Chapter 4 Continuous-Time Fourier Transform

4.0 Introduction

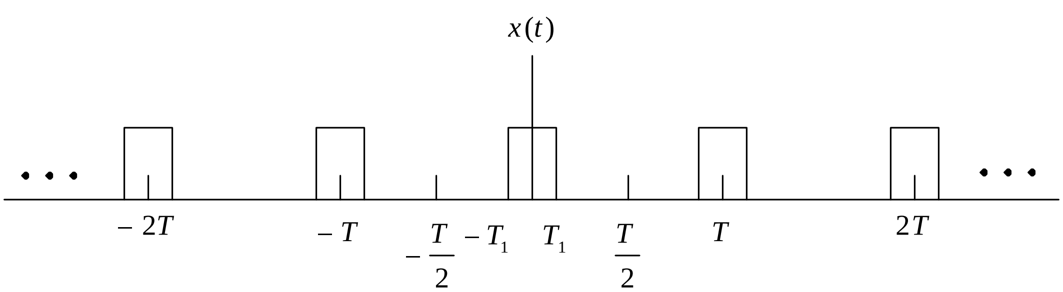
- A periodic signal can be represented as linear combination of complex exponentials which are harmonically related.
- An aperiodic signal can be represented as linear combination of complex exponentials, which are infinitesimally close in frequency. So the representation take the form of an integral rather than a sum
- In the Fourier series representation, as the period increases the fundamental frequency decreases and the harmonically related components become closer in frequency. As the period becomes infinite, the frequency components form a continuum and the Fourier series becomes an integral.

4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

Starting from the Fourier series representation for the continuous-time periodic square wave:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}, \tag{4.1}$$



The Fourier coefficients a_k for this square wave are

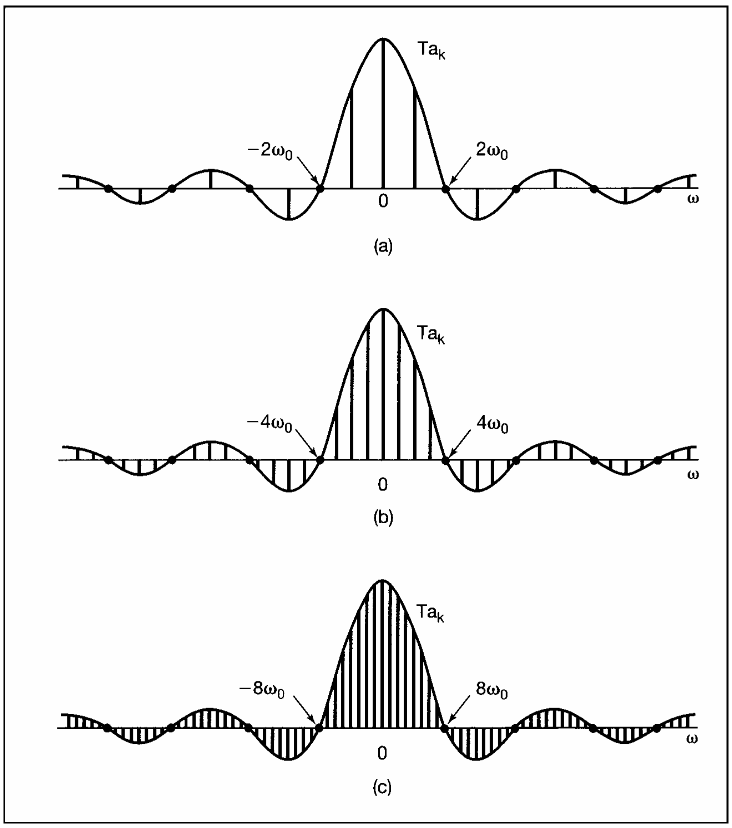
$$a_k = \frac{2\sin(kw_0T_1)}{kw_0T}. \tag{4.2}$$

or alternatively

$$Ta_k = \left.\frac{2\sin(wT_1)}{w}\right|_{w=kw_0}, \tag{4.3}$$

where $2\sin(wT_1)/w$ represent the envelope of Ta_k

- When T increases or the fundamental frequency $w_0 = 2p/T$ decreases, the envelope is sampled with a closer and closer spacing. As T becomes arbitrarily large, the original periodic square wave approaches a rectangular pulse.
- Ta_k becomes more and more closely spaced samples of the envelope, as $T \to \infty$, the Fourier series coefficients approaches the envelope function.

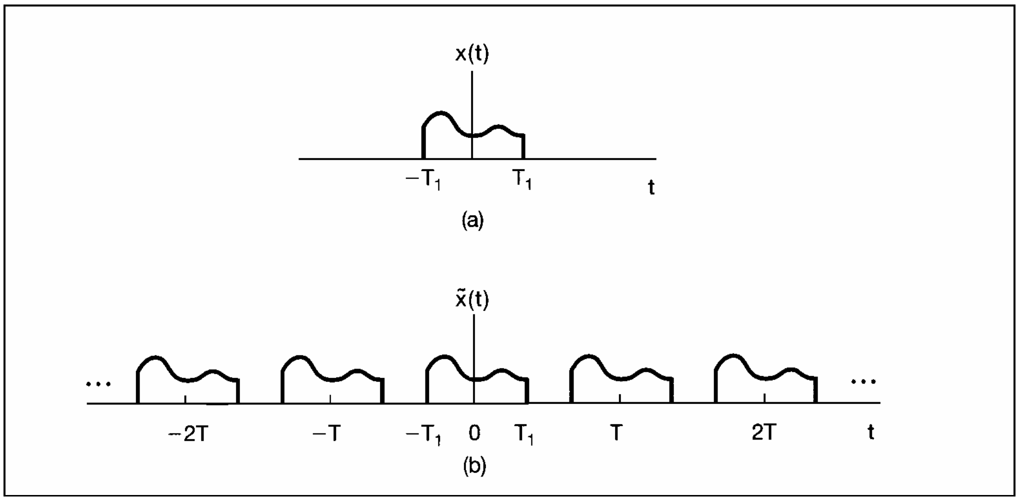


This example illustrates the basic idea behind Fourier's development of a representation for aperiodic signals.

Based on this idea, we can derive the Fourier transform for aperiodic signals.

