of extremal problems for partition systems. For several of these problems, we give exact results and for others, we give a bound on the cardinality of the maximum system. There are still many problems to solve and other possible extensions to be considered. We conclude with a list of questions followed by several conjectures.

9.1 Questions

Question 9.1.1. In Section 2.4.1, the group construction for covering arrays is described. This construction builds a $CA(r(k-1)+1, r, k)$ for many values of k and n, and for $k+1 \leq r \leq 2k$ this construction often gives a good upper bound for $CAN(r, k)$. Can this construction be extended or generalized to give a good upper bound for $CAN(r, k)$ for $r \geq 2k$? This construction uses a "starter vector" which is currently found either by an exhaustive search or a heuristic search. Is it possible to construct the starter vectors directly?

Question 9.1.2. Gargano, Körner and Vaccaro [30] give the asymptotic growth of the maximum size of a set of qualitative independent partitions. Is it possible to find the asymptotic growth of the maximum size of a set of qualitative independent uniform partitions? In particular, if $N'(n, k)$ denotes the largest integer r such that a balanced $CA(n, r, k)$ exists, then what is the value of

$$\limsup_{n\to\infty} \frac{\log_2 N'(n,k)}{n}?$$

Question 9.1.3. Let n and k be positive integers. What are the cores of qualitative independence graphs $QI(n, k)$ for $k > 2$? A more daring version of this question is: for positive integers c and k, is the graph $UQI(ck, k)$ a core of $QI(ck, k)$? And, for positive integers n and k, is the graph $AUQI(n, k)$ a core of $QI(n, k)$?

Question 9.1.4. What are $\alpha(QI(n, k))$, $\chi(QI(n, k))$ and $\chi^*(QI(n, k))$ for $k > 2$?

Question 9.1.5. The natural extension of strength-2 covering arrays on graphs to higher strengths are strength-t covering arrays on t-uniform hypergraphs. Many of the basic results from Sections 5.1 and 5.2 can be extended to strength-t covering arrays on t-uniform hypergraphs. What can be proven for higher strength covering arrays on hypergraphs with a binary alphabet?

Question 9.1.6. We can define a graph, similar to the qualitative independence graphs, that correspond to orthogonal arrays. Let n, k, λ be positive integers with $n = \lambda k^2$. Define a graph $O(n, k)$ to have vertex set $\mathcal{U}_k^n$. Vertices $P, Q \in \mathcal{U}_k^n$ are adjacent in $O(n,k)$ if and only if for all $P_i \in P$ and $Q_j \in Q$, where $i,j \in \{1,\dots,k\}$, $P_i \cap Q_j = \lambda$. Can we find results for the graph $O(n,k)$ similar to the ones for the graph $UQI(n,k)$ given in Chapter 6? Moreover, can we use the graph $O(n,k)$ to find bounds on the size of r in $OA(n,r,k,2)$?

Question 9.1.7. In Example 4.2.6, Section 4.2.2, the eigenvalues of the Kneser graphs are given. These eigenvalues are found by partitioning the vertices of the Kneser graph into an equitable partition with a singleton class. The adjacency matrix for the quotient graph of the Kneser graph, with respect to this equitable partition, has the same eigenvalues as the Kneser graph. In Section 6.4.3, an equitable partition on the vertices of $UQI(n,k)$ with a singleton class is given. This partition is used to reduce the calculations to find the eigenvalues of some uniform qualitative independence graphs. Can this equitable partition be used to find a formula for all the eigenvalues for all the uniform qualitative independence graphs?

Question 9.1.8. What are the eigenvalues of the graph $QI(k^2+1,k)$? What bound on $\omega(QI(k^2+1,k))$ do we get from the ratio bound for maximum cliques?

Question 9.1.9. Can the equitable partition from Section 6.4.3 be generalized to the vertices of the graphs $AUQI(n,k)$ for any positive integers n and k? What are the eigenvalues for $AUQI(n,k)$?

Question 9.1.10. Do the graphs in Section 6.5.1 and 6.5.2 describe an association scheme for $\mathcal{U}_3^n$ where $n = 12, 15$? Do the graphs given in Section 6.5 describe an asymmetric association scheme on $\mathcal{U}_k^n$ for all n and k?

