Solution for ELG 3120 Assignment #3

2.3 Let us define the signals

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$h_1[n] = u[n]$$

We note that $x[n] = x_1[n-2]$ and $h[n] = h_1[n+2]$

Now , we have $y[n] = x[n] * h[n] = x_1[n-2] * h_1[n+2]$

$$= \sum_{k=-\infty}^{\infty} x_1[k-2] h_1[n-k+2]$$

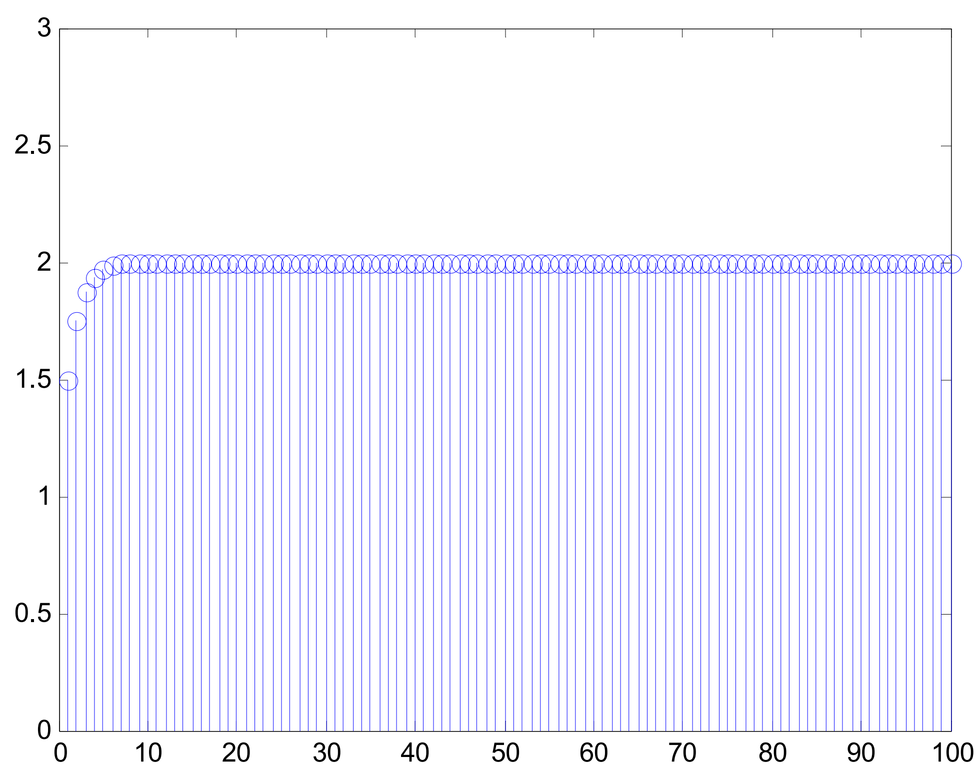
By replacing k with $m+2$ in the above summation, we obtain

$$y[n] = \sum_{m=-\infty}^{\infty} x_1[m] h_1[n-m] = x_1[n] * h_1[n]$$

Using the results of Example 2.3 in the textbook and set $\alpha = \frac{1}{2}$, we get

$$y[n] = 2\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] u[n]$$

The output is plotted below:



2.6 The solution is

$$y[n] = x[n]*h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^{-k} u[-k-1]u[n-k-1]$$

$$= \sum_{k=-\infty}^{-1} \left(\frac{1}{3}\right)^{-k} u[n-k-1]$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k} u[n+k-1]$$