Suppose a signal $x(t)$ with a finite duration, that is, $x(t) = 0$ for $|t| > T_1$, as illustrated in the figure below.

- From this aperiodic signal, we construct a periodic signal $\tilde{x}(t)$, shown in the figure below.



- As $T \to \infty$, $\tilde{x}(t) = x(t)$, for any infinite value of t.
- The Fourier series representation of $\tilde{x}(t)$ is

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} , \qquad (4.4)$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jkw_0 t} dt . \qquad (4.5)$$

- Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since $x(t) = 0$ outside this interval, so we have

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jkw_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jkw_0 t} dt .$$

- Define the envelope $X(jw)$ of Ta_k as

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt . \qquad (4.6)$$

we have for the coefficients a_k,

$$a_k = \frac{1}{T} X(jkw_0)$$

Then $\tilde{x}(t)$ can be expressed in terms of $X(jw)$, that is

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jkw_0) e^{jkw_0 t} = \frac{1}{2p}\sum_{k=-\infty}^{+\infty} X(jkw_0) e^{jkw_0 t} w_0 . \qquad (4.7)$$

- As $T \to \infty$, $\tilde{x}(t) = x(t)$ and consequently, Eq. (4.7) becomes a representation of $x(t)$.
- In addition, $w_0 \to 0$ as $T \to \infty$, and the right-hand side of Eq. (4.7) becomes an integral.

We have the following Fourier transform:

$$x(t) = \frac{1}{2p}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw \quad \textit{Inverse Fourier Transform} \tag{4.8}$$

and

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt \quad \textit{Fourier Transform} \tag{4.9}$$

4.1.2 Convergence of Fourier Transform

If the signal $x(t)$ has finite energy, that is, it is square integrable,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty, \tag{4.10}$$

Then we guaranteed that $X(jw)$ is finite or Eq. (4.9) converges. If $e(t) = \tilde{x}(t) - x(t)$, we have

$$\int_{-\infty}^{\infty} |e(t)|^2 dt = 0. \tag{4.11}$$

An alterative set of conditions that are sufficient to ensure the convergence:

Contition1: Over any period, $x(t)$ must be absolutely integrable, that is

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty, \tag{4.12}$$

Condition 2: In any finite interval of time, $x(t)$ have a finite number of maxima and mi nima.

Condition 3: In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

4.1.3 Examples of Continuous-Time Fourier Transform

Example: consider signal $x(t) = e^{-at}u(t)$, $a > 0$.

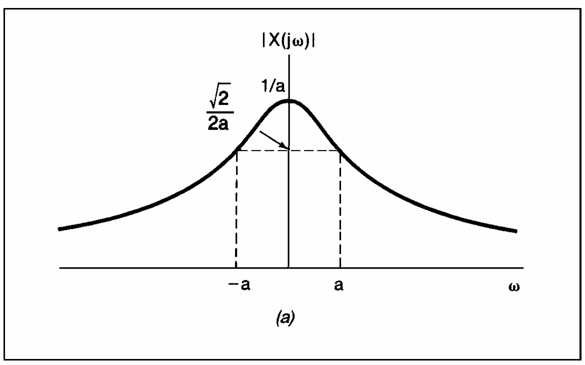
From Eq. (4.9),

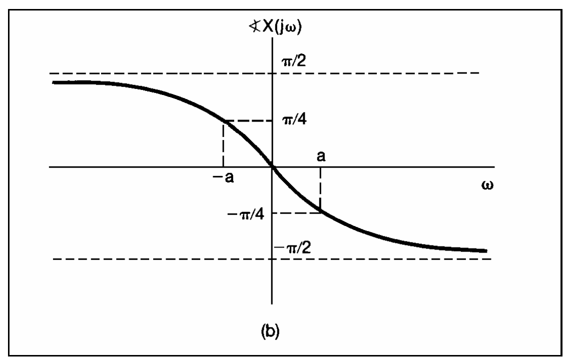
$$X(jw) = \int_0^{\infty} e^{-at} e^{-jwt} dt = -\frac{1}{a + jw} e^{-(a+jw)t} \Big|_0^{\infty} = \frac{1}{a + jw}, \qquad a > 0 \tag{4.12}$$

If a is complex rather then real, we get the same result if $\mathrm{Re}\{a\} > 0$

The Fourier transform can be plotted in terms of the magnitude and phase, as shown in the figure below.

$$|X(jw)| = \frac{1}{\sqrt{a^2 + w^2}}, \qquad \angle X(jw) = -\tan^{-1}\left(\frac{w}{a}\right). \tag{4.13}$$

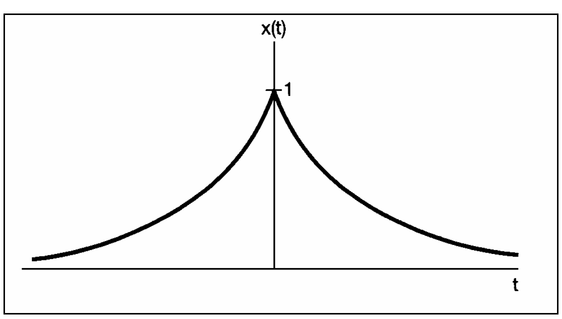


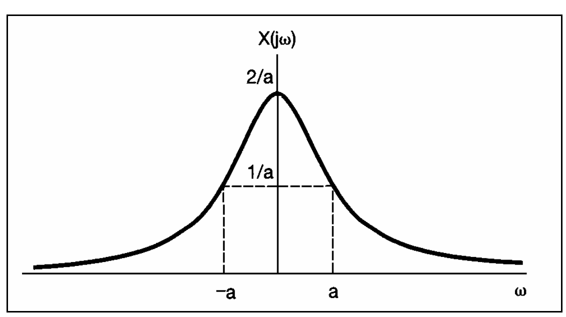


Example: Let $x(t) = e^{-a|t|}, \quad a > 0$

$$X(jw) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-jwt} dt = \int_{-\infty}^{0} e^{at} e^{-jwt} dt + \int_0^{\infty} e^{-at} e^{-jwt} dt = \frac{1}{a - jw} + \frac{1}{a + jw} = \frac{2a}{a^2 + w^2}$$

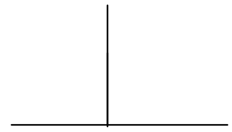
The signal and the Fourier transform are sketched in the figure below.



