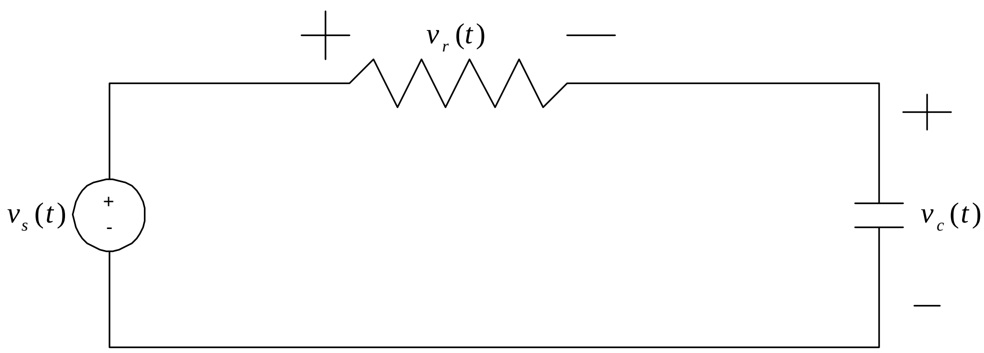
3.10 Examples of continuous-Time Filters Described By Differential Equations

In many applications, frequency-selective filtering is accomplished through the use of LTI systems described by linear constant-coefficient differential or difference equations. In fact, many physical systems that can be interpreted as performing filtering operations are characterized by differential or difference equation.

3.10.1 A simple RC Lowpass Filter

The first-order RC circuit is one of the electrical circuits used to perform continuous-time filtering. The circuit can perform either Lowpass or highpass filtering depending on what we take as the output signal.



If we take the voltage cross the capacitor as the output, then the output voltage is related to the input through the linear constant-coefficient differential equation:

$$RC\frac{dv_c(t)}{dt}+v_c(t)=v_s(t)\,. \tag{3.111}$$

Assuming initial rest, the system described by Eq. (3.111) is LTI. If the input is $v_s(t)=e^{jwt}$, we must have voltage output $v_c(t)=H(jw)e^{jwt}$. Substituting these expressions into Eq. (3.111), we have

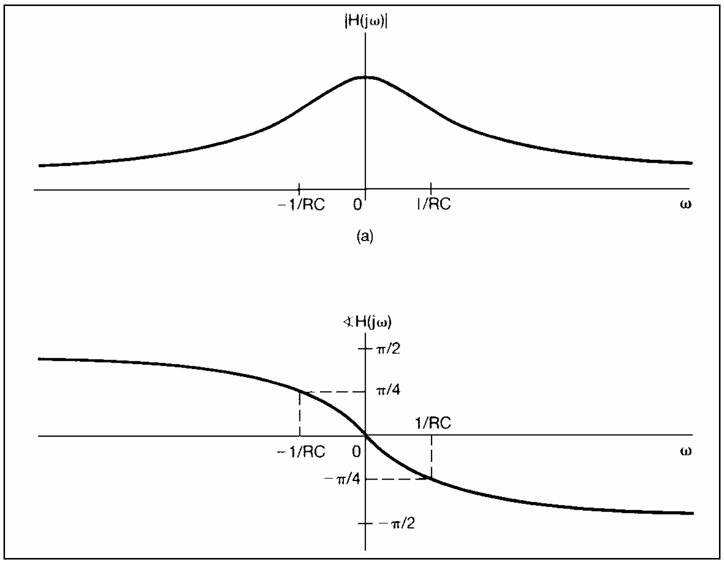
$$RC\frac{d}{dt}\left[H(jw)e^{jwt}\right]+H(jw)e^{jwt}=e^{jwt}\,, \tag{3.112}$$

or

$$RCjwH(jw)e^{jwt}+H(jw)e^{jwt}=e^{jwt}\,, \tag{3.113}$$

Then we have $H(jw) = \dfrac{1}{1+RCjw}$. (3.114)

Te amplitude and frequency response $H(jw)$ is shown in the figure below.



