**Université d'Ottawa**
**Faculté de Génie,**
**École d'Ingénierie et des Technologies de l'Information**

**University of Ottawa**
**Faculty of Engineering,**
**School of Information Technology and Engineering**

# ELG 3120

## Signal and System Analysis

## Midterm

**Thursday, 28 October 2004**

**17:30 – 19:00**

**Professor: Jianping Yao**

- **No textbook and notes (close-book exam)**
- **Initial on the top of each page.**

**Last name:** ____________________

**First name:** ____________________

**Student number:** ____________________

**Question 1:** **/4**

Determine if the following systems are: causal or non-causal, stable or instable, time invariant or time variant, and linear or nonlinear. Justify your answers.

(1) $y(t) = [\sin(4t)]x(t)$

Solution:

(a) It is causal, since the output depends on the input at the same time.

(b) It is stable. Assume $|x(t)| \le M$ , M is a finite number, $|y(t)| = |[\sin(4t)]x(t)| \le |x(t)| \le M$ , that is, if the input is bounded, the output is also bounded. The system is stable.

(c) The system is not time invariant. Assume input is experienced a time delay of $t_0$, the new output should be $y'(t) = [\sin(4t)x(t-t_0)]$. While $y(t-t_0) = [\sin(4(t-t_0))]x(t-t_0)$. Since $y'(t) \ne y(t-t_0)$, the system is not time invariant.

(d) The system is linear. Assume we have two different inputs, $x_1(t)$ and $x_2(t)$, the corresponding outputs are $y_1(t) = \sin(4t)x_1(t)$ and $y_2(t) = \sin(4t)x_1(t)$. If a new input $ax_1(t) + bx_2(t)$, the new output is $y'(t) = [\sin(4t)][ax_1(t) + bx_2(t)] = a\sin(4t)x_1(t) + b\sin(4t)x_2(t) = ay_1(t) + By_2(t)$. The system is linear.

(2) $y[n] = x[n-2] + x[-n+2]$

Solution:

(a) It is not causal, since the output depends on the future input. For example, when $n$ = 0, $y[0] = x[-2] + x[2]$, it is clearly seen at n = 0, the output depends on the future input at $n$ = 2.

(b) It is stable. Assume $|x(t)| \le M$, $M$ is a finite number, $|y(t)| \le 2M$, that is, if the input is bounded, the output is also bounded. The system is stable.

(c) The system is not time invariant. Assume input is experienced a time delay of $n_0$, the new output should be $y'[n] = x[(n-2)-n_0] + x[(-n+2)-n_0] = x[(n-n_0)-2] + x[-(n+n_0)+2]$. While $y[n-n_0] = x[(n-n_0)-2] + x[-(n-n_0)+2]$. Since $y'(t) \ne y(t-t_0)$, the system is not time invariant.

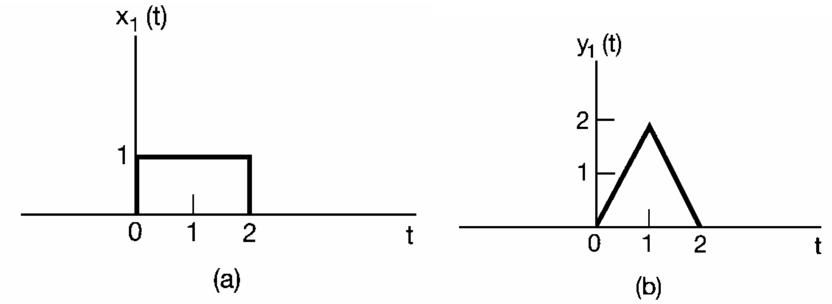
(d) The system is linear. Assume we have two different inputs, $x_1(t)$ and $x_2(t)$, the corresponding outputs are $y_1[n] = x_1[n-2] + x_1[-n+2]$ and $y_2[n] = x_2[n-2] + x_2[-n+2]$. If a new input $ax_1(t) + bx_2(t)$, the new output is $y'[n] = a\{x_1[n-2] + x_1[-n+2]\} + b\{x_2[n-2] + x_2[-n+2]\} = ay_1[n] + by_2[n]$. The system is linear.

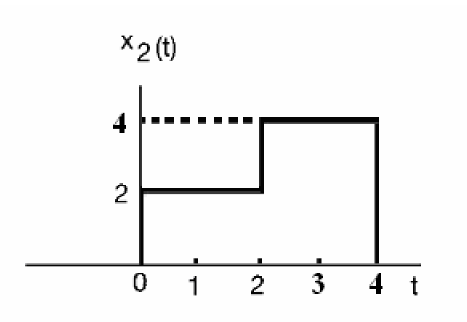
**Question 2:** **/3**

Consider an LTI system whose response to the signal $x_1(t)$ in Fig. 1 (a) is the signal $y_1(t)$ shown in Fig. 1 (b).