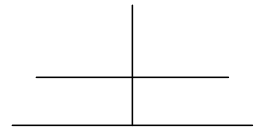


Example: $x(t) = d(t)$. (4.14)

$x(t) = d(t)$



$X(jw) = 1$

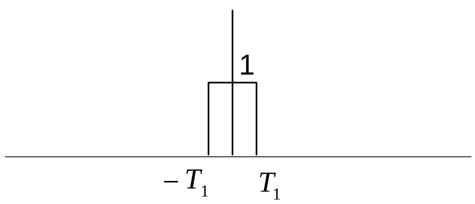


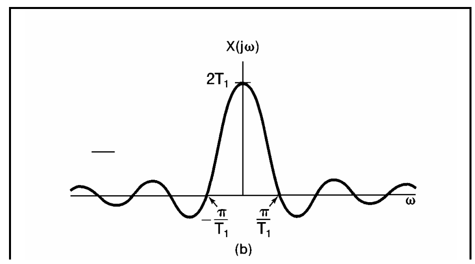
$$X(jw) = \int_{-\infty}^{\infty} d(t) e^{-jwt} dt = 1. \quad (4.15)$$

That is, the impulse has a Fourier transform consisting of equal contributions at all frequencies.

Example: Calculate the Fourier transform of the rectangular pulse signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}. \quad (4.16)$$





$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt = \int_{-T_1}^{T_1} 1 e^{-jwt} dt = 2\frac{\sin wT_1}{w}. \quad (4.17)$$

The Inverse Fourier transform is

$$\hat{x}(t) = \frac{1}{2p} \int_{-\infty}^{\infty} 2\frac{\sin wT_1}{w} e^{jwt} dw, \quad (4.18)$$

Since the signal $x(t)$ is square integrable,

$$e(t) = \int_{-\infty}^{\infty} |x(t) - \hat{x}(t)|^2 dt = 0. \quad (4.19)$$

$\hat{x}(t)$ converges to $x(t)$ everywhere except at the discontinuity, $t = \pm T_1$, where $\hat{x}(t)$ converges to ½, which is the average value of $x(t)$ on both sides of the discontinuity.

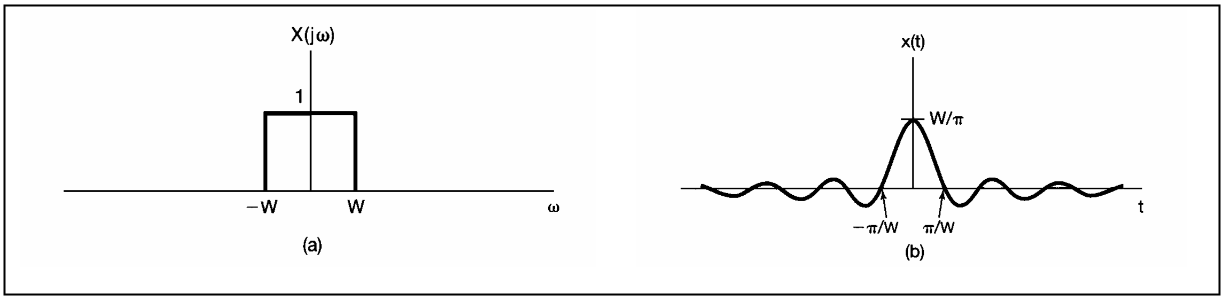
In addition, the convergence of $\hat{x}(t)$ to $x(t)$ also exhibits *Gibbs phenomenon*. Specifically, the integral over a finite-length interval of frequencies

$$\frac{1}{2p} \int_{-W}^{W} 2\frac{\sin wT_1}{w} e^{jwt} dw$$

As $W \to \infty$, this signal converges to $x(t)$ everywhere, except at the discontinuities. More over, the signal exhibits ripples near the discontinuities. The peak values of these ripples do not decrease as W increases, although the ripples do become compressed toward the discontinuity, and the energy in the ripples converges to zero.

Example: Consider the signal whose Fourier transform is

$$X(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases}.$$



The Inverse Fourier transform is

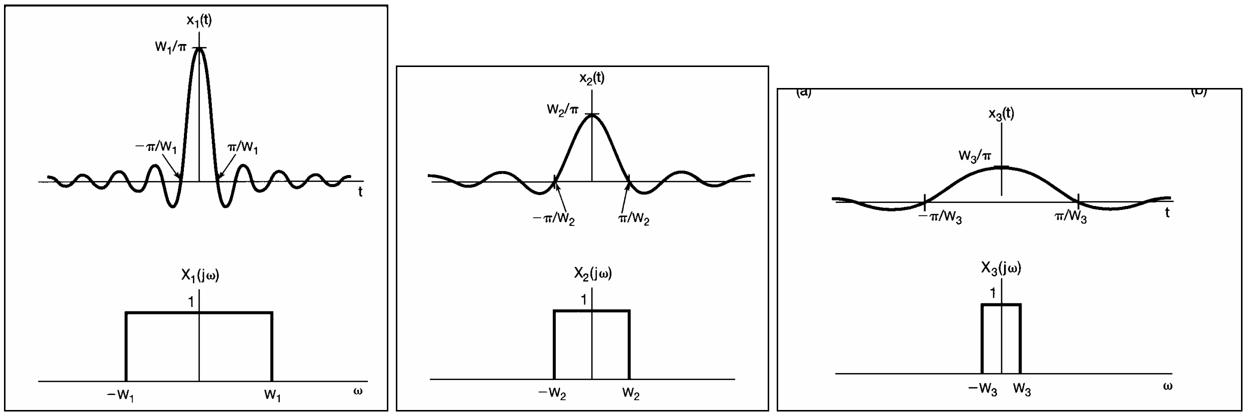
$$x(t) = \frac{1}{2p}\int_{-W}^{W} e^{jwt}\,dw = \frac{\sin Wt}{pt}.$$

Comparing the results in the preceding example and this example, we have

$$\textit{Square wave} \;\underset{\xleftarrow[FT^{-1}]{}}{\xrightarrow{FT}}\; \textit{Sinc function}$$

This means a square wave in the time domain, its Fourier transform is a *sinc* function. However, if the signal in the time domain is a *sinc* function, then its Fourier transform is a square wave. This property is referred to as ***Duality Property***.