(a)

(b)

We can also get the impulse response

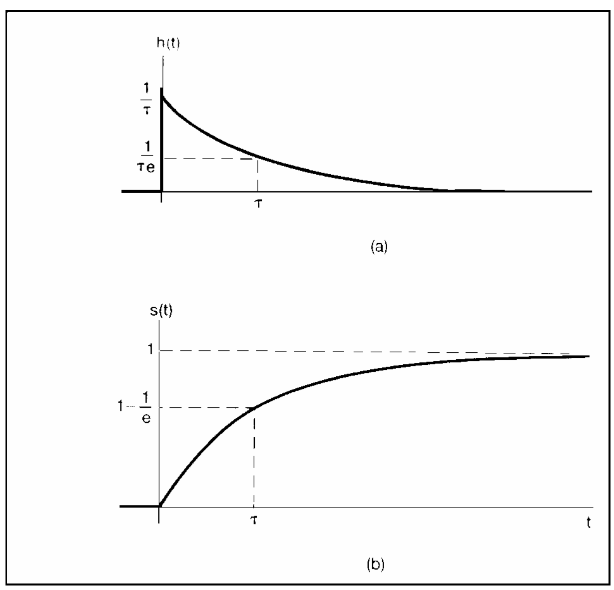
$$h(t) = \frac{1}{RC} e^{-t/RC} u(t), \tag{3.115}$$

and the step response is

$$h(t) = (1 - e^{-t/RC}) u(t), \tag{3.116}$$

The fundamental trade-off can be found by comparing the figures:

- To pass only very low frequencies, $1/RC$ should be small, or RC should be large.
- To have fast step response, we need a smaller RC.
- The type of trade-off between behaviors in the frequency domain and time domain is typical of the issues arising in the design analysis of LTI systems.



(a)

(b)

3.10.2 A Simple RC Highpass Filter

If we choose the output from the resistor, then we get an RC highpass filter.

3.11 Examples of Discrete-Time Filter Described by Difference Equations

A discrete-time LTI system described by the first-order difference equation

$$y[n]-ay[n-1]=x[n]\ . \qquad (3.116)$$

Form the eigenfunction property of complex exponential signals, if $x[n]=e^{jwn}$, then $y[n]=H(e^{jw})e^{jwn}$, where $H(e^{jw})$ is the frequency response of the system.

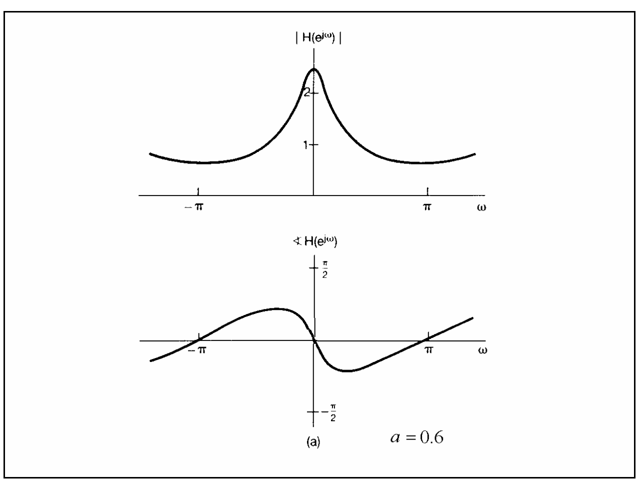
$$H(e^{jw})=\frac{1}{1-ae^{-jw}}\ . \qquad (3.117)$$