(1) Express x2(t) using x1(t).

(2) Sketch the response of the system to the input deposit in Figure (c).

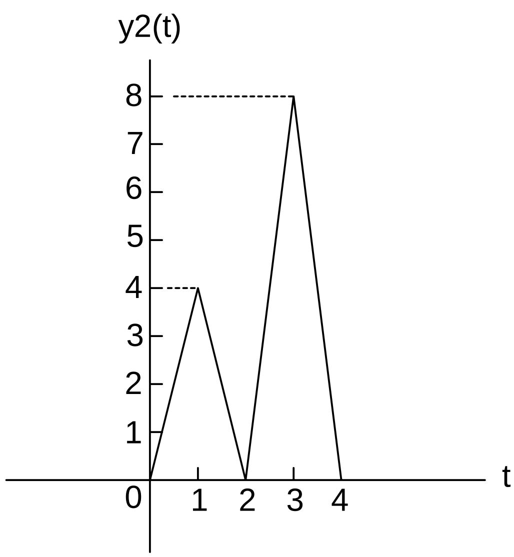



(c)

(1) $x_2(t) = 2x_1(t) + 4x_1(t-2)$

(2) Since the system is LTI system, so the output is $y_2(t) = 2y_1(t) + 4y_1(t-2)$.



**Question 3:** **/5**

Find the output response of the LTI system. The impulse responses of the system is $h(t) = e^{-3t}u(t) + \delta(t-2)$, the input to the system is $x(t) = u(t-2) - u(t-5)$.

Solution:

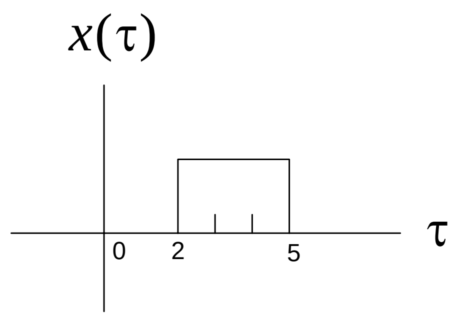
$h(t) = e^{-3t}u(t) + \delta(t-2) = h_1(t) + h_2(t)$, where $h_1(t) = e^{-3t}u(t)$ and $h_2(t) = \delta(t-2)$

The output is

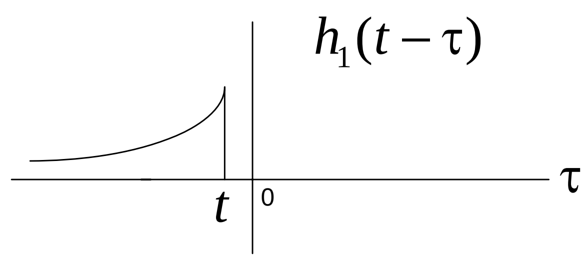
$y(t) = x(t) * h(t) = x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t) = y_1(t) + y_2(t)$

$y_1(t)$:

$$y_1(t) = \begin{cases} 0, & for \;\; t < 2 \\ \int_2^t 1 \cdot e^{-3(t-\tau)} d\tau, & for \;\; 2 < t < 5 \\ \int_2^5 1 \cdot e^{-3(t-\tau)} d\tau, & for \;\; t > 5 \end{cases}$$




$$= \begin{cases} 0, & for \;\; t < 2 \\ \frac{e^{-3t}}{3}\left(e^{3t} - e^{6}\right), & for \;\; 2 < t < 5 \\ \frac{e^{-3t}}{3}\left(e^{15} - e^{6}\right), & for \;\; t > 5 \end{cases}$$




$y_2(t)$:

$y_2(t) = \delta(t-2) * e^{-3t}u(t) = e^{-3(t-2)}u(t-2)$

**Question 4:** **/5**

Consider a causal LTI system whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n]-\frac{1}{2}y[n-1]=x[n]$$

(1) Find the impulse response of the system (without using any transform).

(2) Determine $y[n]$ using convolution if $x[n]=\left(\frac{1}{3}\right)^n u(n)$.