We also note that when the width of $X(jw)$ increases, its inverse Fourier transform $x(t)$ will be compressed. When $W \to \infty$, $X(jw)$ converges to an impulse. The transform pair with several different values of W is shown in the figure below.



4.2 The Fourier Transform for Periodic Signals

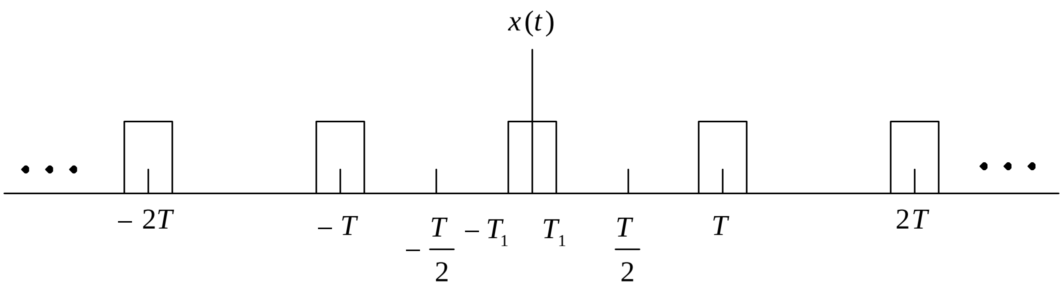
The Fourier series representation of the signal $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} . \tag{4.20}$$

It's Fourier transform is

$$X(jw) = \sum_{k=-\infty}^{\infty} 2pa_k d(w - kw_0) .$$
(4.21)

Example: If the Fourier series coefficients for the square wave below are given



$$a_k = \frac{\sin kw_0 T_1}{pk} , \tag{4.22}$$

The Fourier transform of this signal is

$$X(jw) = \sum_{k=-\infty}^{\infty} \frac{2\sin kw_0 T_1}{k} d(w - kw_0) . \tag{4.23}$$

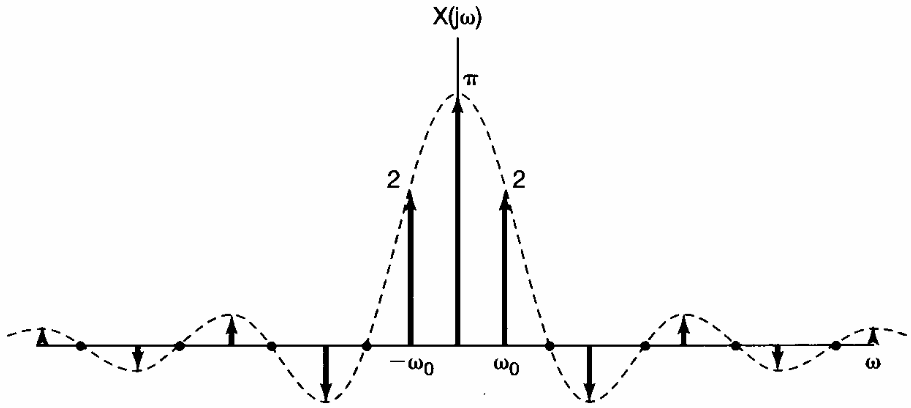
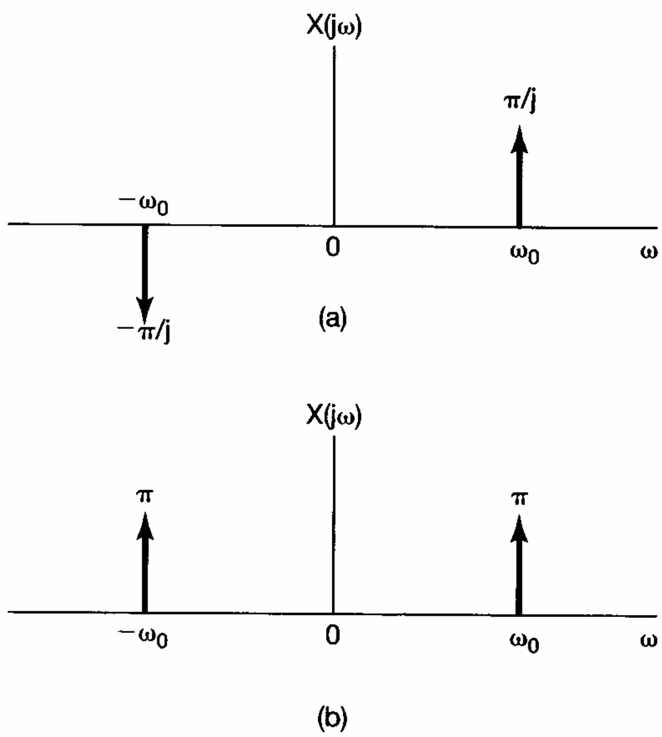


Figure 4.12 Fourier transform of a symmetric periodic square wave.

Example: The Fourier transforms for $x(t) = \sin w_0 t$ and $x(t) = \cos w_0 t$ are shown in the figure below.



Fourier transforms of (a) $x(t) = \sin \omega_0 t$; (b) $x(t) = \cos \omega_0 t$.

Example: Calculate the Fourier transform for signal $x(t)=\sum_{k=-\infty}^{\infty} d(t-kT)$.

The Fourier series of this signal is

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} d(t)e^{-jw_0 t} = \frac{1}{T}.$$

The Fourier transform is

$$X(jw)=\frac{2p}{T}\sum_{k=-\infty}^{\infty} d(w-\frac{2pk}{T}_{0}).$$

The Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2p/T$, as sketched din the figure below.