Question 9.1.11. In the proof of Theorem 7.5.5, for positive integers n,k, where k divides n, a graph $PIP(n,k)$ is defined. The vertex set of this graph is $\mathcal{U}_k^n$ and two partitions are adjacent if and only if they are partially 2-intersecting. Similarly, define a (vertex-transitive) graph $IP(n,k)$ with vertex set all uniform k-partitions of an n-set and two partitions are adjacent if and only if they are intersecting. The graph $IP(n,k)$ is also the graph $G_{\mathcal{I}_n^*(\forall,\forall,k)}$ described in Section 8.2.4. A clique in $IP(n,k)$ is an intersecting uniform k-partition system. The bound from Theorem 7.2.3 can be found with the formula $\omega(IP(n,k)) \leq \frac{|V(IP(n,k))|}{\alpha(IP(n,k))}$ and the fact that $\alpha(IP(n,k)) = \binom{n-1}{\frac{n}{k}-1}$.

Theorem 7.2.3 also states that only trivially intersecting partition systems meet this bound. Is there a way to use the graph $IP(n,k)$ to prove this fact?

Question 9.1.12. Let n, k be positive integers. For $i = 0, 1, \ldots, k$, define a graph G_i on vertex set $\mathcal{U}_k^n$ with partitions $P, Q \in \mathcal{U}_k^n$ adjacent if and only if P and Q have exactly i classes in common. What are the eigenvalues of this graph?

Question 9.1.13. Why is the ratio bound for independent sets (Lemma 4.2.10) tight for the graph $QI(k^2, k)$ for all positive integers k? What other types of designs can be converted into independent sets or cliques in graphs? For what type of design is such a graph regular and when is the ratio bound for independent sets tight for such graphs?

Question 9.1.14. If a partition system is qualitatively independent, then it is a Sperner partition system. Does this relation have a converse? For example, if a 2-partition system is both an intersecting and a Sperner partition system, then it is qualitatively independent. What extra conditions would a general Sperner partition system need to meet to also be a qualitatively independent partition system?

Question 9.1.15. Are there other applications of Sperner partition systems?

Question 9.1.16. In Theorem 7.2.4, we prove that a t-intersecting uniform partition system has cardinality no more than $U(n-tc, k-t)$ for n sufficiently large. What is the exact lower bound for n? In particular, can we set up a graph whose vertex set is $\mathcal{U}_k^n$ and vertices are adjacent if and only if the partitions are t-intersecting, and use eigenvalues of this graph to find the exact lower bound?

Question 9.1.17. Recall the graphs $G_{\mathcal{A}_n^*(X,X,k)}$ (where $A \in \{N, C, D, I\}$ and X can be either $\forall$ or $\exists$) defined in Section 8.2.1. Can we use an equitable partition, similar to the one defined in Section 6.4.3, to find the eigenvalues of these graphs?

9.2 Conjectures

Conjecture 9.2.1. For all $n \geq k^2$, $CAN(QI(n,k),k) = n$. This is equivalent to there being no homomorphism

$$QI(n,k) \to QI(n-1,k).$$

This was given in Conjecture 5.2.6 and Conjecture 5.2.11.

Conjecture 9.2.2. From Theorem 5.4.2, we have for all integers k, $\chi(QI(k^2,k)) \leq \binom{k+1}{2}$; we conjecture that for all integers k, $\chi(QI(k^2,k)) = \binom{k+1}{2}$.

Conjecture 9.2.3. Let k be a positive integer. Consider the graphs $QI(k^2,k)$. For all distinct $i,j \in \{1,\ldots,k^2\}$, let $S_{\{i,j\}}$ denote the set of all partitions in $V(QI(k^2,k))$ with i and j in the same class. Then every maximum independent set in $QI(k^2,k)$ is a set $S_{\{i,j\}}$ for some distinct $i,j \in \{1,\ldots,k^2\}$.

In addition, for all n, the set of characteristic vectors of $S_{\{i,j\}}$ in $UQI(n,k)$ for all distinct $i,j \in \{1,\ldots,n\}$ spans a vector subspace of $\mathbb{R}^{U(n,k)}$ with dimension $\binom{n}{2} - \binom{n}{1}$. Note that in [58] it is shown that this is true for $n = k^2$.

Conjecture 9.2.4. For all k, the graph $QI(k^2,k)$ is a core. We further conjecture that this can be proved using Conjecture 9.2.3 above.

Conjecture 9.2.5. Let n,k,c,r be positive integers with $n = ck+r$ and $0 < r < k$. The largest Sperner partition system in $\mathcal{P}_k^n$ is an almost-uniform Sperner partition system.

Conjecture 9.2.6. For positive integers n,k,c,t with $n = ck$ and $t \leq c$, let $\mathcal{P}$ be a partially t-intersecting uniform k-partition system on an n-set. Then

$$|\mathcal{P}| \leq \binom{n-t}{c-t} \frac{1}{(k-1)!} \binom{n-c}{c} \binom{n-2c}{c} \cdots \binom{c}{c}.$$

Moreover, equality holds if and only if $\mathcal{P}$ is a trivially partially t-intersecting partition system.