**Solution:**

**(1) when** $x[n]=\delta[n]$**,** $y[n]=h[n]$**, so we have**

$$h[n]-\frac{1}{2}h[n-1]=\delta[n] \text{ or } h[n]=\frac{1}{2}h[n-1]+\delta[n]$$

- $n=0,\ h[0]=\frac{1}{2}h[-1]+\delta[0]=\frac{1}{2}\times 0+1=1$

  **(at** $n=-1$**, no input is applied, so the output is zero or** $h[-1]=0$**)**
- $n=1, h[1]=\frac{1}{2}h[0]+\delta[1]=\frac{1}{2}\times 1+0=\frac{1}{2}$
- $n=2, h[2]=\frac{1}{2}h[1]+\delta[2]=\frac{1}{2}\times\frac{1}{2}+0=\left(\frac{1}{2}\right)^2$
- $n=3, h[3]=\frac{1}{2}h[2]+\delta[3]=\left(\frac{1}{2}\right)^2\times\frac{1}{2}+0=\left(\frac{1}{2}\right)^3$

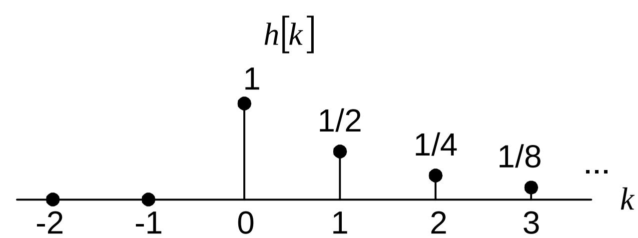
  **…**

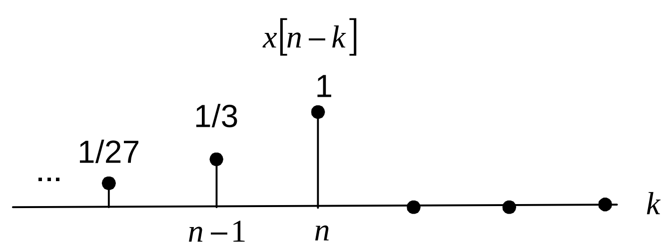
**Therefore, the impulse response is** $h[n]=\left(\frac{1}{2}\right)^n u[n]$

**(2)**

$$y[n]=\begin{cases}0, & \textit{for}\ \ n<0\\ \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{1}{3}\right)^{n-k}, & \textit{for}\ \ n\geq 0\end{cases}$$

$$=\begin{cases}0, & \textit{for}\ \ n<0\\ 2\left[\left(\frac{3}{2}\right)^{n+1}-1\right], & \textit{for}\ \ n\geq 0\end{cases}$$

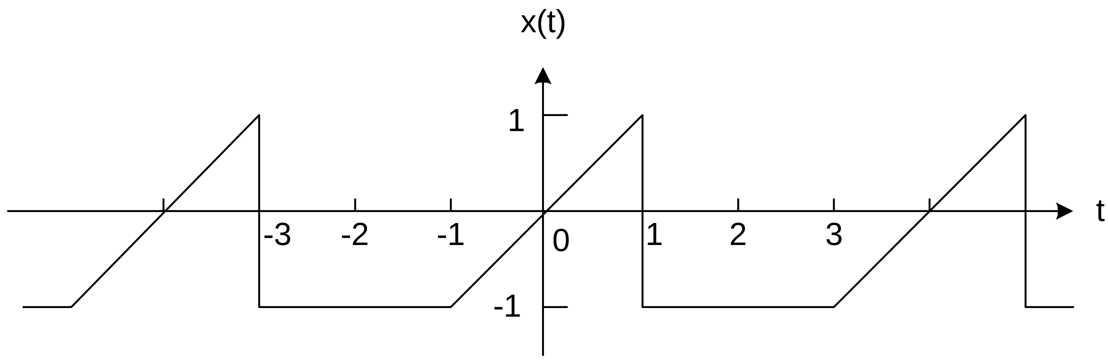


Initials _____

**Question 5:** **/4**

Calculate the Fourier series coefficient of the function shown below.




Solution 1: using the definition directly (note that $\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$ from the formula sheet)

The period is $T = 4$, $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$

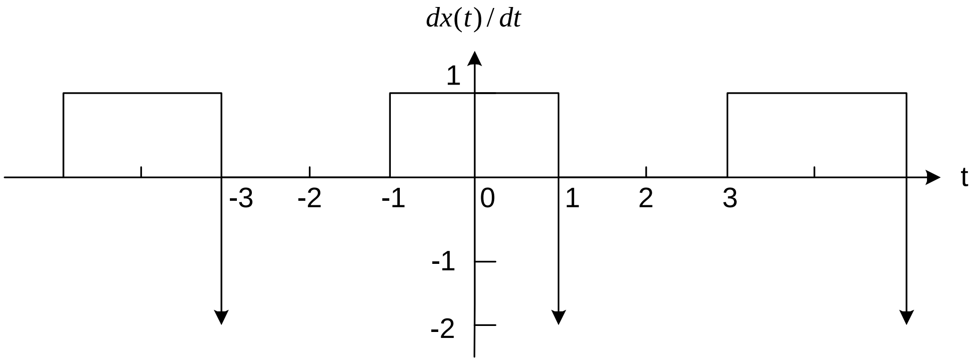
$$a_k = \frac{1}{N}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{N}\int_{-1}^{1} t\cdot e^{-jk\omega_0 t}dt + \frac{1}{N}\int_{1}^{3} 1\cdot e^{-jk\omega_0 t}dt$$