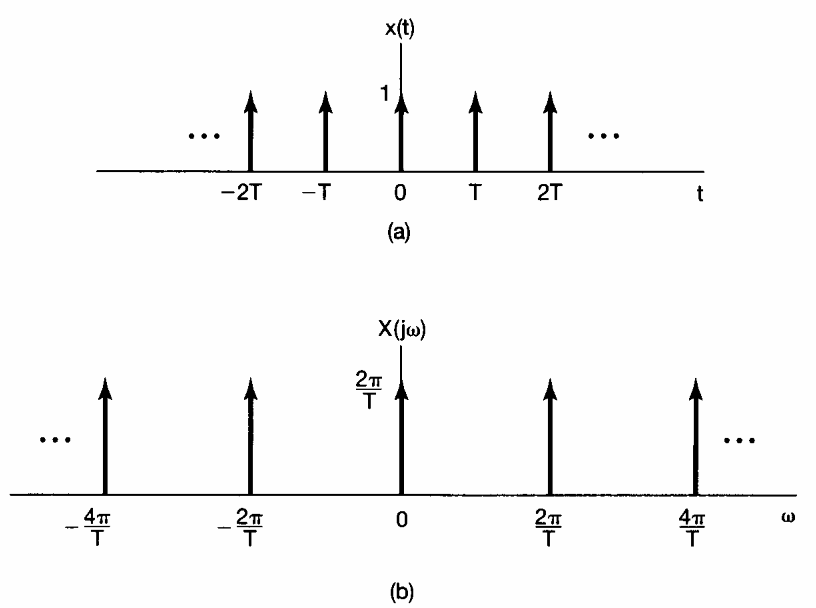


Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

4.3 Properties of The Continuous-Time Fourier Transform

4.3.1 Linearity

If $x(t) \xleftrightarrow{F} X(jw)$ and $y(t) \xleftrightarrow{F} Y(jw)$

Then

$$ax(t)+by(t)\overset{F}{\longleftrightarrow} aX(jw)+bY(jw) \,. \qquad (4.\ 20)$$

4.3.2 Time Shifting

If $x(t)\overset{F}{\longleftrightarrow} X(jw)$

Then

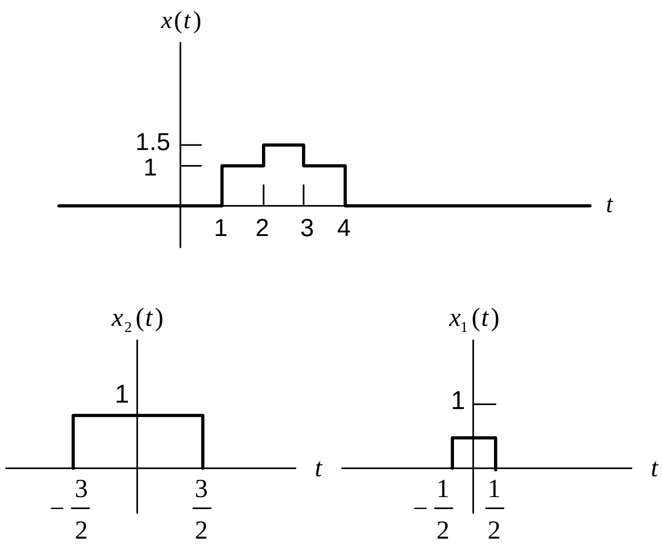
$$x(t-t_0)\overset{F}{\longleftrightarrow} e^{-jwt_0}X(jw) \,. \qquad (4.\ 20)$$

Or

$$F\{x(t-t_0)\}=e^{-jwt_0}X(jw)=\left|X(jw)\right|e^{j[\angle X(jw)-wt_0]} \,. \qquad (4.\ 20)$$

Thus, the effect of a time shift on a signal is to introduce into its transform a phase shift, namely, $-w_0 t$.

Example: To evaluate the Fourier transform of the signal $x(t)$ shown in the figure below.



The signal $x(t)$ can be expressed as the linear combination

$$x(t)=\frac{1}{2}x_1(t-2.5)+x_2(t-2.5) \,. \qquad (4.\ 20)$$

$x_1(t)$ and $x_2(t)$ are rectangular pulse signals and their Fourier transforms are

$$X_1(jw)=\frac{2\sin(w/2)}{w} \text{ and } X_2(jw)=\frac{2\sin(3w/2)}{w}$$

Using the linearity and time-shifting properties of the Fourier transform yields

$$X(jw)=e^{-j5w/2}\left\{\frac{\sin(w/2)+2\sin(3w/2)}{w}\right\}$$

4.3.3 Conjugation and Conjugate Symmetry

If $x(t)\xleftrightarrow{F}X(jw)$

Then

$$x^*(t)\xleftrightarrow{F}X^*(-jw). \qquad (4.\ 20)$$

Since $X^*(jw)=\left[\int_{-\infty}^{+\infty}x(t)e^{-jwt}dt\right]^*=\int_{-\infty}^{+\infty}x^*(t)e^{jwt}dt$,

Replacing w by $-w$, we see that

$$X^*(-jw)=\int_{-\infty}^{+\infty}x^*(t)e^{-jwt}dt, \qquad (4.\ 20)$$

The right-hand side is the Fourier transform of $x^*(t)$.

If $x(t)$ is ***real***, from Eq. (4.20) we can get

$$X(-jw)=X^*(jw). \qquad (4.\ 20)$$

We can also prove that if $x(t)$ is both real and even, then $X(jw)$ will also be real and even. Similarly, if $x(t)$ is both real and odd, then $X(jw)$ will also be purely imaginary and odd.

A real function $x(t)$ can be expressed in terms of the sum of an even function $x_e(t)=Ev\{x(t)\}$ and an odd function $x_o(t)=Od\{x(t)\}$. That is

$$x(t)=x_e(t)+x_o(t)$$

Form the Linearity property,

$F\{x(t)\} = F\{x_e(t)\} + F\{x_o(t)\}$,

From the preceding discussion, $F\{x_e(t)\}$ is real function and $F\{x_o(t)\}$ is purely imaginary. Thus we conclude with $x(t)$ real,

$x(t) \xleftrightarrow{F} X(jw)$

$Ev\{x(t)\} \xleftrightarrow{F} \mathrm{Re}\{X(jw)\}$

$Od\{x(t)\} \xleftrightarrow{F} j\,\mathrm{Im}\{X(jw)\}$

Example: Using the symmetry properties of the Fourier transform and the result $e^{-at}u(t) \xleftrightarrow{F} \dfrac{1}{a+jw}$ to evaluate the Fourier transform of the signal $x(t) = e^{-a|t|}$, where $a > 0$.

Since $x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = 2\left[\dfrac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] = 2Ev\{e^{-at}u(t)\}$,

So $X(jw) = 2\,\mathrm{Re}\left(\dfrac{1}{a+jw}\right) = \dfrac{2a}{a^2+w^2}$

4.3.4 Differentiation and Integration

If $x(t) \xleftrightarrow{F} X(jw)$

Then

$$\frac{dx(t)}{dt} \xleftrightarrow{F} jwX(jw). \qquad (4.\ 20)$$

$$\int_{-\infty}^{t} x(t)dt \xleftrightarrow{F} \frac{1}{jw}X(jw) + pX(0)d(w). \qquad (4.\ 20)$$

Example: Consider the Fourier transform of the unit step $x(t) = u(t)$.