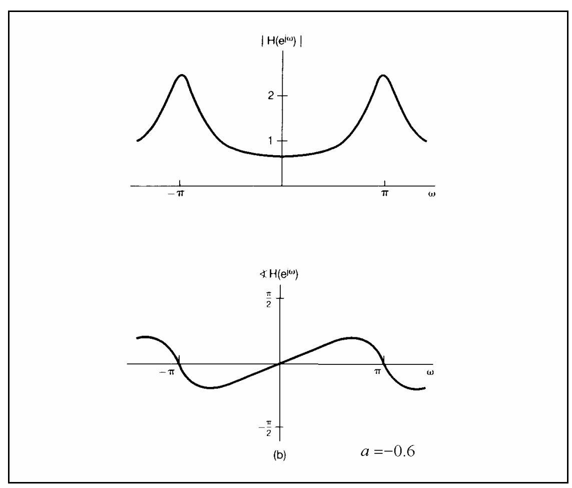
The impulse response of the system is

$$x[n]=a^{n}u[n]. \qquad (3.118)$$

The step response is

$$s[n]=\frac{1-a^{n+1}}{1-a}u[n]. \qquad (3.119)$$





From the above plots we can see that for $a=0.6$ the system acts as a Lowpass filter and $a=-0.6$, the system is a highpass filter. In fact, for any positive value of $a<1$, the system approximates a highpass filter, and for any negative value of $a>-1$, the system approximates a highpass filter, where $|a|$ controls the size of bandpass, with broader pass bands as $|a|$ in decreased.

The trade-off between time domain and frequency domain characteristics, as discussed in continuous time, also exists in the discrete-time systems.

3.11.2.2 Nonrecursive Discrete-Time Filters

The general form of an FIR norecursive difference equation is

$$y[n]=\sum_{k=-N}^{M} b_k x[n-k]. \tag{3.120}$$

It is a weighted average of the $(\mathrm{N}+\mathrm{M}+1)$ values of $x[n]$, with the weights given by the coefficients b_k.

One frequently used example is a *moving-average filter*, where the output of $y[n]$ is an average of values of $x[n]$ in the vicinity of n_0 - the result corresponding a smooth operation or lowpass filtering.

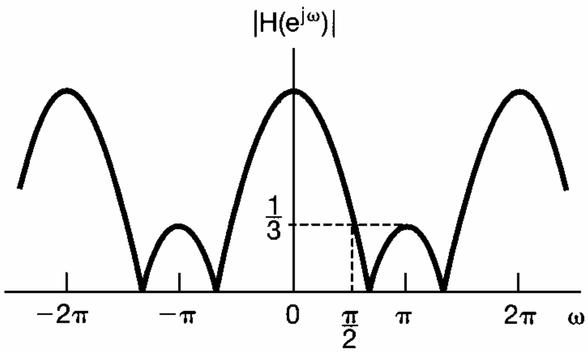
An example: $y[n]=\frac{1}{3}\left(x[n-1]+x[n]+x[n+1]\right)$. (3.121)

The impulse response is

$$h[n]=\frac{1}{3}\left(d[n-1]+d[n]+d[n+1]\right), \tag{3.122}$$

and the frequency response

$$H(e^{jw})=\frac{1}{3}\left(e^{jw}+1+e^{-jw}\right). \tag{3.123}$$



Magnitude of the frequency response of a three-point moving-average lowpass filter.