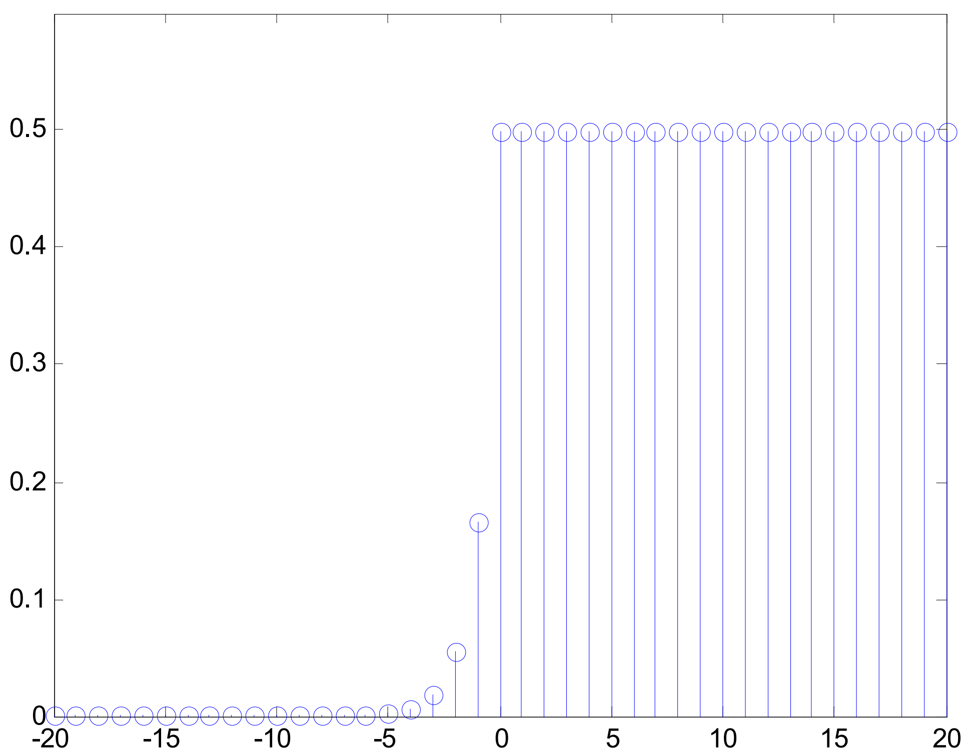
Replacing $k-1$ by p, we have

$$y[n] = \sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^{p+1} u[n+p]$$

For $n \geq 0$, $y[n] = \sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^{p+1} = \frac{1}{2}$

For $n < 0$, $y[n] = \sum_{p=-n}^{\infty} \left(\frac{1}{3}\right)^{p+1} = \left(\frac{1}{3}\right)^{-n+1} \sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^{p} = \left(\frac{1}{3}\right)^{-n+1} \frac{3}{2} = \frac{3^n}{2}$

The output is plotted as following:



2.22

(a) The desired convolution is

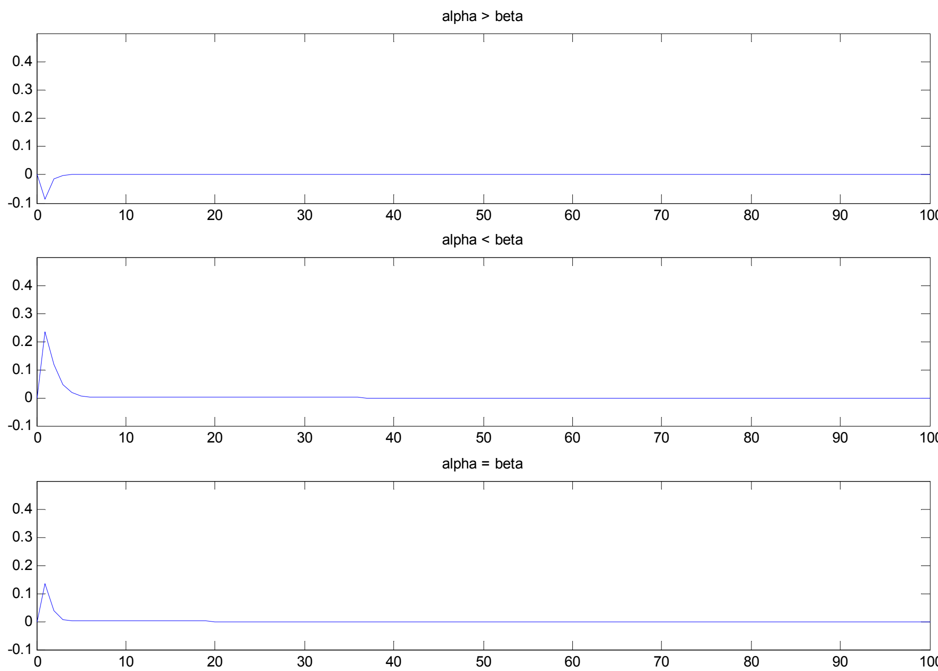
$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\,d\tau$$

$$=\int_{0}^{t}e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau, t\geq 0$$

