CIRCUIT-SAT is NP-hard

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Theorem

The circuit-satisfiability problem is NP-hard.

Proof. (Cormen, Leiserson and Rivest, Introduction to Algorithms.)
The proof uses Karp transformation. Let X be a problem in NP. We will describe a polynomial time algorithm F computing a transformation function f that maps every binary string x to a circuit $K = f(x)$ such that $x \in X$ if and only if $K \in$ CIRCUIT-SAT.

In the next slides we describe how to build $K = f(x)$, and argue that F runs in polynomial time and does the job of mapping "yes" instances of X to "yes" instances of CIRCUIT-SAT and "no" instances of X to "no" instances of CIRCUIT-SAT.

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- The output of algorithm A appears as one of the bits of $c_{T(n)}$.

Configurations and connections used to build K

(figure in Cormen, Leiserson and Rivest - you have a photocopy)

How $K = f(x)$ is built

Build K as shown in the previous picture.
The following values in c_o must be wired to their known values:

- program A.
- initial counter,
- input x,
- initial state of the memory.

The only remaining **inputs** for the circuit K are the bits of y.
Also, all outputs (values in $c_{T(n)}$) are ignored, and the only **output** of K is the bit that represents $A(x, y)$.
Algorithm F then receives x and outputs K, the circuit described above.

Lemma

Let $K = f(x)$ computed by Algorithm F. Then,
K is satisfiable if and only if $x \in X$.

Proof:
$(\Leftarrow)$
Suppose $x \in X$. Then, here exists a certificate y such that $A(x, y) = 1$. If we apply the bits of y to the inputs of K the output of the circuit will be $A(x, y) = 1$. So K is satisfiable.
$(\Rightarrow)$
Suppose K is satisfiable. Then there exists an input y to K such that $K(y) = 1$. But by construction $K(y) = A(x, y)$ and so $A(x, y) = 1$, and $x \in X$.

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Algorithm F runs in polynomial time in $n = |x|$.

Proof:

- First we claim the number of bits to represent each configuration c_i is polynomial on n:
  First, the program for A has constant size (independent on $|x|$), $|x| = n$, $|y| \in O(n^k)$. Since A runs in $O(n^k)$ steps the amount of work storage is also polynomial on n.

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- Each step of the algorithm F that builds K takes polynomial time.