It is know that

$g(t) = d(t) \overset{F}{\longleftrightarrow} 1$

Also note that

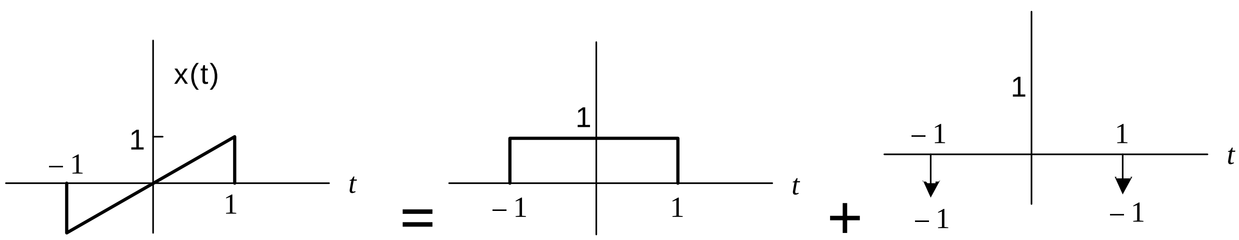
$x(t) = \int_{-\infty}^{t} g(t)\,dt$

The Fourier transform of this function is

$$X(jw) = \frac{1}{jw} + pG(0)d(w) = \frac{1}{jw} + pd(w).$$

where $G(0) = 1$.

Example: Consider the Fourier transform of the function $x(t)$ shown in the figure below.



$$g(t) = \frac{dx(t)}{dt}$$

From the above figure we can see that $g(t)$ is the sum of a rectangular pulse and two impulses.

$$G(jw) = \left(\frac{2\sin w}{w}\right) - e^{jw} - e^{-jw}$$

Note that $G(0) = 0$, using the integration property, we obtain

$$X(jw) = \frac{G(jw)}{jw} + pG(0)d(w) = \frac{2\sin w}{jw^{2}} - \frac{2\cos w}{jw}.$$

It can be found $X(jw)$ is purely imaginary and odd, which is consistent with the fact that $x(t)$ is real and odd.

4.3.5 Time and Frequency Scaling

$x(t) \xleftrightarrow{F} X(jw)$,

Then

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X(\frac{jw}{a}). \tag{4. 20}$$

From the equation we see that the signal is compressed in the time domain, the spectrum will be extended in the frequency domain. Conversely, if the signal is extended, the corresponding spectrum will be compressed.

If $a = -1$, we get from the above equation,

$$x(-t) \xleftrightarrow{F} X(-jw). \tag{4. 20}$$

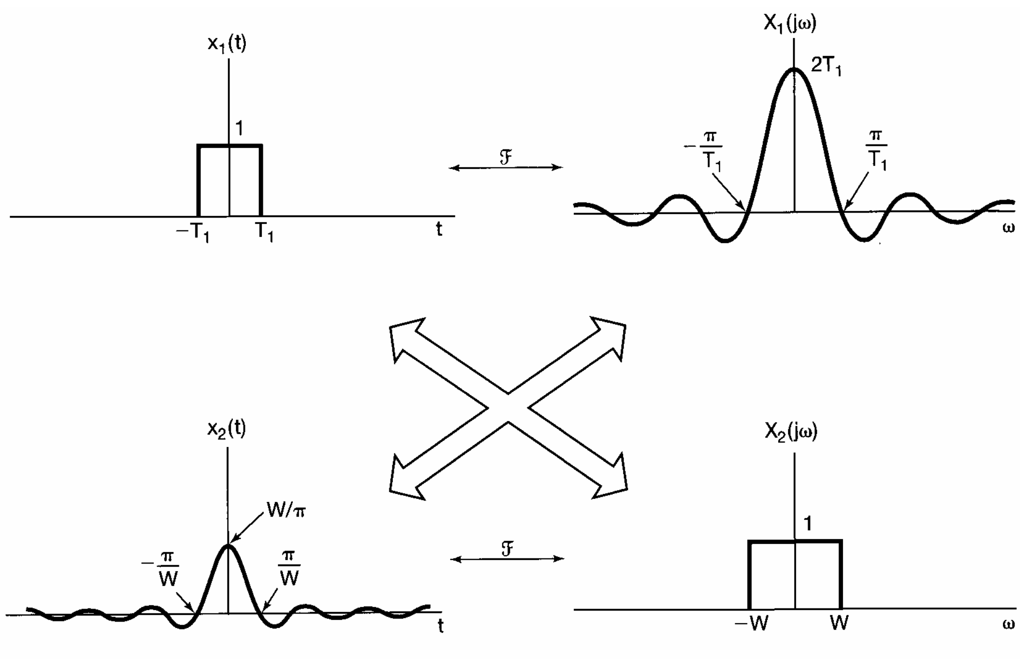
That is, reversing a signal in time also reverses its Fourier transform.

4.3.6 Duality

The *duality* of the Fourier transform can be demonstrated using the following example.

$$x_1(t) = \begin{cases} 1, & t < T_1 \\ 0, & t > T_1 \end{cases} \xleftrightarrow{F} X_1(jw) = \frac{2\sin wT_1}{w}$$

$$x_2(t) = \frac{\sin WT_1}{pt} \xleftrightarrow{F} X_2(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases}$$



The symmetry exhibited by these two examples extends to Fourier transform in general. For any transform pair, there is a dual pair with the time and frequency variables interchanged.