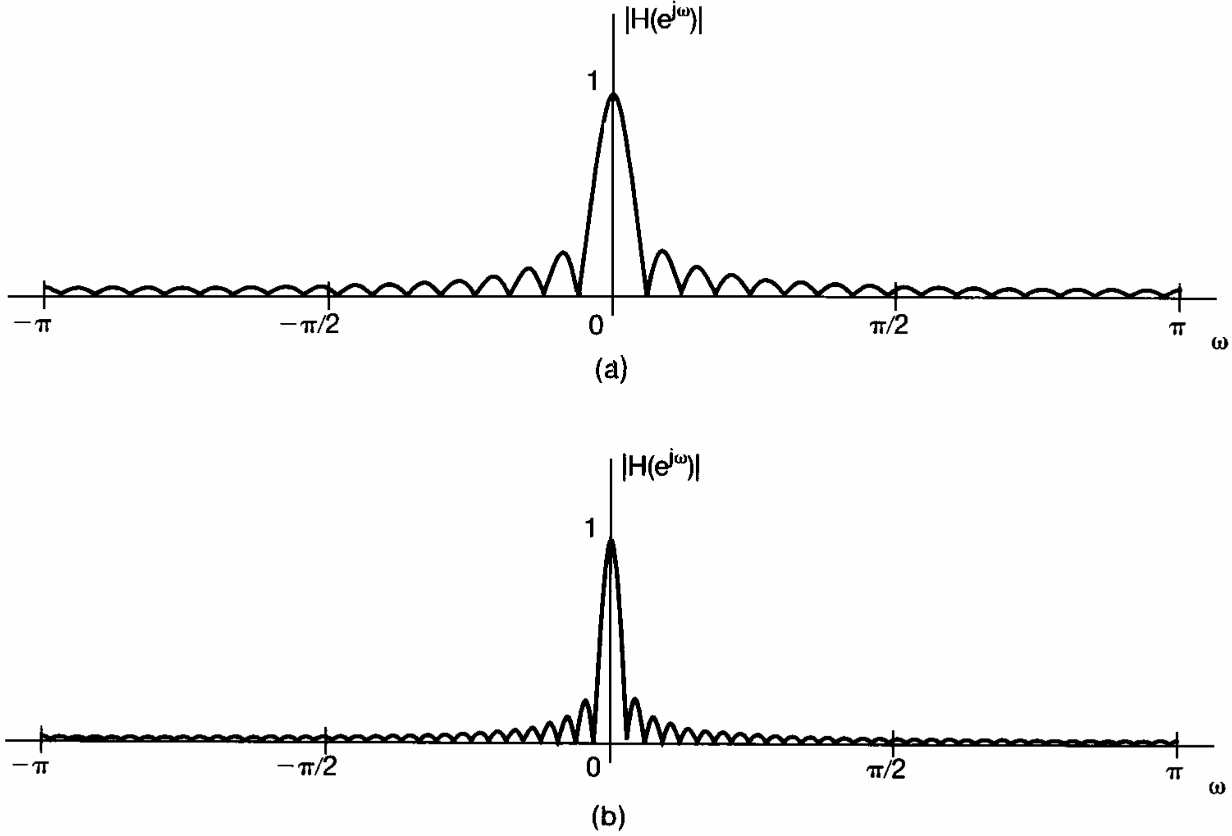
A generalized moving average filter can be expressed as

$$y[n]=\frac{1}{N+M+1}\sum_{k=-N}^{M}b_k x[n-k]\,. \tag{3.124}$$

The frequency response is

$$H(e^{jw})=\frac{1}{M+N+1}\sum_{k=-N}^{M}e^{-jwk}=\frac{1}{M+N+1}e^{jw[(N-M)/2]}\frac{\sin[w(M+N+1)/2]}{\sin(w/2)}\,. \tag{3.125}$$

The frequency responses with different average window lengths are plotted in the figure below.



Magnitude of the frequency response for the lowpass moving-average filter of eq. (3.162): (a) $M = N = 16$; (b) $M = N = 32$.

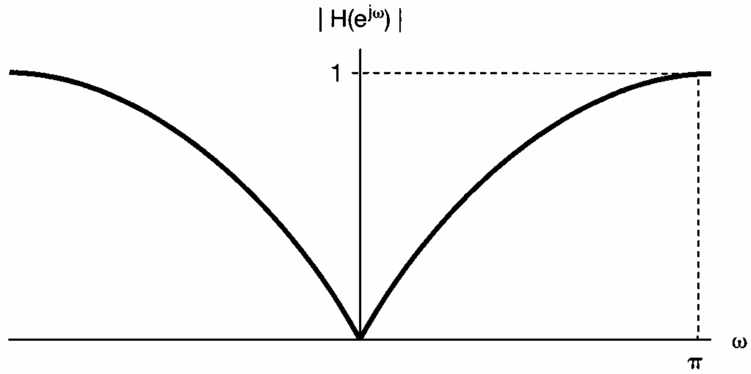
FIR norecursive highpass filter

An example of FIR norecursive highpass filter is

$$y[n] = \frac{x[n] - x[n-1]}{2}. \tag{3.126}$$

The frequency response is

$$H(e^{jw}) = \frac{1}{2}\left(1 - e^{-jw}\right) = je^{jw/2}\sin(w/2). \tag{3.127}$$



Frequency response of a simple highpass filter.