Solution for Midterm

Question 1

(a) $y(t) = x(t-1) + x(2-t)$

Noncausal since the output also depends on the future input

Stable. If the input is bounded as $|x(t)| \le M_x < \infty$,

$$|y(t)| \le |x(t-1)| + |x(2-t)| \le 2M_x < \infty \text{, so the system is stable.}$$

Time-invariant. $y(t) = T[x(t)] = x(t-1) + x(2-t)$, if the input is delayed by τ,

$$y(t,\tau) = x(t-\tau-1) + x(2-t \overset{-}{\cancel{+}} \tau)$$

and if we delay $y(t)$ by τ, we get

$$y(t-\tau) = x(t-\tau-1) + x(2-t+\tau) \cancel{\neq} y(t,\tau)$$

So the system is time-invariant.

Linear. For two input signals $x_1(t)$ and $x_2(t)$,

$$y_1(t) = T[x_1(t)] = x_1(t-1) + x_1(2-t),$$

$$y_2(t) = T[x_2(t)] = x_2(t-1) + x_2(2-t)$$

$$y_3(t) = T[ax_1(t) + bx_2(t)] = ax_1(t-1) + bx_2(t-1) + ax_1(2-t) + bx_2(2-t)$$

$$ay_1(t) + by_2(t) = ax_1(t-1) + bx_2(t-1) + ax_1(2-t) + bx_2(2-t) = y_3(t)$$

So the system is linear.

(b) $y[n] = [\cos(3n)]x[n]$

Causal since the output only depends on the current input

Stable. If the input is bounded as $|x[n]| \le M_x < \infty$,

$$|y[n]| \le |\cos(3n)||x[n]| \le M_x < \infty \text{, so the system is stable.}$$

Time-variant. $y[n] = T[x[n]] = (\cos(3n))x[n]$, if the input is delayed by N,

$$y[n,N] = (\cos(3n))x[n-N]$$

and if we delay $y[n]$ by N, we get

$$y[n-N] = (\cos(3(n-N)))x[n-N] \neq y[n,N]$$

So the system is time-variant.

Linear. For two input signals $x_1[n]$ and $x_2[n]$,

$$y_1[n] = T[x_1[n]] = (\cos(3n))x_1[n],$$

$$y_2[n] = T[x_2[n]] = (\cos(3n))x_2[n]$$

$$y_3[n] = T[ax_1[n] + bx_2[n]] = a(\cos(3n))x_1[n] + b(\cos(3n))x_2[n]$$

$$ay_1[n] + by_2[n] = a(\cos(3n))x_1[n] + b(\cos(3n))x_2[n] = y_3[n]$$

So the system is linear.

Question 2

(1) $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$

For $t \le 0$, $y(t) = 0$

For $0 < t < 4$, $y(t) = \int_0^t e^{-a\tau} d\tau = \dfrac{1-e^{-at}}{a}$

For $t \ge 4$, $y(t) = \int_{t-4}^t e^{-a\tau} d\tau = \dfrac{e^{-at}\left(e^{4a}-1\right)}{a}$

(2) $y[n] = \frac{1}{4} y[n-1] + x[n]$

The output depends on the initial condition. For this question, we assume the initial rest condition and we get the output result as following.

$y[0] = \frac{1}{4} y[-1] + x[0] = 1$

$y[1] = \frac{1}{4} y[0] + x[1] = \frac{5}{4}$

$y[2] = \frac{1}{4} y[1] + x[2] = \frac{21}{16} = \frac{5}{16}$

$y[3] = \frac{1}{4} y[2] + x[3] = \frac{1}{4} y[2]$

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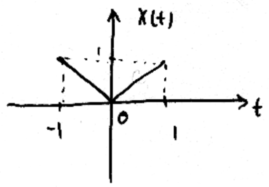
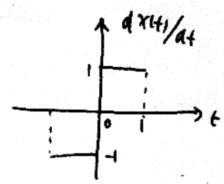
For $n \ge 3$, $y[n] = y[2]\left(\frac{1}{4}\right)^{n-2} = \frac{\cancel{21}}{16}\left(\frac{1}{4}\right)^{n-2}$

Question 3

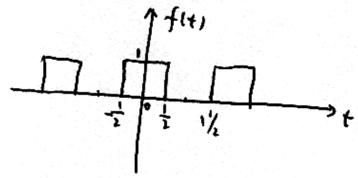
(1) (a) In one period, the wave forms of $x(t)$ and $\frac{dx(t)}{dt}$ are plotted as following





(b) $a_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{1}{2}\left(\int_{-1}^{0} -t\,dt + \int_0^1 t\,dt\right) = \frac{1}{2}$

(c) First we assume that $f(t) =$



Based on the result from example 3.5, the Fourier Series coefficients for $f(t)$ is

$$\begin{cases} C_k = \dfrac{\sin \pi \frac{k}{2}}{k\pi} & k \neq 0 \\ C_0 = \frac{1}{2} & k = 0 \end{cases}$$

And we know that $g(t) = \frac{dx(t)}{dt} = 2f(t-\frac{1}{2}) - 1$

so the fourier series coefficients for $g(t)$ are

$$b_k = \begin{cases} 2C_k\, e^{-jk\frac{\pi}{2}} & k \neq 0 \\ 2C_0 - 1 & k = 0. \end{cases}$$

$$= \begin{cases} 2\dfrac{\sin \pi \frac{k}{2}}{k\pi}\, e^{-jk\frac{\pi}{2}} & k \neq 0 \\ 0 & k = 0. \end{cases}$$

Since $g(t) = \frac{dx(t)}{dt}$, then the fourier series coefficients for $x(t)$ are

$$a_k = \frac{b_k}{jk\cdot\frac{2\pi}{T}} = \frac{2\sin\frac{\pi k}{2}}{k\pi} e^{-jk\frac{\pi}{2}} \cdot \frac{1}{jk\pi} = \frac{2\sin\frac{\pi k}{2}\, e^{-jk\frac{\pi}{2}}}{j(k\pi)^2}, \quad k \neq 0$$

(2) $N=8$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{8}\left(1 + e^{-jk\frac{2\pi}{8}} + e^{-jk\frac{2\pi}{4}} + e^{-jk\frac{2\pi}{8}\cdot 3}\right)$$

$$= \frac{1}{8}\left(1 + e^{-jk\frac{\pi}{4}} + e^{-jk\frac{\pi}{2}} + e^{-jk\frac{\pi}{4}3}\right)$$

$$= \frac{1}{8}\left(1 + e^{-j\frac{\pi}{4}k}\right)\left(1 + e^{-j\frac{\pi}{2}k}\right)$$