Example: Consider using duality and the result $e^{-|t|} \xleftrightarrow{F} X(jw) = \dfrac{2}{1+w^2}$ to find the Fourier transform $G(jw)$ of the signal

$$g(t) = \frac{2}{1+t^2}.$$

Since $e^{-|t|} \xleftrightarrow{F} X(jw) = \dfrac{2}{1+w^2}$, that is,

$$e^{-|t|} = \frac{1}{2p}\int_{-\infty}^{\infty}\left(\frac{2}{1+w^2}\right)e^{jwt}dw,$$

Multiplying this equation by $2p$ and replacing t by $-t$, we have

$$2pe^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+w^2}\right)e^{-jwt}dw$$

Interchanging the names of the variables t and w, we find that

$$2pe^{-|w|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-jwt}dw \;\Rightarrow\; F^{-1}\left(\frac{2}{1+t^2}\right) = 2pe^{-|w|}.$$

Based on the duality property we can get some other properties of Fourier transform:

$$\boxed{-jtx(t) \xleftrightarrow{F} \frac{dX(jw)}{dw}}$$

$$\boxed{e^{jw_0t}x(t) \xleftrightarrow{F} X(j(w-w_0))}$$

$$\boxed{-\frac{1}{jt}x(t) + px(0)d(t) \xleftrightarrow{F} \int_{-\infty}^{w} x(h)dh}$$

4.3.7 Parseval's Relation

If $x(t) \stackrel{F}{\longleftrightarrow} X(jw)$,

We have

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2p} \int_{-\infty}^{\infty} |X(jw)|^2 dw}$$

Parseval's relation states that the total energy may be determined either by computing the energy per unit time $|x(t)|^2$ and integrating over all time or by computing the energy per unit frequency $|X(jw)|^2 / 2p$ and integrating over all frequencies. For this reason, $|X(jw)|^2$ is often referred to as the ***energy-density spectrum***.

4.4 The convolution properties

$$\boxed{y(t) = h(t) * x(t) \stackrel{F}{\longleftrightarrow} Y(jw) = H(jw)X(jw)}$$

The equation shows that the Fourier transform maps the convolution of two signals into product of their Fourier transforms.

$H(jw)$, the transform of the impulse response, is the frequency response of the LTI system, which also completely characterizes an LTI system.

Example: The frequency response of a differentiator.

$$y(t) = \frac{dx(t)}{dt}.$$

From the differentiation property,

$$Y(jw) = jwX(jw),$$

The frequency response of the differentiator is

$$H(jw) = \frac{Y(jw}{X(jw)}) = jw.$$

Example: Consider an integrator specified by the equation:

$$y(t) = \int_{-\infty}^{t} x(t)dt \ .$$

The impulse response of an integrator is the unit step, and therefore the frequency response of the system:

$$H(jw) = \frac{1}{jw} + pd(w) \ .$$

So we have

$$Y(jw) = H(jw)X(jw) = \frac{1}{jw} X(jw) + pX(0)d(w) \ ,$$

which is consistent with the integration property.

Example: Consider the response of an LTI system with impulse response

$$h(t) = e^{-at}u(t) \ , \qquad a > 0$$

to the input signal

$$x(t) = e^{-bt}u(t) \ , \qquad b > 0$$

To calculate the Fourier transforms of the two functions:

$$X(jw) = \frac{1}{b + jw} \ , \text{ and}$$

$$H(jw) = \frac{1}{a + jw} \ .$$

Therefore,

$$Y(jw) = \frac{1}{(a + jw)(b + jw)} \ ,$$

using partial fraction expansion (assuming $a \neq b$), we have

$$Y(jw) = \frac{1}{b - a}\left[\frac{1}{a + jw} - \frac{1}{b + jw}\right]$$

The inverse transform for each of the two terms can be written directly. Using the linearity property, we have

$$y(t) = \frac{1}{b-a}\left[e^{-at}u(t) - e^{-bt}u(t)\right].$$

We should note that when $a = b$, the above partial fraction expansion is not valid. However, with $a = b$, we have

$$Y(jw) = \frac{1}{(a + jw)^2},$$

Considering $\frac{1}{(a + jw)^2} = j\frac{d}{dw}\left[\frac{1}{a + jw}\right]$, and

$$e^{-at}u(t) \xleftrightarrow{F} \frac{1}{a + jw}, \text{ and}$$

$$te^{-at}u(t) \xleftrightarrow{F} j\frac{d}{dw}\left[\frac{1}{a + jw}\right],$$

so we have

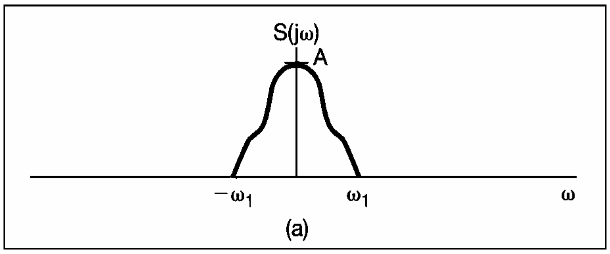
$$Y(t) = te^{-at}u(t).$$

4.5 The Multiplication Property

$$\boxed{r(t) = s(t)p(t) \longleftrightarrow R(jw) = \frac{1}{2p}\int_{-\infty}^{+\infty} S(jq)P(j(w-q))dq}$$

Multiplication of one signal by another can be thought of as one signal to scale or modulate the amplitude of the other, and consequently, the multiplication of two signals is often referred to as ***amplitude modulation***.

Example: Let $s(t)$ be a signal whose spectrum $S(jw)$ is depicted in the figure below.



Also consider the signal

$p(t) = \cos\omega_0 t$,

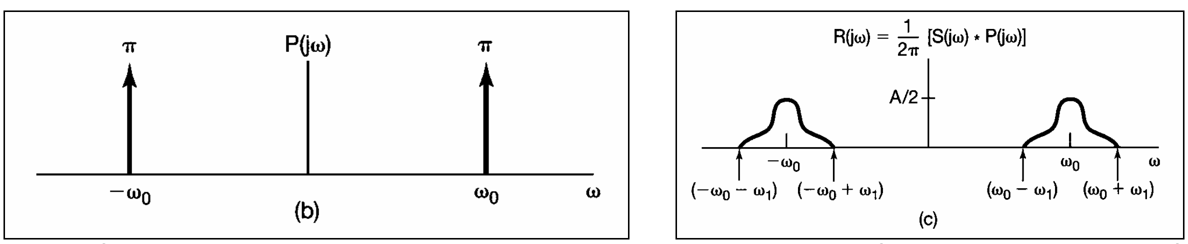
then

$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$.

The spectrum of $r(t) = s(t)p(t)$ is obtained by using the multiplication property,

$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\omega)P(j(\omega - \theta))d\theta$$
,
$$= \frac{1}{2}S(j\omega - \omega_0) + \frac{1}{2}S(j\omega + \omega_0)$$

which is sketched in the figure below.