Note that Corollaries 7.5.6, 7.5.7, 7.5.8 and 7.5.11 prove the bound in this conjecture for several specific values of t, k and n.

Appendix A

Tables of Bounds for $CAN(r, k)$

Tables 7 and 8 gives a list of starter vectors that improve the previously best known upper bounds for covering arrays.

Table 7: New upper bounds and corresponding starter vectors for $k \leq 12$.

$k=6$	starter vector	new bound	old bound
$r=9$	0 1 1 2 1 1 3 5 3	46	48
$r=10$	0 1 1 1 1 2 4 3 1 2	51	52
$k=7$			
$r=10$	0 1 1 1 3 4 1 3 2 6	61	63
$r=11$	0 1 1 1 1 2 1 4 6 5 3	67	73
$r=12$	0 1 1 1 1 1 2 1 4 6 5 3	73	76
$k=8$			
$r=11$	0 1 1 2 2 4 2 5 6 3 2	78	80
$r=12$	0 1 1 1 2 6 2 6 1 1 6 5	85	99
$r=13$	0 1 1 1 1 2 1 3 7 5 1 3 4	92	102
$r=14$	0 1 1 1 1 1 2 1 3 7 5 1 3 4	99	104
$k=9$			
$r=13$	0 1 1 1 3 2 1 6 2 5 5 3 4	105	120
$r=14$	0 1 1 1 1 2 8 5 6 2 1 3 6 7	113	131
$r=15$	0 1 1 1 1 1 2 3 2 7 1 5 4 2 5	121	135
$r=16$	0 1 1 1 1 1 1 2 3 2 7 1 5 4 2 5	129	145
$r=17$	0 1 1 1 1 1 1 1 2 1 2 6 8 5 3 6 7	137	148
$r=18$	0 1 1 1 1 1 1 1 1 2 1 2 6 8 5 3 6 7	145	151
$k=10$			
$r=15$	0 1 1 1 1 4 2 8 1 9 4 5 6 8 4	136	166
$r=16$	0 1 1 1 1 1 2 9 5 7 1 5 5 3 2 8	145	177
$r=17$	0 1 1 1 1 1 1 2 4 9 5 8 7 2 1 8 5	154	180
$r=18$	0 1 1 1 1 1 1 1 2 4 9 5 8 7 2 1 8 5	163	180
$r=19$	0 1 1 1 1 1 1 1 1 2 1 4 9 7 8 2 6 8 5	172	180
$r=21$	0 1 1 1 1 1 1 1 1 1 1 1 2 5 6 8 4 8 6 5 2	190	202
$r=22$	0 1 1 1 1 1 1 1 1 1 1 1 1 2 5 6 8 4 8 6 5 2	199	202
$k=11$			
$r=17$	0 1 1 1 1 1 3 10 6 9 4 10 9 7 1 4 5	171	221
$r=18$	0 1 1 1 1 1 1 2 5 8 3 1 3 10 6 5 8 2	181	225
$r=19$	0 1 1 1 1 1 1 1 2 5 8 3 1 3 10 6 5 8 2	191	231
$r=20$	0 1 1 1 1 1 1 1 1 2 5 8 3 1 3 10 6 5 8 2	201	231
$r=21$	0 1 1 1 1 1 1 1 1 1 2 5 8 3 1 3 10 6 5 8 2	211	231
$r=22$	0 1 1 1 1 1 1 1 1 1 1 2 5 8 3 1 3 10 6 5 8 2	221	231
$k=12$			
$r=18$	0 6 5 10 7 3 9 1 3 2 3 7 6 6 3 1 4 8	199	255
$r=19$	0 8 3 3 6 7 10 9 7 10 5 9 5 7 4 2 10 9 3	210	276
$r=20$	0 7 9 2 10 10 2 9 8 9 4 9 2 3 1 4 5 6 1 9	221	276
$r=21$	0 5 1 9 1 4 2 7 9 4 7 6 5 5 2 3 7 7 2 7 9	232	276
$r=22$	0 1 1 3 10 6 9 4 10 9 1 1 1 4 8 6 3 2 7 8 4 11	243	288
$r=23$	0 1 1 1 3 10 6 9 4 10 9 1 1 1 1 9 2 11 5 11 1 6 5	254	288
$r=24$	0 1 1 1 1 3 10 6 9 4 10 9 1 1 1 1 6 5 2 11 1 5 6 2	265	288
$r=25$	0 7 10 7 8 2 2 1 1 10 1 5 5 5 1 7 10 1 6 1 6 4 9 9 10	276	288

Table 8: New upper bounds and corresponding starter vectors for $13 \leq k \leq 18$.