$$= \frac{1}{4}\int_{-1}^{1} te^{-jk\frac{\pi}{2}t}dt + \frac{1}{N}\int_{1}^{3} e^{-jk\frac{\pi}{2}t}d = \frac{1}{4}\left[\frac{e^{-jk\frac{\pi}{2}t}}{\left(-jk\frac{\pi}{2}\right)^2}\right]\left(-jk\frac{\pi}{2}t-1\right)\Bigg|_{-1}^{1} + \frac{e^{-jk\frac{\pi}{2}t}}{\left(-jk\frac{\pi}{2}\right)}\Bigg|_{1}^{3}$$

$$= \left\{\frac{1}{4}\left[\frac{e^{-jk\frac{\pi}{2}}}{\left(-jk\frac{\pi}{2}\right)^2}\right]\left(-jk\frac{\pi}{2}-1\right) - \frac{1}{4}\left[\frac{e^{jk\frac{\pi}{2}}}{\left(-jk\frac{\pi}{2}\right)^2}\right]\left(jk\frac{\pi}{2}-1\right)\right\} + \left\{\frac{e^{-jk\frac{\pi}{2}}}{\left(-jk\frac{\pi}{2}\right)} - \frac{e^{-jk\frac{\pi}{2}3}}{\left(-jk\frac{\pi}{2}\right)}\right\}$$

$$= \frac{1}{jk\omega_0}\left[\frac{\sin k\frac{\pi}{2}}{k\pi} + \frac{e^{-jk\frac{\pi}{2}t}}{4}\right]$$

Solution 2: Using the differentiation property: $x(t) \leftrightarrow a_k$ , $\dfrac{dx(t)}{dt} \leftrightarrow jk\omega_0 a_k$




Let $g(t) = dx(t)/dt$ , and $g(t) \leftrightarrow b_k + c_k$ , where $b_k$ is the Fourier series of the square wave and $c_k$ is the Fourier series of the impulse train function.

$$b_k = \frac{\sin k\omega_0 T_1}{k\pi} = \frac{\sin k\frac{\pi}{2}}{k\pi}, \text{ for } k \neq 0$$

$$c_k = \frac{1}{4}\int_{-1}^{3} \delta(t-1) e^{-jk\frac{\pi}{2}t} dt = \frac{e^{-jk\frac{\pi}{2}t}}{4}\int_{-1}^{3} \delta(t-1) dt = \frac{e^{-jk\frac{\pi}{2}t}}{4}$$

$$b_k + c_k = \frac{\sin k\frac{\pi}{2}}{k\pi} + \frac{e^{-jk\frac{\pi}{2}t}}{4}$$

$$a_k = \frac{1}{jk\omega_0}\left(b_k + c_k\right) = \frac{1}{jk\omega_0}\left[\frac{\sin k\frac{\pi}{2}}{k\pi} + \frac{e^{-jk\frac{\pi}{2}t}}{4}\right], \text{ for } k \neq 0$$

When $k = 0$ , $a_0 = \frac{1}{T} \times area\ within\ one\ period = \frac{1}{4} \times (-2) = -\frac{1}{2}$

**Question 6:** **/4**

Suppose we are given the following information about a signal $x[n]$

1. $x[n]$ is real and even
2. $x[n]$ has a period $N = 8$ and Fourier series coefficients $a_k$
3. $a_9 = 4$
4. $\frac{1}{8}\sum_{n=0}^{7}|x[n]|^2 = 32$

Show that $x[n] = A\cos(Bn + C)$, and specify the numerical values for the A, B and C.

