Université d'Ottawa · University of Ottawa

Faculté de génie
École d'ingénierie
et de technologie de l'information

Faculty of Engineering
School of Information Technology
and Engineering

ELG 3120A

SIGNAL AND SYSTEM ANALYSIS

Final Exam – Fall 2002

Thursday 09:30 - 12:30

SITE Building, Room: B0138

Prof. Jianping Yao

Time allowed: 3 hours
No calculators permitted
Textbook and notes not allowed (close book exam)
Attempt all the questions (50 marks)

Last name:

First name:

Student number:

Question 1 (6 marks)

1. Using the differentiation and shifting properties of Fourier Transform to calculate the Fourier Transform of the signal $x(t)$ shown in Figure 1 (Do not use the direct definition of Fourier Transform to calculate the Fourier Transform of $x(t)$).

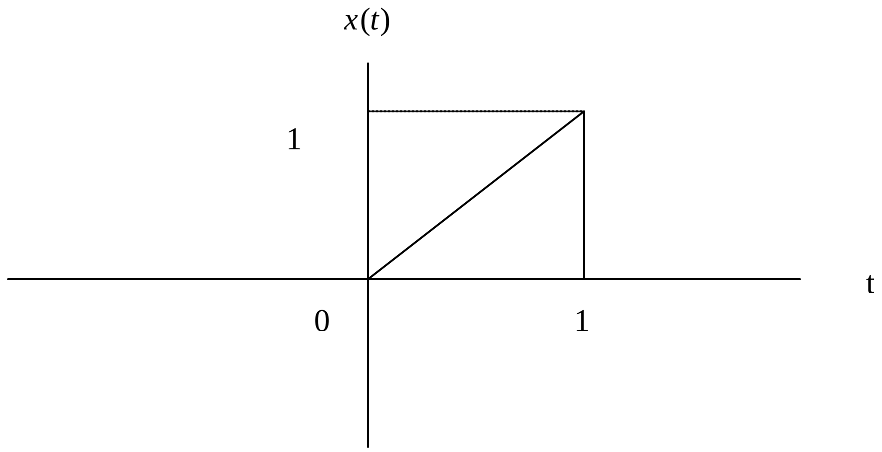

Figure 1

Question 2 (6 = 3 + 3)

The input and the output of a stable and causal LTI system are related by the differential equation

$$\frac{d^2 y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

(a) Find the impulse response of the system.
(b) What is the output response if an input $x(t) = e^{-2t}u(t)$ is applied to the system?

Question 3 (6)

Find the frequency response $H(e^{jw})$ and the impulse response $h[n]$ of the system having the output $y(t)$ for the input $x(t)$.

$$x[n] = \left(\frac{1}{2}^{n}\right)u[n], \text{ and } y[n] = \frac{1}{4}\left(\frac{1}{2}\right)^{n} u[n] + \left(\frac{1}{4}\right)^{n} u[n]$$

Question 4 (7 = 2 + 2 + 2 + 1)

Consider a discrete-time LTI system described by the following difference equation:

$$y[n] = \frac{1}{2}(x[n] + x[n-1])$$

(a) Determine its transfer function $H(e^{jw})$ and impulse response $h[n]$.

(b) Find the expression for the magnitude response $\left|H(e^{jw})\right|$.

(c) Sketch its magnitude response.

(d) Based on the magnitude response plot, determine if the filter is a low-pass, high-pass or band-pass filter.

Question 5 (8 = 3 + 3 +2)

Consider an RLC circuit shown in Figure 2 with input $x(t)$ and output $y(t)$.

(a) Write the differential expression for this system.

(b) Find the frequency response $H(jw)$.

(c) If $L = 10$ mH, $C = 100\ mF$ and $R = 1\Omega$, determine if the system is over-damped, critically damped or under-damped.