$k = 13$			
$r = 21$	0 6 9 5 7 6 2 8 8 9 6 2 3 10 5 3 5 9 8 1 8	253	325
$r = 22$	0 3 9 11 4 1 8 7 3 7 1 3 3 4 1 1 4 2 6 11 6 4	265	325
$r = 23$	0 6 7 10 2 5 4 11 8 2 4 1 9 3 5 3 3 1 10 10 11 4 6	277	325
$r = 24$	0 7 3 7 7 3 10 2 8 3 5 4 2 4 5 8 9 2 11 7 8 7 3 10	289	325
$r = 25$	0 1 2 4 10 11 5 1 10 2 7 6 4 3 6 8 6 7 10 3 10 1 2 8 8	301	325
$r = 26$	0 7 4 9 2 2 3 7 10 6 6 10 10 7 2 7 9 4 5 4 2 4 11 10 4 4	313	325
$k = 14$			
$r = 23$	0 1 5 4 1 9 4 6 1 1 6 1 5 10 7 8 8 1 4 5 1 12 6	300	437
$r = 24$	0 5 4 12 9 2 7 10 10 10 2 1 6 2 8 12 9 7 8 8 1 8 10 7	313	437
$r = 25$	0 8 8 2 12 1 12 10 6 1 6 6 10 4 7 4 3 6 12 1 3 4 9 11 7	326	437
$r = 26$	0 7 2 9 10 7 6 3 11 1 10 2 6 8 3 5 10 5 5 6 12 5 3 2 6 5	339	437
$r = 27$	0 8 4 5 10 12 4 8 3 11 4 3 8 9 2 1 5 12 5 8 7 5 2 2 3 3 12	352	437
$r = 28$	0 5 3 11 6 5 5 10 3 10 6 1 4 11 9 4 12 6 8 1 8 1 10 1 11 12 4 4	365	437
$r = 29$	0 2 4 1 1 2 5 10 3 1 2 9 8 1 1 1 1 9 3 11 10 9 10 1 1 4 12 12 8	378	437
$k = 15$			
$r = 27$	0 11 8 9 11 1 7 12 4 4 11 9 4 2 6 12 10 6 7 6 6 5 1 8 11 4 12	379	450
$r = 28$	0 12 7 1 1 5 13 11 12 8 3 4 5 10 11 2 13 1 4 12 4 3 3 10 4 3 4 3	393	450
$r = 29$	0 2 13 8 12 8 1 13 1 8 11 3 8 2 9 8 13 5 7 7 13 10 11 6 11 6 5 1 12	407	450
$r = 30$	0 1 7 13 11 2 2 2 12 8 3 5 8 9 6 9 8 8 11 11 5 4 4 2 13 9 2 13 3 6	421	450
$r = 31$	0 9 8 1 7 1 5 2 5 3 1 4 3 12 12 1 2 6 1 2 1 13 1 6 1 11 7 7 10 3 10	435	450
$k = 16$			
$r = 29$	0 5 2 6 2 5 11 4 13 1 3 14 9 6 4 8 9 10 12 3 8 3 14 10 9 9 2 9 6	436	496
$r = 30$	0 11 8 9 13 7 3 8 2 10 4 14 6 3 2 12 4 8 6 4 6 13 3 6 6 12 12 4 2 1	451	496
$r = 31$	0 9 5 5 13 2 10 14 8 1 12 4 9 14 6 1 4 6 10 6 6 7 1 14 4 7 13 7 6 3 5	466	496
$r = 32$	0 7 7 3 7 5 1 7 12 3 5 10 11 11 8 10 1 13 6 2 10 11 5 12 5 8 7 13 14 11 9 4	481	496
$k = 17$			
$r = 33$	0 13 4 10 9 5 8 9 4 14 8 13 2 11 6 14 13 10 14 2 4 9 13 5 5 5 7 10 4 7 13 11 4	529	561
$r = 34$	0 8 4 5 7 10 1 7 6 4 14 6 13 1 5 10 10 11 8 5 2 15 3 6 2 13 8 11 8 3 12 2 8 12	545	561
$k = 18$			
$r = 35$	0 2 13 12 8 4 5 12 9 12 4 8 8 15 10 10 16 5 5 4 6 13 15 13 1 11 8 13 14 5 13 10 7 10 8	596	666
$r = 36$	0 3 16 12 4 5 4 1 15 15 16 11 13 6 9 16 3 2 10 8 14 1 14 8 13 15 15 14 10 4 1 10 2 6 9 4	613	666
$r = 37$	0 6 14 3 5 8 12 15 4 15 5 2 11 7 2 3 2 7 15 5 12 12 10 8 4 1 5 5 7 7 10 16 10 16 9 2 4	630	666

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Index