Then $y(t)=\dfrac{e^{-\beta t}\left\{e^{-(\alpha-\beta)t}-1\right\}}{\beta-\alpha}u(t)$ for $\alpha\neq\beta$

$y(t)=te^{-\beta t}u(t)$ for $\alpha=\beta$

The output is plotted as follows



(b) The desired convolution is

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\,d\tau$$

$$=\int_{0}^{2}h(t-\tau)d\tau-\int_{2}^{5}h(t-\tau)d\tau$$

And this can be written as

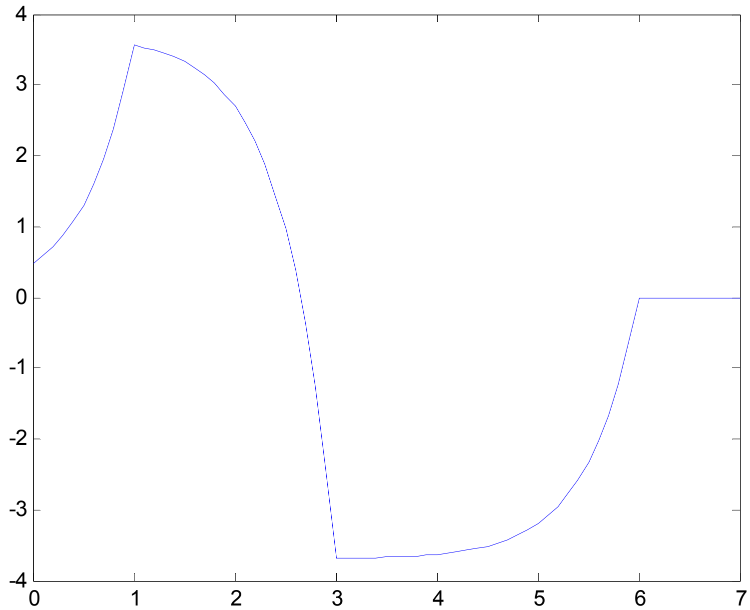
$$y(t)=\int_{0}^{2}e^{2(t-\tau)}d\tau-\int_{2}^{5}e^{2(t-\tau)}d\tau=\frac{1}{2}\left[e^{2t}-2e^{2(t-2)}+e^{2(t-5)}\right] \quad \text{for } t\leq 1$$

$$y(t)=\int_{t-1}^{2}e^{2(t-\tau)}d\tau-\int_{2}^{5}e^{2(t-\tau)}d\tau=\frac{1}{2}\left[e^{2}-2e^{2(t-2)}+e^{2(t-5)}\right] \quad \text{for } 1\leq t\leq 3$$

$$y(t)=-\int_{t-1}^{5}e^{2(t-\tau)}d\tau=\frac{1}{2}\left[e^{2(t-5)}-e^{2}\right] \quad \text{for } 3\leq t\leq 6$$

$$y(t)=0 \quad \text{for } t\geq 6$$

The output is plotted below



(c) The desired convolution is

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\,d\tau$$

$$=\int_{0}^{2}\sin(\pi\tau)\,h(t-\tau)\,d\tau$$

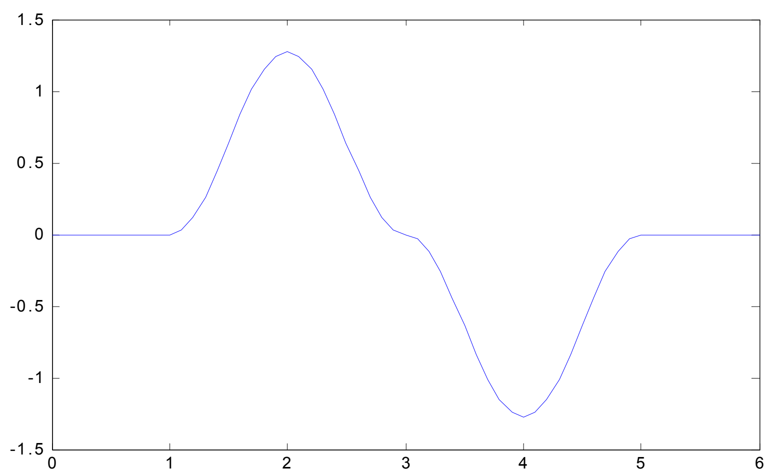
So $y(t)=0 \quad \text{for } t<1$

$$y(t)=\frac{2}{\pi}\left[1-\cos\{\pi(t-1)\}\right] \quad \text{for } 1\le t\le 3$$