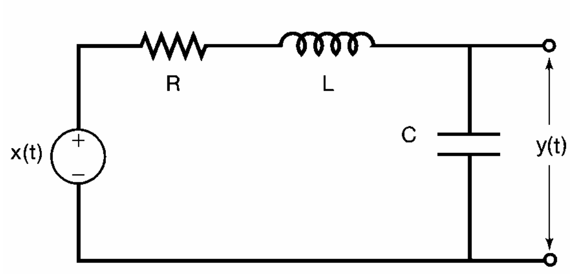

Figure 2

Question 6 (4)

Sketch the amplitude response as a Bode diagram (straight line approximation) for the LTI system described by the transfer function.

$$H(jw) = \frac{(10 + jw)}{(1 + jw)(100 + jw)}$$

Question 7 (6 = 3 + 3)

Determine the Nyquist rate for each of the following two signals $x(t)$ and $y(t)$.

(a) $x(t) = 1 + \cos(100pt) + \cos(200pt)$

(b) $y(t) = x^2(t)$ with the same $x(t)$ in (a).

Question 8 (7 =2 +2+ 3)

Consider a continuous-time LTI system for which the input $x(t)$ and $y(t)$ are related by the differential equation

$$\frac{d^2y(t)}{dt^2}-\frac{dy(t)}{dt}-2y(t)=x(t)$$

(a) Determine the system transfer function $H(s)$.
(b) Sketch the pole-zero pattern of $H(s)$.
(c) Determine $h(t)$ for each of the following cases: (i) The system is stable. (ii) The system is causal, (iii) The system is neither stable nor causal.

Table of Formulas

Convolutions:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-t)dt$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Continuous-time Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt$$

$$a_0 = \frac{1}{T}\int_T x(t)dt \qquad w_0 = \frac{2p}{T}$$

Discrete-time Fourier series:

$$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2p}{N})n} \quad a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(\frac{2p}{N})n}$$

Orthogonal function decomposition:

$$\hat{V}(t) = \sum_{j=1}^{m} a_j f_j(t)$$

$$a_j = \frac{<V(t), f_j(t)>}{\int_T |f_j(t)|^2 dt} = \frac{<V(t), f_j(t)>}{<f_j(t), f_j(t)>}$$

$$<V(t), f_j(t)> = \int_T V(t) f_j^*(t)dt$$

Continuous-time Fourier transform and inverse Fourier transform:

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

$$x(t) = \frac{1}{2p}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

Periodic signals:

$$X(jw) = \sum_{k=-\infty}^{+\infty} 2p\, a_k d(w - kw_0)$$

Continuous-time first and second order lowpass systems in standard form:

$$H(jw) = \frac{1}{1 + jwt}$$

$$H(jw) = \frac{w_n^2}{(jw)^2 + 2zw_n(jw) + w_n^2}$$

Discrete-time Fourier transform and inverse Fourier transform

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$$

$$x[n] = \frac{1}{2p}\int_{2p} X(e^{jw})e^{jwn}dw$$

Impulse-train sampling:

$$x_p(t) = x(t) \times p(t)$$

$$X_p(jw) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(w - kw_s)), \; w_s = \frac{2p}{T}$$

Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{+\infty} \|x(t)\|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \|X(j\omega)\|^2 d\omega$		

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{array}{l}X(e^{j\omega})\\Y(e^{j\omega})\end{array}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4 T	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{Re}\{X(e^{j\omega})\} = \mathcal{Re}\{X(e^{-j\omega})\} \\ \mathcal{Im}\{X(e^{j\omega})\} = -\mathcal{Im}\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{Ev}\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{Od}\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{Re}\{X(e^{j\omega})\}$ $j\mathcal{Im}\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$		

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $\omega_0 \quad a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & N_1 < \|n\| \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \|a\| < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n]\begin{cases} 1, & \|n\| \le N_1 \\ 0, & \|n\| > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \text{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \|\omega\| \le W \\ 0, & W < \|\omega\| \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \|a\| < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n + r - 1)!}{n!(r-1)!} a^n u[n], \quad \|a\| < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to \infty} sX(s)$$

9.6 SOME LAPLACE TRANSFORM PAIRS

As we indicated in Section 9.3, the inverse Laplace transform can often be easily evaluated by decomposing $X(s)$ into a linear combination of simpler terms, the inverse transform of each of which can be recognized. Listed in Table 9.2 are a number of useful Laplace

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{Re}\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{Re}\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{Re}\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{Re}\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} > 0$