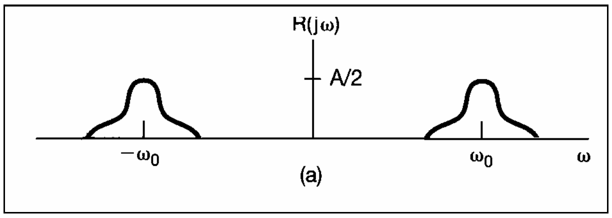


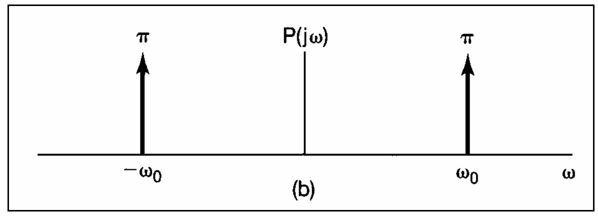
From the figure we can see that the signal is preserved although the information has been shifted to higher frequencies. This forms the basic for ***sinusoidal amplitude modulation*** systems for communications.

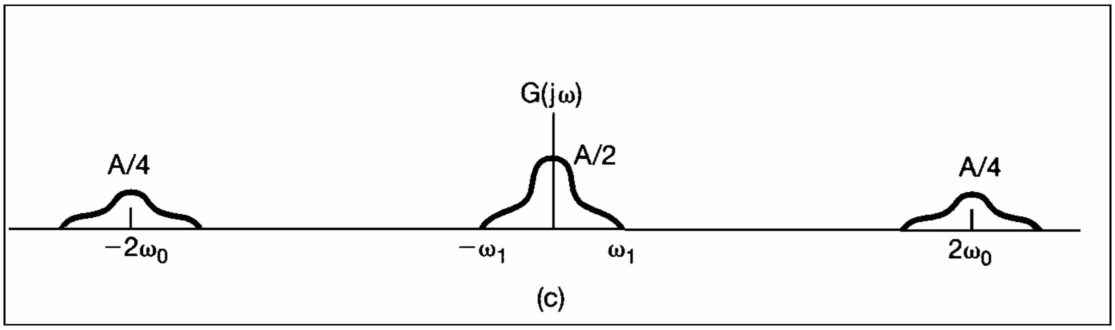
Example: If we perform the following multiplication using the signal $r(t)$ obtained in the preceding example and $p(t) = \cos\omega_0 t$, that is,

$g(t) = r(t)p(t)$

The spectrum of $P(j\omega)$, $R(j\omega)$ and $G(j\omega)$ are plotted in the figure below.







If we use a lowpass filter with frequency response $H(jw)$ that is constant at low frequencies and zero at high frequencies, then the output will be a scaled replica of $S(jw)$. Then the output will be scaled version of $s(t)$ - the modulated signal is recovered.

4.6 Summary of Fourier Transform Properties and Basic Fourier Transform Pairs

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
4.3.2	Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases}X(j\omega)=X^*(-j\omega)\\ \Re e\{X(j\omega)\}=\Re e\{X(-j\omega)\}\\ \Im m\{X(j\omega)\}=-\Im m\{X(-j\omega)\}\\ \lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert\\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega)\end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$ $x_o(t)=\mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(j\omega)\rvert^2d\omega$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t \rvert < T_1 \\ 0, & T_1 < \lvert t \rvert \le \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t \rvert < T_1 \\ 0, & \lvert t \rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

4.7 System Characterized by Linear Constant-Coefficient Differential Equations

An LTI system described by the following differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}, \tag{4. 67}$$

which is commonly referred to as an Nth-order differential equation.

The frequency response of this LTI system

$$H(jw) = \frac{Y(jw)}{X(jw)}, \tag{4. 68}$$

where $X(jw)$, $Y(jw)$ and $H(jw)$ are the Fourier transforms of the input $x(t)$, output $y(t)$ and the impulse response $h(t)$, respectively.

Applying Fourier transform to both sides, we have

$$F\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = F\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}, \tag{4. 69}$$

From the linearity property, the expression can be written as

$$\sum_{k=0}^{N} a_k F\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k F\left\{\frac{d^k x(t)}{dt^k}\right\}. \tag{4. 70}$$

From the differentiation property,

$$\sum_{k=0}^{N} a_k (jw)^k Y(jw) = \sum_{k=0}^{M} b_k (jw)^k X(jw) \qquad \Rightarrow \qquad H(jw) = \frac{Y(jw)}{X(jw)} = \frac{\sum_{k=0}^{M} b_k (jw)^k}{\sum_{k=0}^{N} a_k (jw)^k} \tag{4. 71}$$

$H(jw)$ is a rational function, that is, it is a ratio of polynomials in (jw).

Example: Consider a stable LTI system characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t), \text{ with } a > 0 .$$

The frequency response is

$$H(jw) = \frac{1}{jw + a}.$$

Te impulse response of this system is then recognized as

$$h(t) = e^{-at}u(t).$$

Example: Consider a stable LTI system that is characterized by the differential equation

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

The frequency response of this system is

$$H(jw) = \frac{(jw) + 2}{(jw)^2 + 4(jw) + 3} = \frac{jw + 2}{(jw + 1)(jw + 3)}.$$

Then, using the method of partial-fraction expansion, we find that

$$H(jw) = \frac{1/2}{jw + 1} + \frac{1/2}{jw + 3}.$$

The inverse Fourier transform of each term can be recognized as

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$

Example: Consider a system with frequency response of $H(jw) = \dfrac{jw + 2}{(jw + 1)(jw + 3)}$ and suppose that the input to the system is

$$x(t) = e^{-t}u(t),$$

find the output response.

The output in the frequency domain is give as

$$Y(jw) = H(jw)X(jw) = \left[\frac{jw + 2}{(jw + 1)(jw + 3)}\right]\left[\frac{1}{jw + 1}\right] = \frac{jw + 2}{(jw + 1)^2 (jw + 3))},$$

Using partial-fraction expansion, we have

$$Y(jw) = \frac{1/4}{jw+1} + \frac{1/2}{(jw+1)^2} + \frac{1/4}{(jw+3))},$$

By inspection, we get directly the inverse Fourier transform:

$$h(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t).$$