$$y(t)=\frac{2}{\pi}\left[\cos\{\pi(t-3)\}-1\right] \quad \text{for } 3\le t\le 5$$

$$y(t)=0 \quad \text{for } t\ge 5$$

The output is plotted below



(d) The desired convolution is

Let $h(t) = h_1(t) - \frac{1}{3}\delta(t-2)$

Where $h_1(t) = \begin{cases} 4/3 & 0 \le t \le 1 \\ 0 & otherwise \end{cases}$

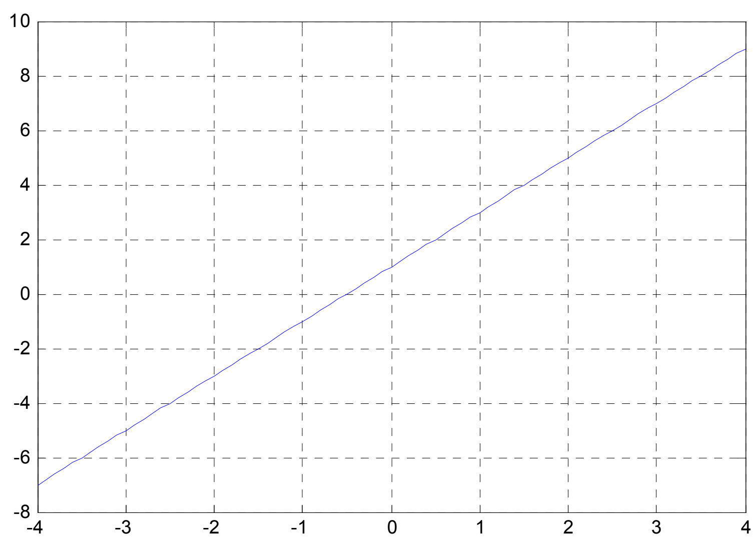
So $y(t) = h(t) * x(t) = h_1(t) * x(t) - \frac{1}{3}x(t-2)$

And $h_1(t) * x(t) = \int_{t-1}^{t} \frac{4}{3}(a\tau + b)\,d\tau = \frac{4}{3}\left[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)\right]$

Then

$$y(t) = \frac{4}{3}\left[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)\right] - \frac{1}{3}\left[a(t-2) + b\right]$$
$$= at + b = x(t)$$

The output is plotted below



(e) The desired convolution is

$x(t)$ periodic implies $y(t)$ periodic. We only give one period.

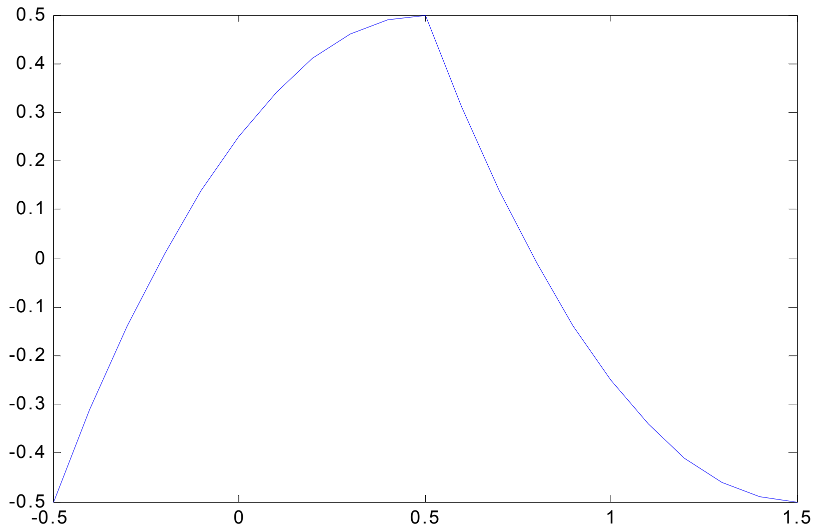
For $-\frac{1}{2} \le t \le \frac{1}{2}$ we have

$$y(t) = \int_{t-1}^{-\frac{1}{2}} (t - \tau - 1)\,d\tau + \int_{-\frac{1}{2}}^{t} (1 - t + \tau)\,d\tau = \frac{1}{4} + t - t^2$$