**Solution:**

From Condition 3, $a_9 = a_{8+1} = a_1 = 4$

From condition 1, $x[n]$ is even, $a_k$ is also even, so we have $a_k = a_{-k}$, so $a_{-1} = 4$

From Condition 4, $\frac{1}{8}\sum_{n=0}^{7}|x[n]|^2 = \sum_{n=-1}^{6}|a_k|^2 = |a_{-1}|^2 + |a_0|^2 + ... + |a_6|^2 = 4^2 + |a_0|^2 + 4^2 + ... + |a_6|^2 = 32$

So $a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = 0$

Therefore $x[n] = a_{-1}e^{-j\frac{\pi}{4}n} + a_1 e^{j\frac{\pi}{4}n} = 4e^{-j\frac{\pi}{4}n} + 4e^{j\frac{\pi}{4}n} = 8\cos\left(\frac{\pi}{4}n\right)$

A = 8, B = $\frac{\pi}{4}$, C = 0.

Summations:

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \qquad |a| < 1$$

$$\sum_{k=n_1}^{\infty} a^k = \frac{a^{n_1}}{1-a} \qquad |a| < 1$$

$$\sum_{k=0}^{n_1} a^k = \frac{1-a^{n_1+1}}{1-a} \qquad a \neq 1$$

$$\sum_{k=n_1}^{n_2} a^k = \frac{a^{n_1} - a^{n_2+1}}{1-a} \qquad a \neq 1$$

Convolutions :

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Continuous-time Fourier Series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

$$a_0 = \frac{1}{T}\int_T x(t)dt \qquad \omega_0 = \frac{2\pi}{T}$$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k),$$

$$a_k = A_k e^{j\theta_k} \quad k \geq 1$$

Discrete-time Fourier Series:

$$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(\frac{2\pi}{N})n}$$

Other:

$$\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

**TABLE 3.1** PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

| Property | Section | Periodic Signal | Fourier Series Coefficients |
|---|---|---|---|
| | | $\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$ | $a_k$<br>$b_k$ |
| Linearity | 3.5.1 | $Ax(t) + By(t)$ | $Aa_k + Bb_k$ |
| Time Shifting | 3.5.2 | $x(t - t_0)$ | $a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$ |
| Frequency Shifting | | $e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$ | $a_{k-M}$ |
| Conjugation | 3.5.6 | $x^*(t)$ | $a^*_{-k}$ |
| Time Reversal | 3.5.3 | $x(-t)$ | $a_{-k}$ |
| Time Scaling | 3.5.4 | $x(\alpha t)$, $\alpha > 0$ (periodic with period $T/\alpha$) | $a_k$ |
| Periodic Convolution | | $\int_T x(\tau)y(t-\tau)d\tau$ | $Ta_k b_k$ |
| Multiplication | 3.5.5 | $x(t)y(t)$ | $\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$ |
| Differentiation | | $\frac{dx(t)}{dt}$ | $jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$ |
| Integration | | $\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$) | $\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$ |
| Conjugate Symmetry for Real Signals | 3.5.6 | $x(t)$ real | $\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$ |
| Real and Even Signals | 3.5.6 | $x(t)$ real and even | $a_k$ real and even |
| Real and Odd Signals | 3.5.6 | $x(t)$ real and odd | $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | | $\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{cases}$ | $\mathcal{Re}\{a_k\}$<br>$j\mathcal{Im}\{a_k\}$ |

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

**TABLE 3.2** PROPERTIES OF DISCRETE-TIME FOURIER SERIES

| Property | Periodic Signal | Fourier Series Coefficients |
| --- | --- | --- |
| | $\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period $N$ and fundamental frequency $\omega_0 = 2\pi/N$ | $\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period $N$ |
| Linearity | $Ax[n] + By[n]$ | $Aa_k + Bb_k$ |
| Time Shifting | $x[n - n_0]$ | $a_k e^{-jk(2\pi/N)n_0}$ |
| Frequency Shifting | $e^{jM(2\pi/N)n} x[n]$ | $a_{k-M}$ |
| Conjugation | $x^*[n]$ | $a^*_{-k}$ |
| Time Reversal | $x[-n]$ | $a_{-k}$ |
| Time Scaling | $x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period $mN$) | $\frac{1}{m} a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$ |
| Periodic Convolution | $\sum_{r=\langle N\rangle} x[r]y[n-r]$ | $Na_k b_k$ |
| Multiplication | $x[n]y[n]$ | $\sum_{l=\langle N\rangle} a_l b_{k-l}$ |
| First Difference | $x[n] - x[n-1]$ | $(1 - e^{-jk(2\pi/N)})a_k$ |
| Running Sum | $\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$ | $\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$ |
| Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$ |
| Real and Even Signals | $x[n]$ real and even | $a_k$ real and even |
| Real and Odd Signals | $x[n]$ real and odd | $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | $\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$ | $\Re e\{a_k\}$ <br> $j\Im m\{a_k\}$ |

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N\rangle} \lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$$