For $\frac{1}{2} \le t \le \frac{3}{2}$ we have

$$y(t) = \int_{t-1}^{\frac{1}{2}} (1 - t + \tau)\,d\tau + \int_{\frac{1}{2}}^{t} (t - 1 - \tau)\,d\tau = t^2 - 3t + \frac{7}{4}$$

The output is plotted below



2.29

(a) Causal because $h(t)=0$ for $t<0$

Stable because $\int_{-\infty}^{\infty}|h(t)|dt=\frac{e^{-8}}{4}<\infty$

(b) Noncausal because $h(t)\neq 0$ for $t<0$

Unstable because $\int_{-\infty}^{\infty}|h(t)|dt=\infty$

(c) Noncausal because $h(t)\neq 0$ for $t<0$

Stable because $\int_{-\infty}^{\infty}|h(t)|dt=\frac{e^{100}}{2}<\infty$

(d) Noncausal because $h(t)\neq 0$ for $t<0$

Stable because $\int_{-\infty}^{\infty}|h(t)|dt=\frac{e^{-2}}{2}<\infty$

(e) Noncausal because $h(t)\neq 0$ for $t<0$

Stable because $\int_{-\infty}^{\infty}|h(t)|dt=\frac{1}{3}<\infty$

(f) Causal because $h(t)=0$ for $t<0$

Stable because $\int_{-\infty}^{\infty}|h(t)|dt=1<\infty$

(g) Causal because $h(t)=0$ for $t<0$

Unstable because $\int_{-\infty}^{\infty}|h(t)|dt=\infty$

2.33. (a) (i) From Example 2.14, we know that

$$y_1(t) = \left[\frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t}\right] u(t).$$

(ii) We solve this along the lines of Example 2.14. First assume that $y_p(t)$ is of the form Ke^{2t} for $t > 0$. Then using eq. (P2.33-1), we get for $t > 0$

$$2Ke^{2t} + 2Ke^{2t} = e^{2t} \qquad \Rightarrow \qquad K = \frac{1}{4}.$$

We now know that $y_p(t) = \frac{1}{4}e^{2t}$ for $t > 0$. We may hypothesize the homogeneous solution to be of the form

$$y_h(t) = Ae^{-2t}.$$

Therefore,

$$y_2(t) = Ae^{-2t} + \frac{1}{4}e^{2t}, \qquad \text{for } t > 0.$$

Assuming initial rest, we can conclude that $y_2(t) = 0$ for $t \leq 0$. Therefore,

$$y_2(0) = 0 = A + \frac{1}{4} \qquad \Rightarrow \qquad A = -\frac{1}{4}.$$

Then,

$$y_2(t) = \left[-\frac{1}{4}e^{2t} + \frac{1}{4}e^{-2t}\right] u(t).$$

(iii) Let the input be $x_3(t) = \alpha e^{3t}u(t) + \beta e^{2t}u(t)$. Assume that the particular solution $y_p(t)$ is of the form

$$y_p(t) = K_1 \alpha e^{3t} + K_2 \beta e^{2t}$$

for $t > 0$. Using eq. (P2.33-1), we get

$$3K_1\alpha e^{3t} + 2K_2\beta e^{2t} + 2K_1\alpha e^{3t} + 2K_2\beta e^{2t} = \alpha^{3t} + \beta e^{2t}.$$

Equating the coefficients of e^{3t} and e^{2t} on both sides, we get

$$K_1 = \frac{1}{5} \qquad \text{and} \qquad K_2 = \frac{1}{4}.$$

Now hypothesizing that $y_h(t) = Ae^{-2t}$, we get

$$y_3(t) = \frac{1}{5}\alpha e^{3t} + \frac{1}{4}\beta e^{2t} + Ae^{-2t}$$

for $t > 0$. Assuming initial rest,

$$y_3(0) = 0 = A + \alpha/5 + \beta/4 \qquad \Rightarrow \qquad A = -\left(\frac{\alpha}{5} + \frac{\beta}{4}\right).$$

Therefore,

$$y_3(t) = \left\{\frac{1}{5}\alpha e^{3t} + \frac{1}{4}\beta e^{2t} - \left(\frac{\alpha}{5} + \frac{\beta}{4}\right) e^{-2t}\right\} u(t).$$

Clearly, $y_3(t) = \alpha y_1(t) + \beta y_2(t)$.

(iv) For the input-output pair $x_1(t)$ and $y_1(t)$, we may use eq. (P2.33-1) and the initial rest condition to write

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t), \qquad y_1(t) = 0 \text{ for } t < t_1. \tag{S2.33-1}$$

For the input-output pair $x_2(t)$ and $y_2(t)$, we may use eq. (P2.33-1) and the initial rest condition to write

$$\frac{dy_2(t)}{dt} + 2y_2(t) = x_2(t), \qquad y_2(t) = 0 \text{ for } t < t_2. \tag{S2.33-2}$$

Scaling eq. (S2.33-1) by α and eq. (S2.33-2) by β and summing, we get

$$\frac{d}{dt}\{\alpha y_1(t) + \beta y_2(t)\} + 2\{\alpha y_1(t) + \beta y_2(t)\} = \alpha x_1(t) + \beta x_2(t),$$

and

$$y_1(t) + y_2(t) = 0 \text{ for } t < \min(t_1, t_2).$$

By inspection, it is clear that the output is $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ when the input is $x_3(t) = \alpha x_1(t) + \beta x_2(t)$. Furthermore, $y_3(t) = 0$ for $t < t_3$, where t_3 denotes the time until which $x_3(t) = 0$.

(b) (i) Using the result of (a-ii), we may write

$$y_1(t) = \frac{K}{4}\left[e^{2t} - e^{-2t}\right] u(t).$$

(ii) We solve this along the lines of Example 2.14. First assume that $y_p(t)$ is of the form $KYe^{2(t-T)}$ for $t > T$. Then using eq. (P2.33-1), we get for $t > T$

$$2Ke^{2(t-T)} + 2Ke^{2(t-T)} = e^{2t} \qquad \Rightarrow \qquad K = \frac{1}{4}.$$

We now know that $y_p(t) = \frac{K}{4}e^{2(t-T)}$ for $t > T$. We may hypothesize the homogeneous solution to be of the form

$$y_h(t) = Ae^{-2t}.$$

Therefore,

$$y_2(t) = Ae^{-2t} + \frac{K}{4}e^{2(t-T)}, \qquad \text{for } t > T.$$

Assuming initial rest, we can conclude that $y_2(t) = 0$ for $t \leq T$. Therefore,

$$y_2(T) = 0 = Ae^{-2T} + \frac{K}{4} \qquad \Rightarrow \qquad A = -\frac{K}{4}e^{2T}.$$

Then,

$$y_2(t) = \left[-\frac{K}{4}e^{-2(t-T)} + \frac{K}{4}e^{2(t-T)}\right] u(t-T).$$

Clearly, $y_2(t) = y_1(t-T)$.

(iii) Consider the input-output pair $x_1(t) \rightarrow y_1(t)$ where $x_1(t) = 0$ for $t < t_0$. Note that

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t), \qquad y_1(t) = 0, \text{ for } t < t_0.$$

Since the derivative is a time-invariant operation, we may now write

$$\frac{dy_1(t-T)}{dt} + 2y_1(t-T) = x_1(t-T), \qquad y_1(t) = 0, \text{ for } t < t_0.$$

This suggests that if the input is a signal of the form $x_2(t) = x_1(t-T)$, then the output is a signal of the form $y_2(t) = y_1(t-T)$. Also, note that the new output $y_2(t)$ will be zero for $t < t_0 + T$. This supports time-invariance since $x_2(t)$ is zero for $t < t_0 + T$. Therefore, we may conclude that the system is time-invariant.